MATHEMATICS III NOTES

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Mathematics III Notes, First Edition

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Layout design by: AC Creations © 2013

The course content was prepared during Spring, 2013.
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Date 4/2/13

Differential Equations.

Section - I.

(only for review, no ques for components from this).

FIRST ORDER D.E. $\left(\frac{dy}{dx} = f(x,y)\right)$

★ TYPE - 1

If $f(x,y) = g(x) \cdot h(y)$ i.e variables are separable.

- Then, write this form:-

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

$$\Rightarrow \frac{dy}{h(y)} = g(x)\,dx.$$

- Then integrate both sides

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

This will give desired solution.

✲ TYPE-II
First order homogeneous D.E.

•• Homogeneous f^n of 2 variables:

If f^n $f(x,y)$ is homogeneous, then, if we ~~sub~~ replace $x \to tx$ & $y \to ty$, we get $f(tx, ty)$.

If we can express:-

$$f(tx,ty) = t^n f(x,y),$$

then $f(x,y)$ is a homogeneous f^n of degree n

ex:- ✓ x^2+y^2 : degree $= 2$.

✓ x^2+xy : degree $= 2$.

✓ $\sqrt{x^2+y^2}$: degree $= 1$.

$\to x \to tx$ & $y \to ty$

$\Rightarrow \sqrt{t^2x^2+t^2y^2} = t\sqrt{x^2+y^2} = t^{1}\sqrt{x^2+y^2}$

✓ $\dfrac{x}{y}$: degree $= 0$.

Similarly

✓ $\sin\left(\dfrac{x}{y}\right)$: degree $= 0$

Method of solving:-

Given a f^n $f(x,y) = \dfrac{dy}{dx} \longrightarrow (1)$.

S1: Check if $f(x,y)$ is homogeneous using above definition.

S2: Put $y = vx$.

Then, $\dfrac{dy}{dx} = v \cdot 1 + x\dfrac{dv}{dx} \longrightarrow (2)$.

(Product rule)

Note: Division of 2 homogeneous expressions gives a homogeneous expression.

S3: Compare (1) & (2), we get

By substituting $y = vx$

$$f(x, vx) = v + x\frac{dv}{dx}$$

S4: Now, separate the variables & solve.

using type - I. by integrⁿ

Q. Verify the eqⁿ is homogeneous & solve.

$$x^2 y' = 3(x^2+y^2)\tan^{-1}\left(\frac{y}{x}\right) + xy$$

$$\Rightarrow y' = 3\left(1+\left(\frac{y}{x}\right)^2\right)\tan^{-1}\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = 3\left(1+\left(\frac{y}{x}\right)^2\right)\tan^{-1}\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right) \quad \left(\div x^2\right)$$

We know $\frac{y}{x}$ = homogeneous of degree = 0 $\left(\because \frac{y}{x} \begin{matrix} y \to ty \\ x \to tx \end{matrix} = \frac{ty}{tx} = t^0\left(\frac{y}{x}\right)\right)$

So, Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow 3(1+v^2)\tan^{-1}(v) + \cancel{v} = \cancel{v} + x\frac{dv}{dx}$$

$$\Rightarrow 3(1+v^2)\tan^{-1}(v) = x\frac{dv}{dx}.$$

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Now, using variable separable method,

$$\frac{dx}{x} = \frac{dv}{3(1+v^2)\tan^{-1}(v)}$$

Integrating both sides

$$\int \frac{dx}{x} = \frac{1}{3}\int \left(\frac{1}{1+v^2}\right)\frac{1}{(\tan^{-1}v)}\,dv$$

Put $\tan^{-1}(v) = t$

$$\Rightarrow \frac{dv}{1+v^2} = dt$$

$$\Rightarrow \ln x = \frac{1}{3}\int \frac{dt}{t}$$

$$\Rightarrow \ln x = \frac{1}{3}\ln(t) + \ln c$$

$$\Rightarrow \ln x = \frac{1}{3}\ln(c\tan^{-1}(v))$$

$\frac{y}{x}$

$$\Rightarrow \ln x = \frac{1}{3}\ln\left(c\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\Rightarrow 3\ln x = \ln\left(c\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\Rightarrow (\ln x^3) = \ln\left(c\tan^{-1}\left(\frac{y}{x}\right)\right)$$

or $\boxed{x^3 = c\tan^{-1}\left(\frac{y}{x}\right)}$

Solution of D.E

~~or~~ $\frac{x^3}{c} = \tan^{-1}\left(\frac{y}{x}\right)$ or $y = x\tan\left(\frac{x^3}{c}\right)$

Ans

$\tan^{-1} v = m$

$\Rightarrow v = \tan m = \frac{y}{x}$

$\Rightarrow \sec m = \frac{\sqrt{x^2+y^2}}{x}$

$\Rightarrow m = \sec^{-1}\left(\frac{\sqrt{x^2+y^2}}{x}\right)$

Q $xy' = \sqrt{x^2+y^2}$

$\Rightarrow y' = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2}}$

$y' = \sqrt{1+\left(\frac{y}{x}\right)^2} \longrightarrow (1)$

(homogeneous in deg. = 0)

& put $y = vx$

or $v = \frac{y}{x}$ & $\frac{dy}{dx} = y' = v + x\frac{dv}{dx} \longrightarrow (2)$

Using (1) & (2)

$\Rightarrow \sqrt{1+v^2} = v + x\frac{dv}{dx}$

$\Rightarrow \sqrt{1+v^2} - v = x\frac{dv}{dx}$

$\Rightarrow \frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}-v} \longrightarrow (3)$

M1 $\quad v = \tan z \Rightarrow dv = \sec^2 z\, dz$

$\Rightarrow \int \frac{dx}{x} = \int \frac{\sec^2 z\, dz}{(\sec z - \tan z)}$

$\int \frac{dx}{x} = \int \frac{(\sec^2 z)(\sec z + \tan z)}{(\sec z - \tan z)(\sec z + \tan z)}\, dz$

$\Rightarrow \ln x = \int (\sec^2 z)(\sec z + \tan z)\, dz$

$\Rightarrow \ln x + \ln c = \int \sec^3 z\, dz + \int (\sec z)(\sec z \tan z)\, dz$

$\sec z = u$

$\Rightarrow \ln(cx) = \underbrace{\int \sec^3 z\, dz}_{\text{solve separately}} + \int u\, du$

- - - - lengthy method. (PTO)

M2 $\frac{dx}{x} = \frac{dv}{\sqrt{1+v^2} - v}$

$\Rightarrow \frac{dx}{x} = \frac{(\sqrt{1+v^2} + v)\,dv}{(1+v^2) - v^2}$

$\Rightarrow \int \frac{dx}{x} = \int \sqrt{1+v^2}\,dv + \int v\,dv$

$\because \left(\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}) \right)$

$\Rightarrow \frac{v}{2}\sqrt{1+v^2} + \frac{1}{2}\log(v+\sqrt{1+v^2}) + \frac{v^2}{2} = \log x + c$

$\Rightarrow v\sqrt{1+v^2} + \log(v+\sqrt{v^2+1}) + v^2 = 2\log x + c_1$

$\Rightarrow \frac{y}{x}\frac{\sqrt{x^2+y^2}}{x} + \log\left(\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}\right) + \frac{y^2}{x^2} = 2\log x + c_1$

$\Rightarrow y\sqrt{x^2+y^2} + x^2\log(y+\sqrt{x^2+y^2}) - x^2\log x + y^2$

$= 2x^2\log x + c_1 x^2$

$\Rightarrow y\sqrt{x^2+y^2} + x^2\log(y+\sqrt{x^2+y^2}) - 3x^2\log x + y^2 = c_1 x^2.$

Alternative solⁿ

Section-8

EXACT EQUATION

Consider an eq^n:-

$$f(x,y) = c \longrightarrow (1)$$

(like ~~s.t.~~, $x^2+y^2 = a^2$: family of curves)

Now, differentiating (1), we get.

$$df = 0 \rightarrow (2)$$

$$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \longrightarrow (3).$$

So, sol^n of D.E. (3) is (1).

~~eq~~ $\because$ f is a f^n of x & y, So, $\frac{\partial f}{\partial x}$ is also a f^n of x & y.

Hence, it can be written as.

General form of DE, say: $\boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$ (M, N taken arbitrarily) $\rightarrow (4)$

To find $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M_{(x,y)}$

& $\frac{\partial f}{\partial y} = N_{(x,y)}$ $\Big\} \longrightarrow (5)$

Using (5) in (4), we get

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \quad (\text{just like eq}^n \text{ (3)})$$

$$\Rightarrow \quad df = 0.$$

$$\Rightarrow \quad f(x,y) = c \quad (\text{we get req}^d \ f^n)$$

$\rightarrow$ sol^n of eq^n (4).

Note ✩ Given a D.E, if we can find a f^n $f(x,y)$ s.t $\frac{\partial f}{\partial x} = M_{(x,y)}$ & $\frac{\partial f}{\partial y} = N(x,y)$ for some $M(x,y)$ & $N(x,y)$ (for DE $M dx + N dy = 0$)

Then, the DE is an exact DE.

Procedure

S1. Check if the given D.E is exact or not.

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$

$\hookrightarrow$ necessary & sufficient cond^n to prove the D.E. $M(x,y)dx + N(x,y)dy = 0$. is an EXACT D.E.

S2) From (5), we know,

$\frac{\partial f}{\partial x} = M(x,y)$: Integrate partially w.r.t x (treat y as constt.)

$$\Rightarrow f = \int M_{(x,y)}\, \partial x + g(y)$$

$\rightarrow$ Integr^n constt is taken as a f^n of y

Aliter :- Use 2^{nd} eq^n from (5).

$\frac{\partial f}{\partial y} = N(x,y)$: Integrate partially w.r.t. y (treat x as constt).

So, $f = \int N(x,y)\,dy + g(x)$

Use any of the methods to get $f(x,y)$ Ans.

ex Consider eq^n :- $y\,dx + x\,dy = 0$.
Its of the form
$M\,dx + N\,dy = 0$

M1
Inspection

$\Rightarrow \int d(xy) = \int 0. \Rightarrow f(x,y)$

Sol^n of the form

$\Rightarrow xy = c$ $f(x,y) = c$.

M2
Using method (←)

$\left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy = 0$.

Then, $d\left(f(x,y)\right) = 0$

So, find f s.t

$\frac{\partial f}{\partial x} = y$ (OR) $\frac{\partial f}{\partial y} = x$

$\Rightarrow f = xy$ $\int \partial f = \int x\,\partial y$.

$\Rightarrow f_{(x,y)} = xy$.

$$\underset{M}{\frac{1}{y}}\,dx - \underset{N}{\frac{x}{y^2}}\,dy = 0$$

ex

Check for exactness.

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2} \qquad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$

They are same, so exact

Method 2 By steps

$$\frac{\partial f}{\partial x} = \frac{1}{y}$$

$$\Rightarrow \partial f = \frac{1}{y}\partial x$$

$$\Rightarrow f_{(x,y)} = \frac{x}{y}.$$

M1 Inspection • It is $d\left(\frac{x}{y}\right) = 0 \Rightarrow \frac{x}{y} = c$

→ It is $f(x,y)$

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2} = \frac{1}{y}dx - \frac{x}{y^2}dy$$

Q, Given $x^2y^3 = c$ find DE.

Idea: Differentiate both sides :- $d(x^2y^3) = d(c)$

$$\Rightarrow \underset{M(x,y)}{2xy^3}\,dx + \underset{N(x,y)}{3y^2x^2}\,dy = 0$$

Complete method done in part (4).

Q. Determine following D.E is exact & solve it.

(1) $(y + y\cos xy)\,dx + (x + x\cos xy)\,dy = 0$.

(2) $(\sin x \tan y + 1)\,dx - (\cos x \sec^2 y)\,dy = 0$

(3) $\frac{1}{y}\sin\left(\frac{x}{y}\right)dx + \frac{x}{y^2}\sin\left(\frac{x}{y}\right)dy = 0$

(4) $2x\left(1+\sqrt{x^2-y}\right)dx - \sqrt{x^2-y}\,dy = 0$

(5) $dx = \frac{y}{1-x^2y^2}\,dx + \frac{x}{1-x^2y}\,dy$

(1) $M_{(x,y)} = y + y\cos xy$, $N_{(x,y)} = x + x\cos xy$.

$\Rightarrow \frac{\partial M}{\partial y} = 1 + y(-\sin(xy))x + \cos xy$ $\quad\quad \frac{\partial N}{\partial x} = 1 + x(-\sin xy)y + \cos xy$

$= 1 - xy\sin xy + \cos xy$ $\quad\quad = 1 - xy\sin xy + \cos xy$

They are same, so, exact

So, solving, $\frac{\partial f}{\partial x} = y + y\cos xy$

$\Rightarrow \partial f = y\,\partial x + y\cos xy\,\partial x$

$\Rightarrow f = y\left[x + \frac{\sin xy}{y}\right] + g(y)$

$\Rightarrow f_{(x,y)} = xy + \sin xy + g(y)$

the f^n

Now, $g'(y) = 0, \Rightarrow \int g'(y)\,dy = \int 0\,dy$

$\Rightarrow g(y) = c.$

So, $f(x,y) = xy + \sin(xy) + c$ ✓

(2) $M(x,y) = \sin x \tan y + 1 \qquad N(x,y) = -\cos x \sec^2 y$

$\Rightarrow \dfrac{\partial M}{\partial y} = \sin x \sec^2 y \qquad \dfrac{\partial N}{\partial x} = +\sin x \sec^2 y$

Same, so, exact.

$$\frac{\partial f}{\partial x} = \sin x \tan y + 1$$

$$\int \partial f = \int (\sin x \tan y + 1)\,\partial x$$

$$\Rightarrow f(x,y) = \tan y(-\cos x) + x + g(y)$$

$$f(x,y) = -\cos x \tan y + x + g(y)$$

Now, $\dfrac{\partial f}{\partial y} = -\cos x \sec^2 y + g'(y) \longrightarrow (1)$

Comparing with $N(x,y)$, $g'(y) = 0$ & $g(y) = c$.

So, sol^n is $\underline{-\cos x \tan y + x = c_2}$.

(3) $M(x,y) = \dfrac{-1}{y}\sin\dfrac{x}{y} \qquad N(x,y) = \dfrac{x}{y^2}\sin\left(\dfrac{x}{y}\right)$

$\dfrac{\partial M}{\partial y} = \dfrac{-1}{y}\cos\left(\dfrac{x}{y}\right)\left(\dfrac{-x}{y^2}\right) + \sin\left(\dfrac{x}{y}\right)\left(\dfrac{1}{y^2}\right) \qquad \dfrac{\partial N}{\partial x} = \dfrac{x}{y^2}\cos\left(\dfrac{x}{y}\right)\left(\dfrac{1}{y}\right) + \sin\left(\dfrac{x}{y}\right)\left(\dfrac{1}{y^2}\right)$

$= \dfrac{x}{y^3}\cos\left(\dfrac{x}{y}\right) + \left(\dfrac{1}{y^2}\right)\sin\left(\dfrac{x}{y}\right) \qquad = \dfrac{x}{y^3}\cos\left(\dfrac{x}{y}\right) + \left(\dfrac{1}{y^2}\right)\sin\left(\dfrac{x}{y}\right)$

Same, so, exact

$$\frac{\partial f}{\partial x} = \frac{-1}{y}\sin\left(\frac{x}{y}\right)$$

$$\Rightarrow \int \partial f = \int \left(-\frac{1}{y}\right)\sin\left(\frac{x}{y}\right)\partial x$$

$$= \frac{\left(-\frac{1}{y}\right)\cos\left(\frac{x}{y}\right)}{\left(\frac{1}{y}\right)} = \cos\left(\frac{x}{y}\right) + g(y).$$

$\Rightarrow f(x,y) = \cos\left(\frac{x}{y}\right) + g(y)$

$\frac{\partial f}{\partial y} = -\sin\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + g'y$

$= \frac{x}{y^2}\sin\left(\frac{x}{y}\right) + g'(y) \longrightarrow (1)$

Comparing (1) with $N(x,y)$, we get

$\Rightarrow \frac{x}{y^2}\sin\left(\frac{x}{y}\right) + g'(y) = \frac{x}{y^2}\sin\left(\frac{x}{y}\right)$

$\Rightarrow g'(y) = 0, \Rightarrow g(y) = c$

So, $f(x,y) = \cos\left(\frac{x}{y}\right) + c$

So, $\int^n = f(x,y) = c_1$

or $\cos\left(\frac{x}{y}\right) + c = c_1$

$\Rightarrow \cos\left(\frac{x}{y}\right) = c_2$ Ans

(4) $M(x,y) = 2x\left(1+\sqrt{x^2-y}\right)$ $\quad N(x,y) = -\sqrt{x^2-y}$

$\frac{\partial M}{\partial y} = 0 + \left(\frac{-2x}{2\sqrt{x^2-y}}\right)$ $\quad \frac{\partial N}{\partial x} = \frac{-(2x)}{2\sqrt{x^2-y}}$

$= \frac{-x}{\sqrt{x^2-y}}$ $\quad \Rightarrow \frac{\partial N}{\partial x} = \frac{-x}{\sqrt{x^2-y}}$

Same, So, exact

$\frac{\partial f}{\partial x} = 2x\left(1+\sqrt{x^2-y}\right)$

$\Rightarrow \partial f = \left(2x + 2x\sqrt{x^2-y}\right)\partial x$

$\Rightarrow \int \partial f = \int 2x\,\partial x + \int 2x\sqrt{x^2-y}\,\partial x$

$x^2 - y = t \Rightarrow 2x\,dx = dt$

$\Rightarrow f = x^2 + \int t^{1/2}\,dt$

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$$\Rightarrow f = x^2 + \int t^{1/2}\, dt$$

$$= x^2 + \frac{t^{3/2}}{3/2} + g(y)$$ const. of integration

$$\Rightarrow f(x,y) = x^2 + \frac{2}{3}(x^2-y)^{3/2} + g(y)$$

Now, $g'(y) = 0$

$$\left[\because \frac{\partial}{\partial x} g(y) = 0\right]$$

$$\Rightarrow \int g'(y)\, dy = \int 0\, dy$$

$$\Rightarrow g(y) = c, \text{ a constant}$$

So, $f(x,y) = x^2 + \frac{2}{3}(x^2-y)^{3/2} + c$

or $d\left(x^2 + \frac{2}{3}(x^2-y)^{3/2} + c\right) = 0$

$$\Rightarrow x^2 + \frac{2}{3}(x^2-y)^{3/2} + c = c_1$$

or $x^2 + \frac{2}{3}(x^2-y)^{3/2} = c_2$

Hence, the f^n

(5) $dx = \frac{y}{1-x^2y^2}\,dx + \frac{x}{1-x^2y^2}\,dy$.

$\Rightarrow (1-x^2y^2-y)\,dx - x\,dy = 0$.

$M(x,y) = 1-y-x^2y^2 \qquad N(x,y) = -x$

$\Rightarrow \frac{\partial M}{\partial y} = -1-x^2 \qquad \frac{\partial N}{\partial x} = -1$

Not equal, so, not exact

$dx = \frac{1}{1-x^2y^2}(y\,dx + x\,dy)$ → It is exact DE

$\Rightarrow \frac{1}{1-x^2y^2}(d(xy)) = \frac{d(xy)}{1-(xy)^2}$

$\int dx = \int \frac{d(xy)}{1-(xy)^2}$.

$\int dx \equiv \int \frac{dm}{1-m^2} \quad ; m = xy$

$\Rightarrow x = \frac{1}{2}\log\left(\frac{1+m}{1-m}\right) + c$ (By formula)

$\Rightarrow x = \frac{1}{2}\log\left(\frac{1+xy}{1-xy}\right) + c$

Q. Check exactness & solve.

$$3x^2(1+\log y)\,dx + \left(\frac{x^3}{y} - 2y\right)dy = 0$$

$M(x,y) = 3x^2(1+\log y)$ $\quad N(x,y) = \frac{x^3}{y} - 2y$

$\Rightarrow \frac{\partial M}{\partial y} = 3x^2\left[0 + \frac{1}{y}\right]$ $\quad \frac{\partial N}{\partial x} = \frac{3x^2}{y}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{3x^2}{y}$ $\quad \frac{\partial N}{\partial x} = \frac{3x^2}{y}$

Same, so, exact.

Now

$$\frac{\partial f}{\partial x} = M(x,y), \quad \frac{\partial f}{\partial y} = N(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \left(\frac{x^3}{y} - 2y\right)$$

$$\Rightarrow \int \partial f = \int\left(\frac{x^3}{y} - 2y\right)\partial y$$

$$\Rightarrow \int \partial f = \left(x^3 \log y - y^2\right) + g(x)$$

$$\Rightarrow f(x,y) = x^3\log y - y^2 + g(x) \quad (1)$$

Now $\frac{\partial f(x,y)}{\partial x} = 3x^2(\log y) - 0 + g'(x)$

Comparing with $M(x,y)$

we get $g'(x) = 3x^2 \Rightarrow g(x) = x^3 + C_1$

Now, solution $= f(x,y) = c$.

$$\Rightarrow \left(x^3 \log y - y^2\right) + \left(x^3 + C_1\right) = c.$$

$\Rightarrow x^3 \log y - y^2 + x^3 = C_2$

or $x^3(1+\log y) - y^2 = C_2$

Solution

Q. $\frac{y\,dx + x\,dy}{1-x^2y^2} + x\,dx = 0$

$\Rightarrow \frac{1}{1-x^2y^2}(y\,dx + x\,dy) + x\,dx = 0$

$\Rightarrow \frac{1}{1-x^2y^2} d(xy) + x\,dx = 0$

$\Rightarrow \int -x\,dx = \int \frac{d(xy)}{1-(xy)^2}$

$\Rightarrow -\frac{x^2}{2} = \log\left(\frac{1+xy}{1-xy}\right) + C$

Just like before (formula)

$\Rightarrow \frac{x^2}{2} + \log\left(\frac{1+xy}{1-xy}\right) = C_1$

Q. $\frac{y\,dx - x\,dy}{(x+y)^2} + dy = dx$

$\Rightarrow \frac{\frac{y\,dx - x\,dy}{y^2}}{\left(\frac{x+y}{y}\right)^2} + dy = dx$

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$$\Rightarrow \frac{d\left(\frac{x}{y}\right)}{\left(1+\frac{x}{y}\right)^2} + dy = dx$$

let $\frac{x}{y} = m$

$$\Rightarrow \int \frac{dm}{(1+m)^2} + \int dy = \int dx$$

$$\Rightarrow \frac{-1}{(1+m)} + y = x + C$$

$$\Rightarrow \frac{-1}{1+\left(\frac{x}{y}\right)} + y = x + C$$

$$\Rightarrow \frac{-y}{x+y} + y = x + C$$

$$\Rightarrow y\left(1 - \frac{1}{x+y}\right) = x + C$$

or $y = \frac{y}{x+y} + x + C$. Ans

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Section - 9

INTEGRATING FACTORS

→ req^d to convert a Non-exact 1'st order D.E to an exact 1st order D.E.

i.e, [Integrating factor] × [Non exact D.E] = [Exact D.E]

So, we have an exact D.E now which can be solved, like done in previous section (Section-8)

eg $y\,dx + (x^2y - x)dy = 0 \longrightarrow (1)$

Here, $\underbrace{\frac{\partial M}{\partial y} = 1 \quad \& \quad \frac{\partial N}{\partial x}}_{\text{unequal. So, not exact}} = 2xy - 1.$

Suppose, $\times \left(\frac{1}{x^2}\right)$ both sides

new $M(x,y)$ 　　　　　　　　New $N(x,y)$

$\Rightarrow \left(\frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0 \longrightarrow (2)$

Now, $\underbrace{\frac{\partial M}{\partial y} = \frac{1}{x^2} \quad \& \quad \frac{\partial N}{\partial x}}_{\text{Same, So, they become exact Now}} = \frac{1}{x^2}.$

So, $\frac{1}{x^2}$ is the Integrating factor

Now, solve for $f(x,y)$ for (2). That is sol^n for (1)

☆ Procedure to find μ (Integrating factor)

We have our non exact D.E as:

$$M_{(x,y)}\,dx + N(x,y)\,dy = 0 \longrightarrow (1)$$

S1) Find $g(x) = \dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N(x,y)}$

only a f^n of x.

using this formula, if we get a f^n of both x & y, i.e, its not independent of y, then, use $h(y)$ formula

(or)

$$h(y) = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{-M(x,y)}$$

only a f^n of y.

———∞———

Q. Solve:-

(a) $(xy-1)\,dx + (x^2 - xy)\,dy = 0. \longrightarrow (1)$

(b) $e^x\,dx + (e^x \cot y + 2y\,\mathrm{cosec}\,y)\,dy = 0 \longrightarrow (2)$

(c) $(y\log y - 2xy)\,dx + (x+y)\,dy = 0 \longrightarrow (3)$

(d) $(3x^2 - y^2)\,dy - 2xy\,dx = 0. \longrightarrow (4)$

(a) $M(x,y) = xy - 1$ $\qquad N(x,y) = x^2 - xy$

$\dfrac{\partial M}{\partial y} = x$ $\qquad \dfrac{\partial N}{\partial x} = 2x - y.$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. So, its not exact.

Now, finding $g(x)$ (or $h(y)$)

$$g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N(x,y)} = \frac{x-(2x-y)}{x^2 - xy}$$

$$= \frac{-x+y}{x(x-y)}$$

$$= \frac{-\cancel{(x-y)}}{x\cancel{(x-y)}}$$

$$\Rightarrow g(x) = -\frac{1}{x}$$

Method ①: NOT valid everywhere.

$\times\, g(x)$ by ①, we get.

$$\Rightarrow -\frac{1}{x}\left[(xy-1)dx + (x^2-xy)dy\right] = -\frac{1}{x}(0)$$

$$\Rightarrow \underbrace{\left(-y+\frac{1}{x}\right)}_{M(x,y)}dx + \underbrace{(-x+y)}_{N(x,y)}dy = 0.$$

Now, $\frac{\partial M}{\partial y} = -1$ $\qquad$ $\frac{\partial N}{\partial x} = -1$

Same, so, exact.

Method 2: Valid everywhere.

Finding Integrating factor :-

$$\mu = \text{Integrating factor} = e^{\int g(x)dx} \left(\text{or } e^{\int h(y)dy}\right)$$

So, $IF = \mu = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log(1/x)}$

$= \frac{1}{x}$.

$\times \left(\frac{1}{x}\right)$ by (1), we get

$\Rightarrow \frac{1}{x}\left[(xy-1)dx + (x^2 - xy)dy\right] = 0$

$\Rightarrow \underbrace{\left(y - \frac{1}{x}\right)}_{M(x,y)} dx + \underbrace{(x-y)}_{N(x,y)}dy = 0 \longrightarrow A$

$\underbrace{\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1}_{\text{Same, so, exact.}}$

Now, $\frac{\partial f}{\partial y} = N(x,y)$

$\Rightarrow \frac{\partial f}{\partial y} = (x-y)$

$\Rightarrow \int \partial f = \int (x-y)\partial y$

$\Rightarrow f_{(x,y)} = xy - \frac{y^2}{2} + g(x)$

Some constt.

Now $\frac{\partial f}{\partial x} = y + g'(x) = M(x,y)$

$= y - \frac{1}{x}$

$\Rightarrow g'(x) = \frac{-1}{x}$.

$\Rightarrow g(x) = \log\left(\frac{1}{x}\right) + \log c$.

So solution is $f(x,y) = C_1$

$$\Rightarrow xy - \frac{y^2}{2} - \log\left(\frac{1}{x}\right) + \log(c) = C_1$$

$$\Rightarrow xy - \frac{y^2}{2} + \log\left(\frac{1}{x}\right) = C_2 .$$

Ans

Note :-
By multiplying eqⁿ (1) by μ, we get eqⁿ(A).
Here, the structure of eqⁿ (1) is not changing
(i.e, it remains of the form $Mdx + Ndy = 0$)
So, the solution of eqⁿ (A) will be the
solution of eqⁿ (1) .

(b) $e^x dx + (e^x \cot y + 2y \operatorname{cosec} y) dy = 0 \longrightarrow (2)$

$M(x,y) = e^x$ $\quad\quad$ $N(x,y) = e^x \cot y + 2y \operatorname{cosec} y$.

$\frac{\partial M}{\partial y} = 0$ $\quad\quad$ $\frac{\partial N}{\partial x} = e^x \cot y$.

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

So, eqⁿ (2) is non exact .

Now, finding $g(x)$ (or $h(y)$)

$$h(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M(x,y)} = \frac{0 - e^x \cot y}{-e^x}$$

$$\Rightarrow h(y) = \cot y$$

Now,
$$\mu = e^{\int h(y)dy}$$
$$= e^{\int \cot y\, dy}$$
$$= e^{\log(\sin y)}$$
$$\Rightarrow \mu = \sin y$$

$\times$ (sin y) by (1), we get

$$\underbrace{(e^x \sin y)}_{M(x,y)} dx + (\underbrace{e^x \cos y + 2y}_{N(x,y)}) dy = 0$$

$$\frac{\partial M}{\partial y} = e^x \cos y \quad . \quad \frac{\partial N}{\partial x} = e^x \cos y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So its exact

Now, $\frac{\partial f}{\partial x} = M(x,y)$

$$\Rightarrow \frac{\partial f}{\partial x} = e^x \sin y$$

$$\Rightarrow \int \partial f = \int e^x \sin y\, \partial x$$

$$\Rightarrow f(x,y) = e^x \sin y + g(y)$$

Now

$$\frac{\partial f}{\partial y} = e^x \cos y + g'(y) = N(x,y)$$
$$= e^x \cos y + 2y$$

$$\Rightarrow g'(y) = 2y$$
$$\Rightarrow g(y) = y^2 + C$$

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So, solution to (2) is

$$f(x,y) = C_1$$

$$\Rightarrow e^x \sin y + y^2 + C = C_1$$

$$\Rightarrow e^x \sin y + y^2 = C_2$$

Ans

(C) $\underbrace{(y \log y - 2xy)}_{M(x,y)} dx + \underbrace{(x+y)}_{N(x,y)} dy = 0 \longrightarrow$ (3)

$$\frac{\partial M}{\partial y} = \left[y\left(\frac{1}{y}\right) + \log y\right] - 2x \; ; \quad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So, eq^n (3) is not exact

Now, finding

$$h(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M(x,y)}$$

$$= \frac{(1 + \log y - 2x) - 1}{-y \log y + 2xy}$$

$$= \frac{\log y - 2x}{-y(\log y - 2x)} = \frac{-1}{y}$$

$$\Rightarrow h(y) = \frac{-1}{y}$$

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Finding IF $(\mu) = e^{\int h(y)dy}$

$$= e^{\int(-\frac{1}{y})dy}$$

$$= e^{\log(1/y)} = \left(\frac{1}{y}\right).$$

$(\times)$ $\left(\frac{1}{y}\right)$ by ③, we get

$$\left(\log y - 2x\right)dx + \left(\frac{x}{y} + 1\right)dy = 0 \longrightarrow (C)$$

eqn (c) has to be exact (Solve

So,

$$\frac{\partial f}{\partial y} = N(x, y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \left(\frac{x}{y} + 1\right)$$

$$\Rightarrow \int \partial f = \int\left(\frac{x}{y} + 1\right)\partial y$$

$$\Rightarrow f(x,y) = x\log y + y + g(x)$$

Now, $\frac{\partial f}{\partial x} = \log y + g'(x) = M(x,y)$

$$= \log y - 2x$$

$$\Rightarrow g'(x) = -2x$$

$$\Rightarrow g(x) = -x^2 + C$$

So, Solution is

$$x\log y + y - x^2 = C_2$$

(d) $(3x^2 - y^2)\,dy - 2xy\,dx = 0 \longrightarrow (4)$

$$\Rightarrow \underbrace{(-2xy)}_{M(x,y)}dx + \underbrace{(3x^2 - y^2)}_{N(x,y)}dy = 0$$

$$\frac{\partial M}{\partial y} = -2x \qquad \frac{\partial N}{\partial x} = 6x$$

Not same, so, not exact

Now,

$$h(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M(x,y)}$$

$$= \frac{-2x - 6x}{+2xy}$$

$$\Rightarrow h(y) = \frac{-8x}{+2xy} = \frac{4}{-y}.$$

So,

$$\mu = e^{\int \frac{4}{y}dy} = e^{-4\int \frac{dy}{y}}$$

$$\Rightarrow \mu = e^{(-4)\log y} = y^{-4}$$

$\times (y^4)$ by (4), we get

$$\left(-\frac{2x}{y^3}\right)dx + \left(\frac{3x^2}{y^4} - \frac{1}{y^2}\right)dy = 0 \rightarrow (D)$$

It must be exact (Solve & show)

So, $\frac{\partial f}{\partial x} = M(x,y)$

$$\Rightarrow \int \partial f = \int \left(\frac{-2x}{y^3}\right) \partial x$$

$$\Rightarrow f(x,y) = \frac{-1}{y^3}(x^2) + g(y)$$

$$f(x,y) = \frac{-x^2}{y^3} + g(y)$$

$$\frac{\partial f}{\partial y} = +\frac{3x^2}{y^4} + g'(y) = N(x,y)$$

$$= \frac{3x^2}{y^4} - \frac{1}{y^2}$$

$$\Rightarrow g'(y) = \frac{-1}{y^2}.$$

$$\Rightarrow g(y) = \frac{1}{y} + C.$$

So, solution is :-

$$\boxed{\frac{-x^2}{y^3} + \frac{1}{y} = C_2}$$ Ans

✮ RESULTS (exact differentials)

✓ $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2} = \frac{1}{y}dx - \frac{x}{y^2}dy$

✓ $d\left(\frac{y}{x}\right) = \frac{1}{x}dy - \frac{y}{x^2}dx$

✓ $d(xy) = x\,dy + y\,dx$

✓ $d(x^2+y^2) = 2(x\,dx + y\,dy)$

✓ $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{x^2+y^2} = -\,d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

✓ $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{xy}$

✓ $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{xy}$

→ proof :-

$$d\left(\log\left(\frac{y}{x}\right)\right) = \frac{1}{y/x}\,\frac{x\,dy - y\,dx}{x^2}$$

$$= \frac{x\,dy - y\,dx}{xy}$$

$$= -\,d\left(\log\left(\frac{x}{y}\right)\right)$$

Q

(a) $x\,dy - y\,dx = (1+y^2)\,dy$

(b) $x\,dy = (x^5 + x^3y^2 + y)\,dx$

(c) $(y+x)\,dy = (y-x)\,dx$

H.W (d) $x\,dy + y\,dx = \sqrt{xy}\,dy$

(e) $dy + \frac{y}{x}\,dx = \sin x\,dx$

(f) $x\,dy - y\,dx = x^2y^4(x\,dy + y\,dx)$

(g) $(y - xy^2)\,dx + (x + x^2y^2)\,dy = 0$

(a) $\underbrace{x\,dy - y\,dx}$ $= (1+y^2)\,dy$

↳ sth of the form $-\left(\frac{y\,dx - x\,dy}{y^2}\right)$

So, $\div\, y^2$.

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$$\Rightarrow \frac{x\,dy - y\,dx}{y^2} = \frac{(1+y^2)}{y^2}\,dy$$

$$\Rightarrow -\int d\left(\frac{x}{y}\right) = \int\left(\frac{1}{y^2}+1\right)dy$$

$$\Rightarrow -\frac{x}{y} = \frac{-1}{y} + y + c$$

$$\Rightarrow -\frac{x}{y} + \frac{1}{y} + y = c$$

$$\Rightarrow y + \frac{1}{y}(1-x) = c$$

(b) $x\,dy = (x^5 + x^3y^2 + y)\,dx$

$$= (x^5 + x^3y^2)\,dx + y\,dx$$

$$\Rightarrow x\,dy - y\,dx = (x^5 + x^3y^2)\,dx$$

$\div\ x^2$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = x(x^2 + y^2)\,dx$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2+y^2} = x^3\,dx$$

$$\Rightarrow \int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \int x^3\,dx$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{x^4}{4} + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^4}{4} = c$$ Ans

(c) $(y+x)\,dy = (y-x)\,dx$

$\Rightarrow y\,dy + x\,dy = y\,dx - x\,dx$

$\Rightarrow x\,dx + y\,dy = y\,dx - x\,dy$

$\div$ both sides by (x^2+y^2) ($\because$ we got like x [illegible] $\frac{1}{2}d(x^2+y^2)$

$= \dfrac{\frac{1}{2}d(x^2+y^2)}{x^2+y^2} = \dfrac{y\,dx - x\,dy}{x^2+y^2}$

$\Rightarrow \displaystyle\int \frac{d(x^2+y^2)}{2(x^2+y^2)} = \int d\tan^{-1}\left(\frac{x}{y}\right)$

$\Rightarrow \frac{1}{2}\log(x^2+y^2) = \tan^{-1}\left(\frac{x}{y}\right) + C$

Ans

(d) $x\,dy + y\,dx = \sqrt{xy}\,dy$

$\Rightarrow \dfrac{x\,dy + y\,dx}{\sqrt{xy}} = dy$

$\Rightarrow \displaystyle\int \frac{1}{\sqrt{xy}}\,d(xy) = \int dy$

$\Rightarrow 2\sqrt{xy} = y + C$

(e) $dy + \frac{y}{x}\,dx = \sin x\,dx$

$\Rightarrow x\,dy + y\,dx = x\sin x\,dx$

$\Rightarrow \int d(xy) = \int x\sin x\,dx$

$\Rightarrow xy = x(-\cos x) - \int(-\cos x)$

$xy = -x\cos x + \sin x + C$

$\Rightarrow xy + x\cos x - \sin x = C$

M2

$\int x\sin x\,dx$

$= \int \underbrace{x}_{u}\,\underbrace{d(-\cos x)}_{v}$ (now, apply ILATE)

(f) $x\,dy - y\,dx = x^2y^4(x\,dy + y\,dx)$

$\Rightarrow -\left[\frac{y\,dx - x\,dy}{y^2}\right] = x^2y^2\,d(xy)$

$\Rightarrow \int -d\left(\frac{x}{y}\right) = \int x^2y^2\,d(xy)$

$\Rightarrow -\frac{x}{y} = \frac{x^3y^3}{3} + C$

(g) $(y - xy^2)dx + (x + x^2y^2)dy = 0$

$\Rightarrow y\,dx + x\,dy = xy^2dx - x^2y^2dy$

$\Rightarrow \int \frac{d(xy)}{(xy)^2} = \int \frac{dx}{x} - \int dy$

(by $\div$ by x^2y^2 to make RHS integrable separately)

$\Rightarrow -\frac{1}{xy} = \log x - y + C$. Ans

Section - 10

LINEAR EQUATIONS : FIRST ORDER

- General form:-

$$\boxed{\frac{dy}{dx} + P_{(x)}\, y = Q_{(x)}} \longrightarrow (1)$$

Method.

S1) Find Integrating factor (IF)

$$\mu = IF = e^{\int P(x)\,dx}$$

S2) Solⁿ of (1) is given by:-

$$\boxed{y.(IF) = \int Q_{(x)}(IF)\,dx + c} \quad ☆$$

Q. (1) $y' + y = \dfrac{1}{1+e^{2x}}$

(2) $(1+x^2)dy + 2xy\,dx = \cot x\,dx$

(3) $(x \log x)\dfrac{dy}{dx} + y = 3x^3$

(4) $y' + y\cot x = 2x \operatorname{cosec} x$

(5) $(y - 2xy - x^2)dx + x^2 dy = 0$

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(1) $\frac{dy}{dx} + \underbrace{(1)}_{P(x)} y = \underbrace{\frac{1}{1+e^{2x}}}_{Q(x)}$

It's in Std. form

So,

$$IF = e^{\int P(x)dx} = e^{\int dx} = e^{x}.$$

Now, Sol^n:-

$$y \cdot e^{x} = \int \left(\frac{1}{1+e^{2x}}\right) \cdot \underbrace{e^{x}\, dx}_{\downarrow} + C$$

$$= \int \frac{1}{1+(e^{x})^2} (d(e^{x})) + C$$

$$= \int \frac{dt}{1+t^2} + C$$

$$t = e^{x}.$$

$$= \tan^{-1}(t) + C$$

$$\Rightarrow y\, e^{x} = \tan^{-1}(e^{x}) + C$$

Or $y = e^{-x} \tan^{-1}(e^{x}) + Ce^{-x}$ Ans

(b) $(1+x^2)dy + 2xy\,dx = \cot x\, dx$

To make to std form:-

$\div (1+x^2)dx$

$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{\cot x}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} + \underbrace{\left(\frac{2x}{1+x^2}\right)}_{P(x)} y = \underbrace{\frac{\cot x}{1+x^2}}_{Q(x)}$$

$$IF = e^{\int P(x)\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)}$$
$$= (1+x^2)$$

Sol^n:-

$$y\,(IF) = \int Q(x)(IF)\,dx$$

$$\Rightarrow y\,(1+x^2) = \int \frac{\cot x}{\cancel{(1+x^2)}}\cancel{(1+x^2)}\,dx + C$$

$$\Rightarrow y\,(1+x^2) = \log(\sin x) + C$$

$$\Rightarrow y = \frac{\log(\sin x)}{1+x^2} + \frac{C}{1+x^2}$$

(c) $(x \log x)y' + y = 3x^3$

$$\Rightarrow \frac{dy}{dx} + \underbrace{\left(\frac{1/x}{\log x}\right)}_{\to P(x)} y = \frac{3x^2}{\log x} \to Q(x)$$

$$IF = e^{\int P(x)\,dx} = e^{\int \frac{1/x}{\log x}dx} = e^{\log(\log x)}$$
$$= \log x$$

Sol^n

$$y(\log x) = \int \frac{(3x^2)}{\cancel{(\log x)}}\cancel{(\log x)}\,dx + C$$

$$\Rightarrow y(\log x) = x^3 + C$$

Ans

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(d) $y' + \underbrace{(\cot x)}_{P(x)} y = \underbrace{2x \operatorname{cosec} x}_{Q(x)}$

$IF = e^{\int P(x)\,dx} = e^{\log(\sin x)} = \sin x$

Sol^n

$y(\sin x) = \int 2x(\operatorname{cosec} x)\cdot(\sin x)\,dx + C$

$\Rightarrow y \sin x = x^2 + C$

(e) $(y - 2xy - x^2)\,dx + x^2\,dy = 0$

$\div x^2\,dx$

$\Rightarrow \frac{dy}{dx} + \underbrace{\left(\frac{1-2x}{x^2}\right)}_{P(x)} y = \underbrace{1}_{Q(x)}$

$IF = e^{\int P(x)\,dx} = e^{\int \frac{1-2x}{x^2}dx} = e^{\left(\frac{-1}{x}\right) - \log(x^2)}$

$= \frac{e^{-1/x}}{e^{\log(x^2)}} = \frac{1}{e^{1/x}\cdot x^2}$

Sol^n

$y(IF) = \int Q(x)(IF)\,dx$

$\Rightarrow y\left(\frac{1}{x^2 e^{1/x}}\right) = \int \frac{1}{x^2 e^{1/x}}\,dx$

$= -\int \frac{-1/x^2}{e^{1/x}}\,dx$

$1/x = t$

$\Rightarrow -1/x^2\,dx = d$

$\Rightarrow y\left(\frac{1}{x^2 e^{1/x}}\right) = -\int \frac{d(1/x)}{e^{(1/x)}}$

or $\frac{-1}{x^2}dx = d($

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$$\Rightarrow y\left(\frac{1}{x^2 e^{1/x}}\right) = \left(e^{-1/x}\right) + c$$

$$\Rightarrow y\left(\frac{1}{x^2 e^{1/x}}\right) = e^{-1/x} + c$$

$$\Rightarrow y = x^2 + c x^2 e^{1/x}$$ Ans

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Section - 11

REDUCTION OF ORDER

given 2nd order DE $\xrightarrow{\text{Reduce to}}$ 2 1'st order DE
↓ solve both DEs

General Idea:
Split the general 2nd order D.E. $F(x, y, y', y'')$ ... (1)
into two 1'st order D.E. & solve them. This is possible for 2 special types of 2nd order D.Es.

TYPE-I : Dependent variable missing in (1). (no 'y' terms)

So, eqn (1) reduces to the form:

$$F(x, y', y'') = 0 \quad \text{... (2)}$$; x : Independent var.
y : Dependent var.

To solve (2), use substitution :-

Put $y' = p$

$\Rightarrow y'' = \dfrac{dp}{dx}$

$\Rightarrow$ (2) becomes $F(x, p, p') = 0$ ... (3).

So, (3) is a 1'st order D.E in x & p.
Solve (3) & find p in terms of x.

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Now, since $p = \frac{dy}{dx}$, substitute the value of p in the above sol^n & solve it again to get the sol^n of ②.

$\frac{dy}{dx} =$

TYPE-II : Independent variable missing in ①
(no 'x' terms)

So, eq^n ① reduces to :-

$$F(y, y', y'') = 0 \longrightarrow ③$$

Again, put $y' = p \Rightarrow y'' = \cancel{\frac{dp}{dx}}$ → it'll get introduced (not given initially)

$\Rightarrow \frac{dy}{dx} = p$

So, eq^n ③ reduces to

So, Put $y'' = \frac{dp}{dy} \cdot \frac{dy}{dx}$

$= p \cdot \frac{dp}{dy}$

$$F\left(y, p, p\frac{dp}{dy}\right) = 0 \longrightarrow ④$$

So, ④ is a 1'st order D.E in p & y.
dependent ← p; y → Independent

So, solve ④ & find p in terms of y.
Again, substitute p $= dy/dx$ in above sol^n & solve it.

Q 1) $yy'' + (y')^2 = 0 \longrightarrow$ Type - II
2) $xy'' = y' + (y')^3 \longrightarrow$ Type - I

Ans - 1) $yy'' + (y')^2 = 0. \longrightarrow (1)$

Here, ∃ no x terms. So, type - II

So, put $y' = p$

$$\Rightarrow y'' = p\frac{dp}{dy}$$

Now, substitute in (1)

$$\Rightarrow y\left[p\frac{dp}{dy}\right] + p^2 = 0$$

$\div\ py$

$$\Rightarrow \frac{dp}{dy} + \left(\frac{1}{y}\right)p = 0 \longrightarrow (2)$$

eqn (2) is 1'st order DE

$$\Rightarrow \frac{dp}{dy} = -\frac{p}{y}$$

$$\Rightarrow \frac{dp}{p} = -\frac{dy}{y}$$

$$\Rightarrow \log p = \log\left(\frac{1}{y}\right) + \log c$$

$$\Rightarrow p = \frac{c}{y} \rightarrow (3)$$

Now, $p = \frac{dy}{dx} = \frac{c}{y}$.

$$\Rightarrow \int y\,dy = \int c\,dx$$

$$\Rightarrow \frac{y^2}{2} = cx + c.$$

$\Rightarrow y^2 = 2cx + 2c$

Ans

2) $xy'' = y' + (y')^3 \longrightarrow (1)$

Here, $\exists$ no y term. So, type - I

Now, put $y' = p$

$\Rightarrow y'' = \dfrac{dp}{dx}$

So eq^n (1) reduces to

$$x\frac{dp}{dx} = p + p^3 \longrightarrow (2)$$

$$\Rightarrow \int \frac{dp}{p+p^3} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dp}{p(1+p^2)} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{(1+p^2-p^2)dp}{p(1+p^2)} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{\cancel{(1+p^2)}\,dp}{p\cancel{(1+p^2)}} - \frac{1}{2}\int \frac{2p^{\cancel{2}}\,dp}{\cancel{p}(1+p^2)} = \int \frac{dx}{x}$$

$$\left(\begin{aligned} 1+p^2 &= t \\ \Rightarrow 2p\,dp &= dt \end{aligned}\right)$$

$$\Rightarrow \log p - \frac{1}{2}\log(1+p^2) = \log x + c$$

$$\Rightarrow \log p - \log\sqrt{1+p^2} = \log x + \log c$$

$$\Rightarrow \log\left(\frac{p}{\sqrt{1+p^2}}\right) = \log cx$$

$$\Rightarrow \frac{p}{\sqrt{1+p^2}} = cx$$

$$\Rightarrow p^2 = c^2x^2(1+p^2)$$

$$\Rightarrow p^2(1-c^2x^2) = c^2x^2$$

$$\Rightarrow p^2 = \frac{c^2x^2}{1-c^2x^2}$$

$$\Rightarrow p = \frac{cx}{\sqrt{1-c^2x^2}} \longrightarrow (3)$$

(Solⁿ of (2))

Putting $p = \frac{dy}{dx}$, solving

$$\Rightarrow \frac{dy}{dx} = \frac{cx}{\sqrt{1-c^2x^2}}$$

$$\Rightarrow \int dy = \int \frac{cx}{\sqrt{1-c^2x^2}}\,dx$$

$$1-c^2x^2 = t$$

$$\Rightarrow -2c^2x\,dx = dt$$

$$\Rightarrow y = \frac{1}{-2c}\int \frac{-2c^2x\,dx}{\sqrt{1-c^2x^2}}$$

$$\Rightarrow y = \frac{1}{-2c}\int \frac{dt}{\sqrt{t}}$$

$$= \left(\frac{-1}{2c}\right)\left[\frac{t^{1/2}}{1/2}\right] + C_1$$

$$\Rightarrow y = \frac{-1}{c}\sqrt{1-c^2x^2} + C_1$$

or $c(y-c_1) = -\sqrt{1-c^2x^2}$

Squaring both sides

$$\Rightarrow c^2(y-c_1)^2 = (1-c^2x^2)$$

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$\Rightarrow (y-c_1)^2 = \left(\frac{1}{c^2} - x^2\right)$

$\Rightarrow (y-c_1)^2 = (c_2^2 - x^2)$

$\Rightarrow x^2 + (y-c_1)^2 = c_2^2$. Ans

→ circle family with centre $(0, c_1)$ radius $= c_2$.

→ representⁿ of the given DE → (1)

→ eliminating c_1 & c_2, can give the DE

Q. Solve

(a) $xy'' + y' = 4x$

(b) $yy'' - (y')^2 = 0$

(a) $xy'' + y' = 4x \longrightarrow (1)$

No 'y' terms $\Rightarrow$ Type - I

So, Put $y' = p$

& $y'' = \frac{dp}{dx}$.

$\Rightarrow x\frac{dp}{dx} + p = 4x \longrightarrow (2)$

$\Rightarrow \frac{dp}{dx} + \underbrace{\left(\frac{1}{x}\right)}_{P(x)} p = \underbrace{4}_{Q(x)}$.

Finding IF

$\mu = e^{\int P(x)dx} = e^{\int \frac{1}{x} dx} = x$

So, solving

$$p(IF) = \int Q(IF)\,dx$$

$$\Rightarrow p(x) = \int 4(x)\,dx$$

$$\Rightarrow xp = 2x^2 + C \longrightarrow (3) \quad (\text{Sol}^n \text{ of } (2))$$

Now, putting $p = \frac{dy}{dx}$

$$\Rightarrow x\frac{dy}{dx} = 2x^2 + C$$

$$\Rightarrow \frac{dy}{dx} = 2x + \frac{C}{x}$$

$$\Rightarrow \int dy = \int\left(2x + \frac{C}{x}\right)dx$$

$$\Rightarrow y = x^2 + C\log x + C_1$$

$$\Rightarrow (y - C_1) = x^2 + C\log(x)$$

Ans.

(b) $yy'' - (y')^2 = 0 \longrightarrow (1)$

no x term $\Rightarrow$ Type II

Now, put $y' = p$

$$y'' = p\frac{dp}{dy}$$

Using in (1)

$\Rightarrow y\left(p\frac{dp}{dy}\right) - p^2 = 0$

$\Rightarrow$

$\div\, py$

$\Rightarrow \frac{dp}{dy} - \left(\frac{1}{y}\right)p = 0$

$\cancel{P(x)}$ $\cancel{Q(x)}$

$IF = e^{\int P(y)\,dy} = y$

Sol^n :-

$p(IF) = \int Q(IF)\,dy$

$\Rightarrow p(y) = \int (0)(y)\,dy$

$\Rightarrow py = 0$

Now, put $\frac{dy}{dx} = p$

$\Rightarrow \frac{dy}{dx}\cdot y = 0$

not valid because substituting $\frac{1}{y}$ as $P(x)$ (like done here) is not possible

$\Rightarrow \frac{dp}{dy} = \frac{p}{y}$

$\Rightarrow \log p = \log y + \log c$

$\Rightarrow p = cy$

$\Rightarrow \frac{dy}{dx} = cy \Rightarrow \frac{dy}{y} = c\,dx \Rightarrow \log y = cx + c_1$

$\Rightarrow y = e^{cx + c_1}$

$\Rightarrow y = e^{cx} \cdot c_2$

$\Rightarrow y = c_2 e^{cx}$ Ans

Q. Find the specified particular solⁿ :-

$$yy'' = y^2 y' + (y')^2 \; ; \; y = \frac{-1}{2}, y' = 1 \text{ when } x = 0 \quad \text{—— (1)}$$

Here, no x terms. So type - II

Put $y' = p$

& $y'' = p\frac{dp}{dy}$

So, $y\left(p\frac{dp}{dy}\right) = y^2 p + p^2$

$\div py$ both sides

$\Rightarrow \frac{dp}{dy} = y + \frac{p}{y}$

$\Rightarrow \frac{dp}{dy} - \underbrace{\left(\frac{1}{y}\right)}_{P(y)} p = \underbrace{y}_{Q(y)}$

$IF = e^{-\int \frac{1}{y} dy}$

$= e^{-\log y} = \frac{1}{y}$

So $\int P\,(IF) = \int Q\,[IF]\,dy$

$\Rightarrow \frac{P}{y} = \int y\left(\frac{1}{y}\right) dy$

$\Rightarrow \frac{P}{y} = \int dy$

$\Rightarrow \frac{P}{y} = y + c.$

$\Rightarrow P = y^2 + cy$

Put $P = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = (y^2 + cy) \longrightarrow (2)$

$\Rightarrow \frac{dy}{y^2 + cy} = dx$

$\Rightarrow \frac{1}{c}\int \frac{c\,dy}{y(y+c)} = \int dx$

$\Rightarrow \frac{1}{c}\int \frac{(c+y-y)\,dy}{y(y+c)} = \int dx$

$\Rightarrow \frac{1}{c}\left[\int \frac{dy}{y} - \int \frac{dy}{c+y}\right] = x + c_1$

$\Rightarrow \frac{1}{c}\left[\log y - \log(c+y)\right] = x + c_1$

$\Rightarrow \log\left(\frac{y}{c+y}\right) = cx + cc_1 = c_2 \ldots$

$\Rightarrow \frac{y}{c+y} = e^{cx+c_2}$

$y = (c+y)\,e^{cx+c_2}$

$$\Rightarrow \frac{y}{y+c} = e^{cx} \cdot e^{c_2}$$

$$\Rightarrow \frac{y}{y+c} = c_3 e^{cx} \longrightarrow (4)$$

(4) is a general solⁿ of (1)

Now, finding particular solⁿ :-

$$\Rightarrow \frac{y+c-c}{y+c} = c_3 e^{cx}$$

$$\Rightarrow 1 - \frac{c}{y+c} = c_3 e^{cx}$$

Differentiating w.r.t x both sides

$$\Rightarrow +\cancel{c}\left[\frac{1}{y+c}\right]^2 \frac{dy}{dx} = \cancel{c}\, c_3 e^{cx}$$

$$\Rightarrow p\left(\frac{1}{y+c}\right)^2 = c_3 e^{cx} \longrightarrow (5)$$

Using initial condⁿ in (4) & (5), we get

in (4)

Put $x = 0$

$$\Rightarrow \frac{(-1/2)}{\left(-\frac{1}{2}\right)+c} = c_3 e^0 \Rightarrow \frac{-1}{-1+2c} = c_3$$

$$\Rightarrow c_3 = \frac{1}{1-2c} \longrightarrow (6)$$

in (5)

$$(1)\left[\frac{1}{(-1/2)+c}\right]^2 = c_3 e^0$$

$$\Rightarrow \left[\frac{2}{-1+2c}\right]^2 = c_3 \longrightarrow (7)$$

From 6 & 7

$$\frac{1}{1-2c} = \frac{4}{(1-2c)^2}$$

$\Rightarrow 4 = 1-2c$

$\Rightarrow 2c = -3$

$\Rightarrow c = -\frac{3}{2} \rightarrow 8$

Using 8 in 6

$\Rightarrow c_3 = \frac{1}{1-(-3)}$

$\Rightarrow c_3 = \frac{1}{4} \rightarrow 9$

So, using 8 & 9 in 4

$\Rightarrow \frac{y}{y-\frac{3}{2}} = \frac{1}{4} e^{-\frac{3x}{2}}$

$\Rightarrow \frac{2y}{2y-3} = \frac{1}{8} e^{-\frac{3x}{2}}$: Particular soln

Ans

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Section - 10 (Continuation)

BERNOULLI'S EQN.

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$: General form

- $n \in \mathbb{Z}$
- eq^n becomes a linear eq^n when $n=0$ or $n=1$
- So, it behaves as Bernoulli's eq^n at all values of n except 0 or 1

S1 → Substitution to make it a linear eq^n :- $\boxed{z = y^{1-n}}$

S2 → Now, having the linear eq^n in z & x, it can be solved.

S3 → Then, substitute value of z & get sol^n of original eq^n.

Q. Solve :-

(a) $xy^2y' + y^3 = x\cos x$

(b) $x\,dy + y\,dx = xy^2\,dx$

(a) $xy^2y' + y^3 = x\cos x \longrightarrow (1)$

$\div\ xy^2$

$\Rightarrow y' + \left(\frac{1}{x}\right)y = \cos x\left(\frac{1}{y^2}\right)$

It is of the form of Bernoulli's eq^n

Comparing with general form,

$$P(x) = \frac{1}{x}, \quad Q(x) = \cos x, \quad n = -2$$

Now, substituting $z = y^{1-(-2)} \Rightarrow z = y^3$

Now

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= \left[\frac{d}{dy}(y^3)\right] \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dz}{dx} = 3y^2 \frac{dy}{dx} = 3y^2 y'$$

$$\text{So, } y^2 \frac{dy}{dx} = \frac{1}{3}\frac{dz}{dx} \rightarrow (2)$$

Using (2) in (1)

$$\Rightarrow x\left[\frac{1}{3}\frac{dz}{dx}\right] + z = x\cos x$$

$$\Rightarrow \frac{dz}{dx} + \left(\frac{3}{x}\right)z = 3\cos x \rightarrow (3)$$

It is a regular linear eq^n in z & x

Now,

$$P(x) = \frac{3}{x}, \quad Q(x) = 3\cos x$$

$$IF = e^{\int \frac{3}{x}dx} = \underline{\underline{x^3}}.$$

Now sol^n:-

$$z(IF) = \int Q(IF)\,dx$$

$$\Rightarrow z(x^3) = \int \underbrace{(3\cos x)}_{v}\underbrace{(x^3)}_{u}dx + C$$

★ Bernoulli's formula :-

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$$

$\int v \to v_1$, $\int v_1 \to v_2$, $\int v_2 \to v_3$

$$= 3\left[x^3(\sin x) - (3x^2)(-\cos x) + 6x(-\sin x) - 6(\cos x) + 0\right]$$

$$\Rightarrow z(x^3) = 3\left[x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x\right]$$

$$\Rightarrow z = \frac{3}{x^3}\left[\sin x(x^3 - 6x) + \cos x(3x^2 - 6)\right] \longrightarrow (4)$$

Now, $z = y^3$.

Using this in eqn (4),

$$\Rightarrow y^3 = \frac{3}{x^3}\left[\sin x(x^3 - 6x) + \cos x(3x^2 - 6)\right]$$

Soln ✓

(b) $x\,dy + y\,dx = xy^2\,dx$

$\div\ x\,dx$, both sides

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x}\right)y = (1)y^2 \longrightarrow (1)$$

Its Bernoulli's eq^n :- $n = 2$

So, $z = y^{1-(2)} = y^{-1} = \frac{1}{y}$

$$\Rightarrow \boxed{z = \frac{1}{y}} \Rightarrow \frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx}$$

$$\frac{dz}{dx} = \left(\frac{-1}{y^2}\right)\frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = (-y^2)\frac{dz}{dx}}$$

Using them in (1)

$$\Rightarrow -y^2\frac{dz}{dx} + \frac{y}{x} = y^2$$

$\Rightarrow\ \div(-y^2)$

$$\Rightarrow \frac{dz}{dx} + \left(\frac{-1}{x}\right)\left(\frac{1}{y}\right) = (-1)$$

$$\Rightarrow \frac{dz}{dx} \underbrace{- \frac{1}{x}}_{P(x)}(z) = \underbrace{(-1)}_{(Q(x))}$$: A linear eq^n

$$IF = e^{\int P(x)\,dx} = e^{\int \frac{-1}{x}dx} = \left(\frac{1}{x}\right)$$

Solving it :-

$$z(IF) = \int Q(IF)\,dx + c$$

$$\Rightarrow z\left(\frac{1}{x}\right) = \int (-1)\left(\frac{1}{x}\right) dx + c$$

$$\Rightarrow \frac{z}{x} = -\log x + \log C$$

$$\Rightarrow \frac{z}{x} = \log \frac{c}{x}$$

$$z = \frac{1}{y}$$

$$\Rightarrow \frac{1}{xy} = \log \frac{c}{x}$$

$$\Rightarrow y = \frac{1}{x \log\left(\frac{c}{x}\right)} \quad \underline{\underline{Sol^n}}$$

In usual notation dy/dx,

Note Normally, x is independent var. & y is dependent var. But, sometimes, if $\exists$ lot of y terms, then, try solving by taking :-

y as independent var. &

x as dependent var.

i.e, convert the DE. of the form :-

$$\boxed{\frac{dx}{dy} + P(y)x = Q(y)}$$

So, for Bernoulli's eqⁿ, it becomes

$$\boxed{\frac{dx}{dy} + P(y)\,x = Q(y)\,x^n.}$$

Q. $xy' + 2 = x^3(y-1)y'$

$\div x$

$\Rightarrow y' + \frac{2}{x} = x^2(y-1)y'$

$\Rightarrow y'(1 + x^2 - x^2y) + \frac{2}{x} = 0$

$\Rightarrow 1 - x^2(y-1) = -\frac{2}{x}\left(\frac{dx}{dy}\right)$

$\Rightarrow \frac{2}{x}\frac{dx}{dy} - x^2(y-1) = -1$

$\Rightarrow \frac{2}{x}\frac{dx}{dy} = x^2(y-1) - 1$

$\Rightarrow \frac{dx}{dy} = \frac{x^3(y-1)}{2} - \frac{x}{2}$

$\Rightarrow \frac{dx}{dy} = \frac{x^3y}{2} - \frac{x^3}{2} - \frac{x}{2}.$

$$\Rightarrow \underbrace{\frac{dx}{dy} + \frac{x}{2} = x^3\left(\frac{y-1}{2}\right)}_{\text{Bernoulli's form.}} \quad \text{—— (1)}$$

$P(y) = \frac{1}{2},\ Q(y) = \frac{y-1}{2},\ n = 3.$

So, $z = x^{1-n} = x^{1-(3)} = \frac{1}{x^2}$

$$\Rightarrow \boxed{z = \frac{1}{x^2}} \longrightarrow (2)$$

$$\Rightarrow \frac{dz}{dy} = \frac{dz}{dx}\cdot\frac{dx}{dy}$$

$$= \left(\frac{-2}{x^3}\right)\left(\frac{dx}{dy}\right)$$

$$\Rightarrow \frac{dx}{dy} = \left(\frac{-x^3}{2}\right)\frac{dz}{dy} \longrightarrow (3)$$

Using (2) & (3) in (1)

$$\Rightarrow \left(-\frac{x^3}{2}\right)\frac{dz}{dy} + \frac{x}{2} = x^3\left(\frac{y-1}{2}\right)$$

$$\div \left(\frac{-x^3}{2}\right)$$

$$\Rightarrow \frac{dz}{dy} - \left(\frac{1}{x^2}\right) = 1-y$$

$$\Rightarrow \frac{dz}{dy} + y = \left(1 + \underbrace{\frac{1}{x^2}}_{z}\right)$$

$$\Rightarrow \frac{dz}{dy} + y = 1 + z$$

$$\Rightarrow \frac{dz}{dy} \underbrace{-(1)}_{P(y)} z = \underbrace{(1-y)}_{Q(y)}$$

$$IF = e^{\int -dy} = e^{-y}$$

Sol^n $\quad z(IF) = \int Q(IF)\,dy + C$

$$\Rightarrow z e^{-y} = \int \underset{u}{(1-y)}\,\underset{v}{(e^{-y})}\,dy + C$$

Applying Bernoulli's formula

$$\Rightarrow z e^{-y} = \left[(1-y)(-e^{-y}) - (-1)(e^{-y}) + 0\right] + C$$

$$\Rightarrow z e^{-y} = (y-1)e^{-y} + e^{-y} + C$$

$$\Rightarrow z e^{-y} = y e^{-y} + C$$

$$\Rightarrow z = y + c e^{y} \quad (\times e^{y})$$

Now, $z = \dfrac{1}{x^2}$

$$\Rightarrow \frac{1}{x^2} = y + c e^{y}$$

Sol^n ✓

HW Q. $y - xy' = y' y^2 e^{y}$ $\quad$ (Ans. $x = ye^{y} + cy$)

Lot of y terms. So, better in terms of $\dfrac{dx}{dy}$.

$$y - xy' = y'y^2e^y$$

$$\Rightarrow \frac{dy}{dx}\left(y^2e^y + x\right) = y$$

$$\Rightarrow \frac{dx}{dy} = ye^y + \left(\frac{1}{y}\right)x$$

$$\Rightarrow \frac{dx}{dy} + \underbrace{\left(\frac{-1}{y}\right)}_{P(y)}x = \underbrace{ye^y}_{Q(y)}$$: It's a linear eq^n

$$IF = e^{\int P(y)\,dy}$$

$$= \frac{1}{y}$$

Sol^n

$$x(IF) = \int Q\,(IF)\,dy$$

$$\Rightarrow x\left(\frac{1}{y}\right) = \int (\cancel{y}e^y)\left(\frac{1}{\cancel{y}}\right)dy$$

$$\Rightarrow \frac{x}{y} = e^y + c$$

$$x = ye^y + cy$$

Ans

end of 1'st order models.

§ Second Order D.E.

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✯ General 2nd order DE.

Std. form.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

or, $$y'' + P(x)y' + Q(x)y = R(x) \longrightarrow (1)$$

→ If $R(x) = 0$, eqⁿ (1) becomes

$$\boxed{y'' + P(x)y' + Q(x)y = 0} \longrightarrow (2)$$

- eqⁿ (1) is called COMPLETE eqⁿ of eqⁿ (1) (Non-homogeneous eqⁿ)
- eqⁿ (2) is called REDUCED eqⁿ (Homogeneous eqⁿ)

✯ Finding solⁿ of D.E.

Let $y_g(x, c_1, c_2)$ be a general solⁿ of eqⁿ (2).

& $y_p(x)$ be the fixed particular solⁿ of eqⁿ (1)

& it depends on RHS fⁿ, $R(x)$.

Cant yet be said as general sol^n. A $cond^n$ needs to be satisfied, to make general sol^n

If $y(x)$ is $\boxed{\text{any}}$ sol^n of eq^n (1) & is given by

$$\boxed{y(x) = y_g(x, c_1, c_2) + y_p(x)} \longrightarrow (3)$$

y_g: solving eq^n (2)
got by putting $R(x)=0$ in eq^n (1).

y_p: solving eq^n (1).

The sol^n (3) is called general sol^n of D.E (1)

Theorem (1): If $y_1(x)$ & $y_2(x)$ are 2 sol^{ns} of eq^n (2). Then, $[c_1 . y_1(x) + c_2 . y_2(x)]$ is also a sol^n for eq^n (2) for any constt, c_1 & c_2.

linear $combin^n$ of $y_1(x)$ & $y_2(x)$

Proof done under problems of $Reduc^n$ of order.

Note: If neither $y_1(x)$, nor $y_2(x)$ is a constt multiple of the other i.e

Check for LI.

$$y_1(x) \neq k_1 y_2(x)$$

$$\text{or } y_2(x) \neq k_2 y_1(x)$$

Then, they are Linearly Independent (LI)
Otherwise, they are linearly dependent

Theorem (2): If $y_1(x)$ & $y_2(x)$ are 2 LI sol^{ns} of eq^n (2), then, $\boxed{y = c_1 y_1(x) + c_2 y_2(x)}$ is called general sol^n of eq^n (2).

Theorem (3) Another check for LI.

done just like that

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Theorem (3) If $y_1(x)$ & $y_2(x)$ are 2 solns of eqn (2), then, they are NOT LI iff, their WRONSKIAN:

Another check for LI.

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0.$$

else, they are LI.

Q. By eliminating the arbitrary constts c_1 & c_2, find the D.E

(a) $y = c_1 x + c_2 x^2$

(b) $y = c_1 \sin kx + c_2 \cos kx$.

(c) $y = c_1 x + c_2 \sin x$.

(a) $y = c_1 x + c_2 x^2 \longrightarrow (1)$

here, $y_1(x) = x$, $y_2(x) = x^2$.

Checking Wronskian :-

done Just like that

$$W(y_1, y_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

$W(y_1, y_2) \neq 0$ (provided $x \neq 0$, excluding it from domain)

So, they are LI.

Now, $y' = c_1 + 2c_2 x$

$y'' = 2c_2$. $\Rightarrow \boxed{c_2 = \frac{y''}{2}.} \longrightarrow (2)$

$\Rightarrow y' = c_1 + 2\left(\frac{y''}{2}\right)x \Rightarrow \boxed{c_1 = y' - y''x} \longrightarrow (3)$

So, using ② & ③ in eq^n ①.

$$\Rightarrow y = (y' - y''x)x + \left(\frac{y''}{2}\right)x^2$$

$$\Rightarrow y''\left(\frac{x^2}{2} - x^2\right) + y'x - y = 0$$

$$\Rightarrow + y''\left(-\frac{x^2}{2}\right) + y'x - y = 0$$

$$\Rightarrow y''(x^2) - y'(2x) + 2y = 0 \rightarrow ④$$

So, eq^n ① (given) is the general sol^n of D.E ④.

Arbitrary constt → fixed constt

(b) $y = \underbrace{c_1 \sin kx}_{y_1(x)} + \underbrace{c_2 \cos kx}_{y_2(x)} \rightarrow ①$

So, $y' = k(c_1 \cos kx - c_2 \sin kx)$

$y'' = (c_1 \sin kx + c_2 \cos kx)(-k^2)$

$\Rightarrow y'' = y(-k^2)$

$\Rightarrow k^2 y + y'' = 0 \rightarrow ②$

↳ reqd D.E

eq^n ① is sol^n of D.E. ②.

(c) $y = c_1 x + c_2 \sin x \rightarrow ①$

$y' = c_1 + c_2 \cos x \rightarrow ②$

$y'' = -c_2 \sin x \Rightarrow c_2 \sin x = -y''$.

Using it in (1).

$\Rightarrow y = c_1 x - y''$

$\Rightarrow c_1 = \dfrac{y + y''}{x} \longrightarrow (3)$

Using (3) in (2)

$$\Rightarrow y' = \frac{y+y''}{x} + \left(-\frac{y''}{\sin x}\right)\cos x$$

$$\Rightarrow y'(x\sin x) = y(\sin x) + y''(\sin x - x\cos x)$$

$$\Rightarrow y''(\sin x - x\cos x) - y'(x\sin x) + y(\sin x) = 0$$

$$\Rightarrow y''(1 - x\cot x) - y'(x) + y = 0 \longrightarrow (4)$$

eq^n (1) is general sol^n of DE (4).

—x—

Section - 15

* PROBLEMS

(1) Show that $y = c_1 x + c_2 x^2$ is the general sol^n of $x^2 y'' - 2xy' + 2y = 0 \longrightarrow (1)$ & find the particular sol^n for which $y(1) = 3$ & $y'(1) = 5$

Sol^n Any general sol^n for 2nd ord. homogeneous DE is of the form :-

$$y = c_1 y_1(x) + c_2 y_2(x)$$

Here, its given as,

$$y = c_1 x + c_2 x^2$$

$\Rightarrow y_1(x) = x$ & $y_2(x) = x^2$ (Comparing)

These are 2 independent sol^n of independent eq^n ①

Checking this claim

$y_1(x) = x$ i.e $y = x$ should satisfy eq^n ①

So, $y' = 1$

$y'' = 0$

Using in ①.

$$\Rightarrow x^2(0) - 2(x)(1) + 2(x) = 0$$

$$\Rightarrow 0 = 0$$

So, claim is true

So, $y_1(x) = x$ is a sol^n of ①

Now, checking for $y_2(x) = x^2$ separately.

$$y' = 2x$$

$$y'' = 2.$$

So, eq^n ① becomes

$$x^2(2) - 2x(2x) + 2(x^2) = 0$$

$$\Rightarrow 0 = 0$$

So, $y_2(x) = x^2$ is also a sol^n of ①

Wronskian of $(y_1, y_2) = W(y_1, y_2) =$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0$$

$(\forall \mathbb{R} \sim \{0\})$

$\therefore$ y_1 & y_2 are 2 sol^{ns}, LI.

Then, linear combin^n of y_1 & y_2 should be general sol^n of $y(x)$

$\Rightarrow y(x) = C_1 y_1(x) + C_2 y_2(x)$

$\Rightarrow y(x) = C_1 x + C_2 x^2$ is valid general sol^n of (1).

$\longrightarrow$ (2)

H.P

To find particular sol^n :-

From (2)

$y'(x) = C_1 + 2C_2 x \longrightarrow$ (3)

given, $y(1) = 3$ & $y'(1) = 5$

$\Rightarrow$ Using cond^{ns} in (2) & (3)

$\Rightarrow 3 = C_1(1) + C_2(1)^2$	$5 = C_1 + 2C_2(1)$
$\Rightarrow C_1 + C_2 = 3.$	$\Rightarrow C_1 + 2C_2 = 5.$
$\Rightarrow C_1 = 3 - C_2$	

$\Rightarrow 3 - C_2 + 2C_2 = 5$

$\Rightarrow C_2 = 2$

$\Rightarrow C_1 = 1$

So, particular sol^n is :-

$y(x) = x + 2x^2$ Ans

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(2) Verify that the f^{ns} :-

$y_1(x) = 1$ & $y_2(x) = e^{-x}$

are LI in the interval [0, 2]. & also, these 2 f^{ns} are LI sol^n for DE $y'' + y' = 0 \rightarrow (1)$

Find particular sol^n satisfying $y(2) = 0$, $y'(2) = e^{-2}$

Finding Wronskian

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x} \neq 0 \quad \forall x \text{ in interval } [0,2]$$

$\therefore$ $y_1(x)$ & $y_2(x)$ are LI sol^{ns}

Also, the general sol^n of DE (1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\Rightarrow y(x) = c_1 + c_2 e^{-x} \longrightarrow (2)$$

Finding particular sol^n:-

$$y'(x) = -c_2 e^{-x}.$$

Using $cond^{ns}$ given.

$y(2) = 0 \Rightarrow c_1 + c_2 e^{-2} = 0$

$y'(2) = e^{-2} \Rightarrow -c_2 e^{-2} = e^{-2} \Rightarrow c_2 = -1$

$\Rightarrow c_1 = e^{-2}$

So, particular sol^n is :-

$$y(x) = e^{-2} + (-1)e^{-x}$$

$$\Rightarrow y(x) = e^{-2} - e^{-x}$$

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(3) Show that :-

$y = c_1 e^x + c_2 e^{2x} \rightarrow (1)$ is general sol^n of
$y'' - 3y' + 2y = 0 \rightarrow (2)$ on any interval
& find particular sol^n for which $y(0) = -1$
$y'(0) = 1$

Any general sol^n for 2nd order homogeneous DE has the form

$$y = c_1 y_1(x) + c_2 y_2(x)$$

Comparing with (1), we get

$y_1(x) = e^x$, $y_2(x) = e^{2x}$ are sol^{ns} $\rightarrow$ claim

Checking if our claim is true.

$y = e^x$
$y' = e^x$
$y'' = e^x$
Using in (2)
$\Rightarrow e^x - 3e^x + 2e^x = 0$
$\Rightarrow 0 = 0$

$y = e^{2x}$
$y' = 2e^{2x}$
$y'' = 4e^{2x}$
Using in (2)
$\Rightarrow 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$
$\Rightarrow 0 = 0$

So, both are sol^{ns}.

Check for linear independence :-

Wronskian $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$
$= e^{3x} \neq 0$
$(\forall x)$

So, sol^{ns} are LI.
Hence, $y(x) = c_1 \underbrace{y_1(x)}_{e^x} + c_2 \underbrace{y_2(x)}_{e^{2x}} \rightarrow (3)$ is a general sol^n of eq^n (2).

Finding particular sol^n

$y(0) = -1$, $y'(0) = 1$

Using in ③

$y'(x) = c_1 e^x + 2c_2 e^{2x}$

$\Rightarrow -1 = c_1(1) + c_2(1)$

$\Rightarrow c_1 + c_2 = -1$

$\Rightarrow 1 = c_1(1) + 2c_2(1)$

$\Rightarrow c_1 + 2c_2 = 1$

$\Rightarrow c_2 = 2$

& $c_1 = -3$.

So, using in ③, we get

$y(x) = -3e^x + 2e^{2x}$ is particular sol^n of DE ②.

Ans

Section - 16

Use of one sol^n to find another sol^n

General homogeneous eq^n :-

S2) $y'' + P(x)y' + Q(x)y = 0 \longrightarrow$ ①

S1) Let $y_1(x)$ be a known sol^n of ①.

To find $y_2(x)$ (other LI sol^n) :- by default.

S1) Find $v(x) = \int \frac{1}{[y_1(x)]^2} \left(e^{-\int P(x)dx}\right) dx$

comes from eq^n ①.

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S2) $y_2(x) = v(x) \cdot y_1(x)$

$\therefore$ General solⁿ for eqⁿ ① is :-

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$

→ Given →comes from (S2)

—x—

Q (a) $y'' + y = 0$; $y_1 = \sin x$ → (A)
(b) $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$; $y_1 = x^2$ → (B)
(c) $(1-x^2)y'' - 2xy' + 2y = 0$; $y_1 = x$ → (C)
(d) $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$; $y_1 = x^{-1/2}\sin x$ → (D)

(a) Given $y_1 = \sin x$, a known solⁿ. We find $v(x)$

$$v(x) = \int \frac{1}{[y_1(x)]^2}\left(e^{-\int P(x)\,dx}\right)dx$$

where $P(x) = 0$ (from comparing ① & (A))

$$\Rightarrow v(x) = \int \frac{1}{\sin^2 x}\, e^{-\int 0\,dx}\, dx$$

$$= \int \operatorname{cosec}^2 x\, dx \quad \text{or} \quad \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x}\, dx$$

$$= \int dx + \int \cot^2 x\, dx$$

directly

$$= -\cot x$$

$\Rightarrow \quad y_2(x) = v(x) \cdot y_1(x)$

$$= (-\cot x)(\sin x)$$

$\Rightarrow y_2(x) = -\cos x$

So, general solⁿ (reqd) of eqⁿ (A) is

$y(x) = C_1 y_1(x) + C_2 y_2(x)$

$\Rightarrow y(x) = C_1 \sin x - C_2 \cos x$ — Ans

(b) Given :-

$$y'' + \frac{1}{x} y' - \frac{4}{x^2} y = 0$$

Comparing with (1),

$$y'' + P(x) y' + Q(x) y = 0$$

we get

$$P(x) = \frac{1}{x}, \quad Q(x) = \frac{-4}{x^2}$$

Now

$$v(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int P(x) dx} dx$$

$$= \int \frac{1}{x^4} e^{-\int \frac{1}{x} dx} dx$$

$$= \int \frac{1}{x^4} e^{-\log x} dx$$

$$= \int \frac{1}{x^4} \cdot \frac{1}{x} dx$$

$$= \int x^{-5} dx$$

$$= -\frac{x^{-4}}{4}$$

So, $y_2(x) = V(x)\ y_1(x)$

$$= -\frac{x^{-4}}{4} \cdot x^2$$

$$\Rightarrow y_2(x) = -\frac{1}{4x^2}$$

So, general solⁿ :-

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1 x^2 + c_2\left(-\frac{1}{4x^2}\right)$$

$$\Rightarrow y(x) = c_1 x^2 + c_3\left(\frac{1}{x^2}\right) \quad ; \quad c_3 = -\frac{1}{4}c_2$$

(c) Given $(1-x^2)y'' - 2xy' + 2y = 0$

- Legendre D.E :-

General form :- $(1-x^2)y'' - 2xy' + n(n+1)y = 0$.

with $n = 1$, we have our given eqⁿ (c).

$\div (1-x^2)$

$$\Rightarrow y'' - \left(\frac{2x}{1-x^2}\right)y' + \left(\frac{2}{1-x^2}\right)y = 0$$

Comparing with eqⁿ (1)

$$P(x) = \frac{-2x}{1-x^2}$$

Now, $v(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int P(x)dx}\, dx$

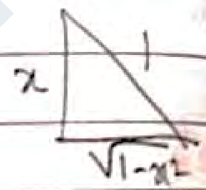
$$\Rightarrow V(x) = \int \frac{1}{x^2}\, e^{-\int \frac{-2x}{1-x^2}dx}$$

$$= \int \frac{1}{x^2}\, e^{-[\log(1-x^2)]}\, dx$$

$$= \int \frac{1}{x^2(1-x^2)}\, dx$$

$$= \int \frac{1-x^2+x^2}{x^2(1-x^2)}\, dx$$

$$= \int \frac{\cancel{1-x^2}}{x^2\cancel{(1-x^2)}}\, dx + \int \frac{\cancel{x^2}}{\cancel{x^2}(1-x^2)}\, dx$$



$$= \frac{-1}{x} + \int \frac{dx}{1-x^2}$$

$x = \sin\theta$

$\Rightarrow dx = \cos\theta\, d\theta$

$$= -\frac{1}{x} + \int \frac{\cancel{\cos\theta}\, d\theta}{\cos^{\cancel{2}}\theta}$$

$$= -\frac{1}{x} + \int \frac{\sec\theta(\sec\theta + \tan\theta)}{(\sec\theta + \tan\theta)}\, d\theta$$

$$= -\frac{1}{x} + \int \frac{(\sec^2\theta + \sec\theta\tan\theta)}{\sec\theta + \tan\theta}\, d\theta$$

$$= -\frac{1}{x} + \log(\sec\theta + \tan\theta)$$

$$= -\frac{1}{x} + \log\left(\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow V(x) = -\frac{1}{x} + \log\left(\frac{1+x}{\sqrt{1-x^2}}\right)$$

directly comes from formula : $\int \frac{dx}{1-x^2} = \frac{1}{2}\log\left(\frac{1-x}{1+x}\right)$

Now,

$$y_2(x) = V(x) \cdot y_1(x)$$

$$\Rightarrow y_2(x) = \left[\frac{-1}{x} + \log\left(\frac{1+x}{\sqrt{1-x^2}}\right)\right] x$$

So,

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\Rightarrow y(x) = c_1 x + c_2 x \left[-\frac{1}{x} + \log\left(\frac{1+x}{\sqrt{1-x^2}} \right) \right]$$

Ans

(d) Given the DE.

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$$

$\div x^2$

$$\Rightarrow y'' + \left(\frac{1}{x}\right) y' + \left(x - \frac{1}{4x}\right) y = 0$$

$$P(x) = \frac{1}{x} \quad \text{(Comparing with eq}^n \text{ (1))}$$

Now, $$V(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int P(x)dx} dx$$

$$= \int \frac{1}{(x^{-1/2} \sin x)^2} e^{-\int \frac{1}{x} dx} dx$$

$$= \int \frac{x}{\sin^2 x} \left(\frac{1}{x}\right) dx$$

$$= \int \operatorname{cosec}^2 x \, dx$$

$$\Rightarrow V(x) = -\cot x$$

Now, $$y_2(x) = V(x) \, y_1(x)$$

$$= (-\cot x)(x^{-1/2} \sin x)$$

$$\Rightarrow y_2(x) = -x^{-1/2} \cos x.$$

So, solⁿ:-

$$y(x) = c_1 y_1(x) + c_2(y_2(x))$$
$$= c_1 x^{-1/2} \sin x - c_2 x^{-1/2} \cos x$$
$$\Rightarrow y(x) = c_1 x^{-1/2} \sin x + c_3 x^{-1/2} \cos x.$$

$\hookrightarrow c_3 = -c_2$

☆ Bessel's D.E

General form:-

$$x^2 y'' + x y'' + (x^2 - n^2) y = 0$$

$\hookrightarrow$ also called n^{th} order Bessels D.E.

$\hookrightarrow$ the form used in part (d) above ($n = \frac{1}{2}$)

Q. Solve

(e) $y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$; $y_1 = x$

(f) $y'' - \left(\frac{x+2}{x}\right) y' + \frac{x+2}{x^2} y = 0$; $y_1 = x$

(g) $xy'' - (2x+1)y' + (x+1)y = 0$; $y_1 = e^x$.

(e) Comparing with eqⁿ (1),

$$P(x) = \frac{-x}{x-1}$$

$$v(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int P(x)\,dx} \, dx$$

$$= \int \frac{1}{x^2} e^{\int \frac{x}{x-1} dx} dx$$

$$= \int \frac{1}{x^2} e^{\int \frac{x-1+1}{x-1} dx} dx$$

$$= \int \frac{1}{x^2} e^{[x + \log(x-1)]} dx$$

$$= \int e^x \cdot \frac{x-1}{x^2} dx$$

$$= \int \underset{\text{II}}{e^x} \underset{(\text{I})}{\frac{1}{x}} dx + \int e^x \left(\frac{-1}{x^2}\right) dx$$

ILATE

$$= \frac{1}{x} e^x \cancel{- \int \left(\frac{-1}{x^2}\right) e^x dx + \int e^x \left(\frac{-1}{x^2}\right) dx}$$

$$\therefore V(x) = \frac{e^x}{x}$$

Now,

$$y_2(x) = V(x)\, y_1(x)$$
$$= \frac{e^x}{\cancel{x}} \cdot \cancel{x}$$

$$\Rightarrow y_2(x) = e^x$$

So, sol^n :-

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$
$$= c_1 x + c_2 e^x$$

Ans

(f) Comparing with eqⁿ ①

$$P(x) = -\left(\frac{x+2}{x}\right)$$

$$V(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int P(x)\,dx}\,dx$$

$$= \int \frac{1}{x^2} e^{-\int \frac{(x+2)}{x}dx}\,dx$$

$$= \int \frac{1}{x^2} e^{x+2\log(x)}\,dx$$

$$= \int e^{x} \cdot \cancel{\frac{1}{x^2}} \cdot \cancel{x^2}\,dx$$

$$\Rightarrow V(x) = e^{x}$$

So, $y_2(x) = V(x)\,y_1(x)$

$$= e^{x} \cdot x$$

$$\Rightarrow y_2(x) = xe^{x}$$

So, solⁿ :-

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\Rightarrow y(x) = c_1 x + c_2 x e^{x} \quad \text{Ans}$$

(g) $xy'' - (2x+1)y' + (x+1)y = 0$

$\div x$

$$\Rightarrow y'' - \left(2 + \frac{1}{x}\right)y' + \left(1 + \frac{1}{x}\right)y = 0$$

Now, we know general form :-

$$y'' + P(x)y'' + Q(x)y = 0$$

Comparing, we get

$$P(x) = -\left(2 + \frac{1}{x}\right)$$

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Now, $v(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int P(x)dx} dx$

$= \int e^{-2x} e^{-\int -(2+\frac{1}{x})dx} dx$

$= \int e^{-2x} e^{2x + \log x} dx$

$= \int e^{-2x} \cdot e^{2x} x \, dx$

$\Rightarrow v(x) = \frac{x^2}{2}$

Now,

$y_2(x) = v(x) \cdot y_1(x)$

$= \frac{x^2}{2} \cdot e^x$

So, solⁿ:-

$y(x) = c_1 y_1(x) + c_2 y_2(x)$

$= c_1 e^x + \frac{c_2 x^2}{2} e^x$

Ans

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Section - 17

- 2nd order, constt. coeff. homogeneous D.E

General concept for coming sections

✯ n^{th} ORDER D.E (non homogeneous eqⁿ)

$$\boxed{\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = R(x)} \quad (1)$$

The general solⁿ of (1) is of the form
$y(x) = y_g(x) + y_p(x)$

→ $y_p(x)$: particular solⁿ of (1) (depends on fⁿ $R(x)$)
* (If $R(x) = 0$, then, $y_p(x) = 0$)

→ $y_g(x)$: general solⁿ of corresponding homogeneous eqⁿ.

i.e, $$\boxed{\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots - a_{n-1}\frac{dy}{dx} + a_n y = 0} \quad (2)$$

✯ Finding $y_g(x)$ (i.e, general solⁿ of eqⁿ (2))

S1. Write eqⁿ (2) in simplified form :-

$$\frac{d^n y}{dx^n} = y^n$$

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \dots a_{n-1} y' + a_n y = 0$$

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Now, replace :

$$y^n = m^n, \; y^{n-1} = m^{n-1}, \; \ldots \; y' = m, \; y = 1$$

& form an auxiliary eq^n :

$$\boxed{m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_{n-1} m + a_n = 0} \quad (3)$$

→ algebraic eq^n of degree 'n'

→ m : any unknown.

→ has 'n' roots

Step (2) Solve eq^n (3) & find the 'n' roots.

→ Case (i) : If all n roots are real & distinct.

So, say, $m = \underbrace{m_1, m_2, m_3, \ldots, m_n}_{n \text{ roots of variable} : m}$.

Then, $\boxed{y_g(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}}$

→ Case (ii) : If 2 roots are equal & all others are distinct.

So, we'll have n roots as :

$m = m_1, m_1, m_3, m_4, \ldots, m_n$

$(m_1 = m_2)$

f^n called as COMPLEMENTARY Function.

Then, $\boxed{y_g(x) = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}}$

→ Case (iii) : If 3 roots are equal & others are distinct.

So, $m = m_1, m_1, m_1, m_4, \ldots, m_n$

$m_1 = m_2 = m_3$

Then, $\boxed{y_g(x) = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}}$

∴ complex roots occur in conjugate pairs

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→ Case (iv): If 2 roots are complex & remaining real & distinct

So, say $m_1 = m_2 = \alpha \pm i\beta$; α → Re(m_1), β → Im(m_1)

& $m_3, m_4, \ldots, m_n$: real & distinct.

So,

$$\boxed{y_g(x) = \left[e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)\right] + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}}$$

→ Case (v): If we get equal pairs of complex roots:

i.e, $m_1 = m_2 = \alpha \pm i\beta$ & $m_3 = m_4 = \alpha \pm i\beta$

& $m_5, m_6, \ldots, m_n$: real & distinct.

Then,

$$\boxed{y_g(x) = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right] + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}}$$

Q (a) $y'' + y' - 6y = 0$

(b) $y'' + 2y' + y = 0$

(c) $2y'' - 4y' + 8y = 0$

given initial cond^ns → find values of c_1 & c_2

(d) $y'' - 5y' + 6y = 0$; $y(1) = e^2$, $y'(1) = 3e^2$

(e) $y'' - 6y' + 5y = 0$; $y(0) = 3$, $y'(0) = 11$

(f) $y'' + 4y' + 5y = 0$; $y(0) = 1$, $y'(0) = 0$

(a) $y'' + y' - 6y = 0$ ⟶ (i)

Its homogeneous eq^n

So, general sol^n given by $y_g(x)$

Auxiliary eq^n of eq^n 1

$y'' = m^2, \; y' = m, \; y = 1$

$\Rightarrow m^2 + m - 6 = 0$

$\Rightarrow (m+3)(m-2) = 0$

$\Rightarrow \underbrace{m = -3, 2}_{\text{Real \& distinct roots}}$

Then, $y_g(x) = y(x) = c_1 e^{-3x} + c_2 e^{2x}$ is the sol^n of given DE

(b) $y'' + 2y' + y = 0 \longrightarrow$ 2

It's a homogeneous eq^n ($\because$ RHS $= 0$)

So, general sol^n ($y_g(x)$) is given by finding Auxiliary eq^n of 2

$y'' = m^2, \; y' = m, \; y = 1$

$\Rightarrow m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1$

2 roots are equal.

So, $y_g(x) = y(x) = (c_1 + c_2 x)e^{-x}$ is sol^n of given DE 2

(c) $2y'' - 4y' + 8y = 0 \longrightarrow$ 3

$\Rightarrow y'' - 2y' + 4y = 0$

It's a homogeneous eq^n.

Auxiliary eq^n of 3 is given by

$y'' = m^2, \; y' = m, \; y = 1$

$\Rightarrow m^2 - 2m + 4 = 0$

$\Rightarrow m = \dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3} \; [\equiv (\alpha \pm i\beta)]$

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The 2 roots are complex. So, sol^n is

$$y_g(x) = e^{\alpha x}\left[C_1 \cos \beta x + C_2 \sin \beta x\right]$$

$$\Rightarrow y_g(x) = e^{x}\left[C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x\right] = y(x)$$

This is the sol^n to given ΔE.

(d) $y'' - 5y' + 6y = 0 \longrightarrow (4)$

Its again homogeneous

So, auxiliary eq^n of (4) can be written as

$m^2 - 5m + 6 = 0$ $\quad [y'' = m^2;\ y' = m;\ y = 1]$

$\Rightarrow (m-3)(m-2) = 0$

$\Rightarrow m = 2, 3$

The roots are real & distinct

So, general sol^n, $y_g(x) = C_1 e^{2x} + C_2 e^{3x} = y(x) \longrightarrow (A)$

Eliminating C_1 & C_2 with the given $cond^{ns}$

$y(1) = e^2$, $y'(1) = 3e^2$

$\Rightarrow C_1 e^2 + C_2 e^3 = e^2$

$\Rightarrow C_1 + eC_2 = 1 \longrightarrow (i)$

From (A),

$y'(x) = 2C_1 e^{2x} + 3C_2 e^{3x}$

$\Rightarrow y'(1) = 3e^2 = 2C_1 e^2 + 3C_2 e^3$

$\Rightarrow 3 = 2C_1 + 3C_2 e \longrightarrow (ii)$

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Solving (i) & (ii)

$\Rightarrow 3 = 2(1-eC_2) + 3eC_2$

$\Rightarrow 1 = eC_2$

$\Rightarrow \boxed{C_2 = e^{-1}}$

$\Rightarrow \boxed{C_1 = 0}$

Hence, the solⁿ (Particular)

$y(x) = e^{-1}e^{3x} = e^{3x-1}$ Ans

(e) $y'' - 6y' + 5y = 0 \longrightarrow (5)$

Its homogeneous eqⁿ

Finding Auxiliary eqⁿ of (5)

$\Rightarrow m^2 - 6m + 5 = 0$

$\Rightarrow m = \frac{6 \pm \sqrt{36-20}}{2} = 3 \pm 2 = 1, 5$

$\Rightarrow m = \underbrace{1, 5}_{\text{Real \& distinct roots}}$

So, $y_g(x) = y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{x} + c_2 e^{5x} \longrightarrow (B)$

Using initial condⁿˢ in eqⁿ (B)

$y(0) = 3$; $y'(0) = 11$

$\Rightarrow y(0) = 3 = c_1 e^0 + c_2 e^{5(0)}$

$\Rightarrow c_1 + c_2 = 3 \longrightarrow (iii)$

From (B),

$y'(x) = c_1 e^x + 5c_2 e^{5x}$

$\Rightarrow y'(0) = 11 = c_1 + 5c_2 \longrightarrow (iv)$

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Solving (iii) & (iv), we get

$$11 = (3 - c_2) + 5c_2$$
$$\Rightarrow \boxed{c_2 = 2}$$
$$\Rightarrow \boxed{c_1 = 1}$$

Hence, particular solⁿ is :-

$$y(x) = e^{x} + 2e^{5x}$$

Ans

(f) $y'' + 4y' + 5y = 0 \longrightarrow (6)$.

Its homogeneous eqⁿ.

Finding auxiliary eqⁿ of (6)

$$\Rightarrow m^2 + 4m + 5 = 0$$
$$\Rightarrow m = \frac{-4 \pm \sqrt{16-20}}{2}$$
$$\Rightarrow m = \underbrace{-2 \pm i}_{\text{Both roots imaginary pair}} \; (\equiv \alpha \pm i\beta)$$

So, general solⁿ, $y_{(g)}(x) = y(x)$

$$y(x) = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$
$$\Rightarrow y(x) = e^{-2x}\left[c_1 \cos x + c_2 \sin x\right] \longrightarrow (c)$$

Using initial condⁿˢ on (c)

$$y(0) = 1, \; y'(0) = 0.$$

$$\Rightarrow y(0) = 1 = e^{-2(0)}\left[c_1 \cos 0 + c_2 \sin 0\right]$$

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$\Rightarrow \boxed{c_1 = 1}$

& , differentiating (c)

$\Rightarrow y'(x) = (-2e^{-2x})(\cos x + c_2 \sin x)$

$\qquad + (e^{-2x})(-\sin x + c_2 \cos x)$

$\Rightarrow y'(0) = 0 = -2e^0(\cos 0 + c_2 \sin 0)$

$\qquad + e^0(-\sin 0 + c_2 \cos 0)$

$\Rightarrow 0 = -2 + c_2$

$\Rightarrow \boxed{c_2 = 2}$

Hence, the particular sol^n is

$y(x) = e^{-2x}[\cos x + 2 \sin x]$

Ans

✯ EULER's D.E

Specific form : $\boxed{x^2 y'' + p x y' + q y = 0}$; p, q : consts. → (1)

→ way to solve:

Put $x = e^z$ (or $z = \log x$) → (2)

So, now, $y \xrightarrow{f^n \text{ of}} z \xrightarrow{f^n \text{ of}} x$

$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz}\left(\frac{1}{x}\right)$ $\quad (\because z = \log x)$

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$$\Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}} \rightarrow (3)$$

Let $D = \frac{d}{dx}$, $\theta = \frac{d}{dz}$

Now, using operator symbols,

$$x\,Dy = \theta y \Rightarrow \cancel{xD}\ \boxed{xD = \theta} \rightarrow (4)$$

So, left with only operators.

Differentiating (3) again.

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx}.1 = \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right).\frac{dz}{dx}$$

a f^n of z
So, can be differen-tiated only w.r.t z. This won't be possible

$$= \frac{d^2y}{dz^2}\left(\frac{1}{x}\right)$$

$\because z = \log x$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

$$\Rightarrow x^2D^2y + xDy = \theta^2 y$$

$D^2 = \frac{d^2}{dx^2}$, $\theta^2 = \frac{d^2}{dz^2}$

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$$\Rightarrow x^2D^2 + xD = \theta^2$$

or $\Rightarrow x^2D^2y + xDy = \theta^2 y$

$$\Rightarrow x^2D^2y + \theta y = \theta^2 y \quad \text{(from 4)}$$

$$\Rightarrow \boxed{x^2D^2 = \theta(\theta-1)} \longrightarrow 5$$

<u>Note</u>:- Remember the results of eq^n 4 & 5 to solve problems.

Q. (a) $x^2y'' + 3xy' + 10y = 0 \longrightarrow$ (a)
(b) $x^2y'' + 2xy' + 3y = 0 \longrightarrow$ (b)

Comparing with Euler's eq^n std. form (eq^n 1),
we get
The eq^{ns} are of that form.
Now, put $x = e^z$ or $z = \log x$
Now, $D = \frac{d}{dx}$, $\theta = \frac{d}{dz}$

$$\left.\begin{aligned} \text{where, } & xD = \theta \\ \& \ & x^2D^2 = \theta(\theta-1) \end{aligned}\right\} \longrightarrow \text{(c)}$$

(a) $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 10y = 0$

$\Rightarrow x^2D^2y + 3xDy + 10y = 0$

$\Rightarrow (x^2D^2 + 3xD + 10)y = 0 \longrightarrow$ (A)

$\Rightarrow (\theta(\theta-1) + 3\theta + 10)y = 0$ (from c)

$\Rightarrow (\theta^2 + 2\theta + 10) y = 0 \longrightarrow (d)$

$\hookrightarrow$ constt. coeff. 2nd ord. homogeneous D.E in y & z ; $\theta = \frac{d}{dz}$, $\theta^2 = \frac{d^2}{dz^2}$

Solving (d), finding auxiliary eq^n

$\Rightarrow m^2 + 2m + 10 = 0$

$\Rightarrow m = \frac{-2 \pm \sqrt{4-40}}{2}$

$m = -1 \pm 3i \; (\equiv \alpha + i\beta)$ } Both Imaginary roots

$\therefore y_g(x) = y(x) = e^{-z}(c_1 \cos 3z + c_2 \sin 3z) \longrightarrow (e)$

Using substitutions & eq^n (e) in eq^n (A)

$\Rightarrow y(x) = e^{-\log x}\left(c_1 \cos(3\log x) + c_2 \sin 3(\log x)\right)$

$\Rightarrow y(x) = \frac{1}{x}\left(c_1 \cos(3\log x) + c_2 \sin(3\log x)\right)$

(b) $x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 3y = 0$

$\Rightarrow x^2 D^2 y + 2xDy + 3y = 0$

$\Rightarrow (x^2D^2)y + 2(xD)y + 3y = 0$

$\Rightarrow [x^2D^2 + 2(xD) + 3]y = 0 \longrightarrow (B)$

$\Rightarrow (\theta^2 + \theta + 3)y = 0 \longrightarrow (f)$

constt coeff, homogeneous D.E, 2nd ord in y & z

So, finding auxiliary eq^n of (f)

$\Rightarrow m^2 + m + 3 = 0$

$\Rightarrow m = \underbrace{\frac{-1 \pm i\sqrt{11}}{2}}$

Both imaginary roots

So, $y_g(x) = y(x) = e^{\alpha z}(C_1 \cos\beta z + C_2 \sin\beta z)$

$= e^{(-1/2)z}\left(C_1 \cos\left(\frac{\sqrt{11}}{2}z\right) + C_2 \sin\left(\frac{\sqrt{11}}{2}z\right)\right) \rightarrow (g)$

Put $z = \log x$ & using in eq^n (g)

$\Rightarrow y(x) = \frac{1}{\sqrt{x}}\left(C_1 \cos\left(\frac{\sqrt{11}}{2}\log x\right) + C_2 \sin\left(\frac{\sqrt{11}}{2}\log x\right)\right)$

Ans

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Section - 18

METHOD OF UNDETERMINED COEFF.

(to find particular solⁿ of
- 2nd order
- constt. coeff.
- non homogeneous D.E)

So, for eqⁿ : $\boxed{y'' + py' + qy = R(x)}$ ⟶ ①

→ p, q : constts
→ $R(x)$: any fⁿ of x.

✓ General solⁿ of eqⁿ ①,

$y(x) = y_g(x) + y_p(x)$

→ $y_g(x)$: general solⁿ of corresponding homogeneous eqⁿ :-

already solved ← $\boxed{y'' + py' + qy = 0}$.
in section-17 (using auxiliary eqⁿ)

☆ Finding Particular Solⁿ $(y_p(x))$:

$y_p(x)$ depends on the nature of $R(x)$ [the RHS fⁿ]

- $y_p(x)$ can be found for the following std. fⁿˢ :-

Case (i) : $R(x) = e^{ax}$ (exponential fⁿ)
Case (ii) : $R(x) = \cos(ax)$ or $\sin(ax)$
Case (iii) : $R(x)$ = polynomial in x : $x^n + a_1 x^{n-1} + \dots + a_1 x + a_0$

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~~Case (iv) : $R(x)$ = combin^n of above fns.~~
~~(eg $x^2 e^x$, $x^2 \sin 2x$, $e^x \sin 4x$ ....)~~

ATOM FORMS
different forms in which RHS of eq^n ① can be got

★ Std. cases for $R(x)$:-

Case(i) : $R(x) = p(x)$: a polynomial.
Case(ii) : $R(x) = p(x)\, e^{kx}$: (polynomial) × (exponential)
Case(iii) : $R(x) = p(x)\, e^{kx} \cos mx$: (poly)×(exp)×(cosine)
Case(iv) : $R(x) = p(x)\, e^{kx} \sin mx$: (poly)×(exp)×(sine)

Note

If $R(x) = \underbrace{R_1(x) + R_2(x) + R_3(x)}_{\text{3 terms for } R(x)}$.

Then, ① can be solved as.

Writing eq^n 3 times & solving for particular sol^n

$$\begin{cases} y'' + py' + qy = R_1(x) \} \text{ particular sol}^n : y_1 \\ y'' + py' + qy = R_2(x) \} \quad " \quad " = y_2 \\ y'' + py' + qy = R_3(x) \} \quad " \quad " = y_3 \end{cases}$$

(Valid only when, $R = R_1 + R_2 + \dots$; not valid when its sub, prod, div or Linear combin^n)

Then, particular sol^n of eq^n ①
is $y_p(x) = y_1 + y_2 + y_3$.

& general sol^n, $y_g(x)$ is solved using
$y'' + py' + qy = 0$
(Independent of $R, R_1, R_2, \dots$)

So,
Sol^n of $y(x) = y_g(x) + y_p(x)$.

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✱ Procedure for solving undetermined coeff$_s$:-

S1) Solve the homogeneous eq^n

$y'' + py' + qy = 0$ & find its general sol^n: $y_g(x)$.

S2) Differentiate atom f^n $R(x)$ repeatedly. ~~linear~~ Isolate independent f^{ns}, whose LC (linear combin^{ns}) are derivatives.

Multiply them by undetermined coeff$_s$:- $d_1, d_2, \ldots, d_k$ to form initial trial sol^n.

S3) If initial trial sol^n duplicates terms found in $y_g(x)$ (calculated in S1), then, multiply the trial sol^n by x, repeatedly, until it ~~go~~ doesn't duplicate.

Use this modified trial sol^n subsequently.

S4) Substitute this final trial sol^n in eq^n ①
& find $d_1, d_2, \ldots, d_k$

S5) Write the ~~gen~~ particular sol^n of ① ~~as~~: using $d_1, d_2, \ldots, d_k$.
~~$y(x) =$~~ y_p

S6) General sol^n ~~of~~ eq^n ① $= y(x) = y_g(x) + y_p(x)$

or, complete sol^n, say.

$y(x)$: general sol^n of non-homogeneous eq^n (eq^n → ①)

$y_g(x)$: general sol^n of homogeneous eq^n

☆ 2nd ORDER -

Q1) $y'' + y = 3\sin x$: Solve by method of undetermined coeffs

$\hookrightarrow$ (A) $\quad \searrow$ atom fn

Sol^n Comparing with eq^n (1),

$$y'' + py' + qy = R(x), \text{ we get}$$

$$R(x) = 3\sin x$$
$$= (3)\sin x$$
$$= (3)(e^0)\sin x$$
$$= (\text{poly})(\text{exp})(\text{sine})$$

So, it comes under case (iv).

Now,

S1) $y'' + py' + qy = 0$

$\Rightarrow y'' + y = 0$

$\Rightarrow$ Auxiliary eq^n :-

$$m^2 + 1 = 0$$
$$\Rightarrow m = \pm i \Rightarrow \alpha = 0, \beta = 1$$

So, $y_g(x) = e^{0x}(c_1\cos x + c_2\sin x)$

$\Rightarrow y_g(x) = c_1\cos x + c_2\sin x \rightarrow (2)$

S2) Atom f^n or RHS f^n = $3\sin x = h$

Differentiating

$\Rightarrow h' = 3\cos x$

$h'' = -3\sin x$

$h''' = -3\cos x$

$h'''' = 3\sin x$

$\Big\}$ f^n type remaining same.

Now, Isolate the independent f^n :- $\{\sin x, \cos x\}$

(3, -3 : consts)

Initial Trial sol^n = $d_1 \sin x + d_2 \cos x \rightarrow (3)$

$\hookrightarrow d_1, d_2$: undetermined coeff.

S3) Comparing eq^n (2) by (3),
we have, they are duplicating.

Now,

$\times\ x$

$\Rightarrow x(d_1 \sin x) + x(d_2 \cos x)$

$= d_1(x \sin x) + d_2(x \cos x) \rightarrow (4)$

Comparing (4) by (2), we find they are not duplicating

So, eq^n (4) is final trial sol^n

S4) Using (4) in eq^n (A)

Now, $y'' + y = 3\sin x \rightarrow (A)$

Trial sol^n = $x(d_1 \sin x + d_2 \cos x)$

$= x\ q(x)$

$\hookrightarrow q(x) = (d_1 \sin x + d_2 \cos x) \rightarrow (5)$

$\Rightarrow [x(q(x))]'' + [x\,q(x)] = 3\sin x$

$\Rightarrow [x\,q'(x) + q(x)]' + x\,q(x) = 3\sin x$

$\Rightarrow x\,q''(x) + q'(x) + q'(x) + x\,q(x) = 3\sin x$

$\Rightarrow x\,q''(x) + 2q'(x) + x\,q(x) = 3\sin x$

$\Rightarrow x(q''(x) + q(x)) + 2q'(x) = 3\sin x$

$\hookrightarrow (B)$

From (5)

$q'(x) = d_1 \cos x - d_2 \sin x$

$q''(x) = -d_1 \sin x - d_2 \cos x = -(d_1 \sin x + d_2 \cos x)$

$= -q(x)$

$\Rightarrow \quad q(x) + q''(x) = 0 \longrightarrow (6)$

Using (6) in (B)

$\Rightarrow 2q'(x) = 3\sin x$

$\Rightarrow 2(d_1 \cos x - d_2 \sin x) = 3\sin x$

$\Rightarrow 2d_1 = 0 \;,\; -2d_2 = 3.$

$\Rightarrow d_1 = 0 \;,\; d_2 = \frac{-3}{2} \longrightarrow (7)$

S5) Substituting (7) in eq^n (4).

Particular sol^n,

$y_p(x) = 0(x\sin x) - \frac{3}{2}(x\cos x) \to (8)$

S6) Complete sol^n of eq^n (A)

$y(x) = y_g(x) + y_p(x)$

$\Rightarrow y(x) = \underbrace{c_1 \cos x + c_2 \sin x}_{\text{from eq}^n (2)} - \underbrace{\frac{3}{2} x\cos x}_{\text{from eq}^n (8)}.$

— x —

Q7) $y'' - 2y' + 2y = e^x \sin x \longrightarrow (1)$

Comparing with std. form

$$y'' + py' + qy = R(x)$$

So, $R(x) = e^x \sin x$

$= 1 \cdot e^x \cdot \sin x$

$= (\text{poly}) \times (\text{exp}) \times (\text{sine})$

So, its of case IV

S1) $y'' - 2y' + 2y = 0 \longrightarrow (2)$

Finding Auxiliary eq^n

$\Rightarrow m^2 - 2m + 2 = 0$

$\Rightarrow (m-1)^2 = -1$

$\Rightarrow m - 1 = \pm i$

$\Rightarrow m = \underbrace{1 \pm i}$

Imaginary root pairs

So, $y(x) = e^x (c_1 \cos x + c_2 \sin x)$

is general sol^n of given homogeneous eq^n (2)

$\Rightarrow y_g(x) = c_1 e^x \cos x + c_2 e^x \sin x \longrightarrow (3)$

S2) Build initial trial sol^n:-

Let RHS, $e^x \sin x = h$

$h' = e^x \cos x + \sin x \, e^x$

$h'' = -e^x \sin x + e^x \cos x + e^x \cos x + \sin x \, e^x$

We find same terms are repeating themselves.

So, no need to do further differentiation

Now, isolating the Independent terms (terms appearing again & again)

$\Rightarrow$ Initial trial solⁿ :-

$$y = d_1 e^x \cos x + d_2 e^x \sin x \longrightarrow (4)$$

$\hookrightarrow d_1, d_2$: undetermined coeff.

S3) Comparing (4) & (3)

We get the same form again.

Since the terms in (4) duplicates term in (3),

$\times$ initial trial solⁿ by x.

By this, any form of duplicⁿ is avoided

$$\Rightarrow y = d_1 (x e^x \cos x) + d_2 (x e^x \cos x) \longrightarrow (5)$$

Comparing with (3), $\exists$ no duplicⁿ.

So, this is final trial solⁿ

$$\Rightarrow y = x e^x (d_1 \cos x + d_2 \sin x)$$

$$\Rightarrow y = e^x \, q(x)$$

$\hookrightarrow q(x) = x [d_1 \cos x + d_2 \sin x]$

S4) Using this in (1) $\rightarrow$ To find values of d_1, d_2

$$\Rightarrow [e^x q(x)]'' - 2[e^x q(x)]' + 2e^x q(x) = e^x \sin x$$

$$\Rightarrow \left(e^x q'(x) + q(x) e^x\right)' - 2[e^x q'(x) + q(x) e^x] + 2e^x q(x) = e^x \sin x$$

$$= e^x q''(x) + q'(x) e^x + q'(x) e^x + q(x) e^x - 2q'(x) e^x - 2q(x) e^x + 2q(x) e^x = e^x \sin x$$

$$\Rightarrow e^x q''(x) + e^x q(x) = e^x \sin x$$

$$\Rightarrow q''(x) + q(x) = \sin x \longrightarrow (6)$$

Now,

$$q(x) = x[d_1 \cos x + d_2 \sin x]$$

$$q'(x) = x[-d_1 \sin x + d_2 \cos x] + [d_1 \cos x + d_2 \sin x]$$

$$q''(x) = x[-d_1 \cos x - d_2 \sin x] + [-d_1 \sin x + d_2 \cos x] + [-d_1 \sin x + d_2 \cos x]$$

$$\Rightarrow q''(x) = -q(x) + 2[-d_1 \sin x + d_2 \cos x]$$

Now, $\frac{q'(x)}{x} = [-d_1 \sin x + d_2 \cos x] + q(x)$

$$\Rightarrow [-d_1 \sin x + d_2 \cos x] = \frac{q'(x)}{x} - q(x)$$

$$q''(x) + q(x) = 2[-d_1 \sin x + d_2 \cos x]$$

Using it in (6)

$$\Rightarrow -2d_1 \sin x + 2d_2 \cos x = \sin x$$

Comparing LHS & RHS

$$\Rightarrow -2d_1 = 1, \quad 2d_2 = 0$$

$$\Rightarrow d_1 = -\frac{1}{2}, \quad d_2 = 0 \rightarrow (7)$$

S5) Substituting (7) in (5), we get req^d particular sol^n for (1)

$$\Rightarrow y_p(x) = e^x \left[x \cdot -\frac{1}{2} \cos x + 0 \cdot \sin x\right]$$

$$\Rightarrow y_p(x) = -\frac{x e^x \cos x}{2} \rightarrow (8)$$

S6) Complete sol^n of eq^n (1), $y(x) = y_g(x) + y_p(x)$

From eq^n (3) & (8)

$$y(x) = (c_1 e^x \cos x + c_2 e^x \sin x) - \frac{x e^x \cos x}{2} \quad \text{Ans}$$

not necessary ∵ $1+x-x^2$ can be taken as single term → polynomial

can also be solved by separating terms.

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Q.3) $y'' - 2y' + y = (1+x-x^2)e^x \longrightarrow (1)$

Comparing with Std. form

$$y'' + py' + qy = R(x)$$

$\Rightarrow R(x) = (1+x-x^2)e^x$

$= (\text{poly}) \times (\text{exp})$

$=$ Case - (ii)

S1) Finding general solⁿ :-

$y_g(x) = y'' - 2y' + y = 0 \longrightarrow (2)$

Finding Auxiliary eqⁿ

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = \underbrace{1, 1}_{\text{real \& equal roots}}$

So, $y_g(x) = (c_1 + c_2 x)e^{mx}$

$\Rightarrow y_g(x) = (c_1 + c_2 x)e^x = c_1 e^x + c_2 x e^x \longrightarrow (3)$

S2) Building trial solⁿ :-

RHS $= (1+x-x^2)e^x = h$, say

$\Rightarrow h' = e^x(1+x-x^2+1-2x)$

$\Rightarrow h' = e^x(2-x-x^2)$

$h'' = e^x(2-x-x^2-1-2x)$

$\Rightarrow h'' = e^x(1-3x-x^2)$

So, initial trial solⁿ $= d_1 x^2 e^x + d_2 x e^x + d_3 e^x \longrightarrow (4)$

(Collecting independent terms)

$\{e^x, xe^x, x^2e^x\}$

multiplying only by x will make the term $x(d_3 e^x)$ duplicating with $c_2 x e^x$.

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S3) Comparing (4) & (3), we find ∃ duplic^n.
So, $\times\, x^2$ by eq^n (4), we get (in order to prevent any duplication)

$$y = x^2 e^x (d_1 x^2 + d_2 x + d_3) \rightarrow (5)$$

$$\Rightarrow y = e^x\, q(x)$$

$$q(x) = d_1 x^4 + d_2 x^3 + d_3 x^2 \rightarrow (6)$$

$$\Rightarrow y = e^x q(x)$$

Using this in eq^n (1)

S4) $$\Rightarrow [e^x q(x)]'' - 2[e^x q(x)]' + e^x q(x) = (1+x-x^2)e^x$$

$$\Rightarrow [e^x(q'(x)+q(x))]' - 2[e^x(q'(x)+q(x))] + e^x q(x) = (1+x-x^2)e^x$$

$$\Rightarrow e^x(\cancel{q'(x)}) + e^x q''(x) + \cancel{e^x q(x)} + \cancel{e^x q'(x)} - 2\cancel{e^x q'(x)} - 2\cancel{e^x q(x)} + \cancel{e^x q(x)} = (1+x-x^2)e^x$$

$$\Rightarrow e^x q''(x) = e^x(1+x-x^2)$$

$$\Rightarrow q''(x) = 1+x-x^2. \rightarrow (7)$$

From (6),

$$q'(x) = 4d_1 x^3 + 3d_2 x^2 + 2d_3 x$$

$$q''(x) = 12 d_1 x^2 + 6 d_2 x + 2d_3.$$

Comparing it with eq^n (7).

$$\left.\begin{aligned} \Rightarrow\ 12 d_1 &= -1 \Rightarrow d_1 = -1/12 \\ 6 d_2 &= 1 \Rightarrow d_2 = 1/6 \\ 2 d_3 &= 1 \Rightarrow d_3 = 1/2 \end{aligned}\right\} \rightarrow (8)$$

S5) Substituting (8) in (5), we get the particular sol^n

$\Rightarrow y_p = x^2(d_1 x^2 e^x + d_2 x e^x + d_3 e^x)$.

$\Rightarrow y_p = \left(\frac{-x^2}{12}e^x + \frac{x}{6}e^x + \frac{e^x}{2}\right)x^2$.

$\Rightarrow y_p(x) = x^2\frac{e^x}{2}\left[\frac{-x^2}{6} + \frac{x}{3} + 1\right] \longrightarrow (9)$

S6) So, complete sol^n is given by :-

$y(x) = y_g(x) + y_p(x)$

So, from (3) & (9)

$\Rightarrow y(x) = (c_1 + c_2 x)e^x + x^2\frac{e^x}{2}\left[\frac{-x^2}{6} + \frac{x}{3} + 1\right]$

Ans

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Q4) $y'' + y' = 10x^4 + 2 \longrightarrow (1)$

Comparing with std. form,

$R(x) = 10x^4 + 2$

Its case(i).

So,

S1) Homogeneous eqⁿ solⁿ

$y'' + y' = 0 \longrightarrow (2)$

$\Rightarrow y_g(x) = c_1 e^{0x} + c_2 e^{-1x}$ [$m_1 = 0, m_2 = -1$ by finding Auxiliary eqⁿ]

$\Rightarrow y_g(x) = c_1 + c_2 e^{-x} \longrightarrow (3)$

S2) Build initial trial solⁿ

$RHS = 10x^4 + 2 = h$

$h' = 40x^3$

$h'' = 120x^2$

$h''' = 240x$

$h'''' = 240(1)$

$h''''' = 0.$

Independent terms: x^4, x^3, x^2, x, x^0

So, isolating independent terms (including RHS)

$\Rightarrow y = d_1x^4 + d_2x^3 + d_3x^2 + d_4x + d_5 \longrightarrow (4)$

S3) Checking duplicⁿ: Comparing (4) & (3)

x^0 is duplicating term

$\therefore \times$ (4) by x

$\Rightarrow y = d_1x^5 + d_2x^4 + d_3x^3 + d_4x^2 + d_5x \longrightarrow (5)$

final trial solⁿ.

S2, S3 not req^d in this case

* Note : Normally, in such type of eq^{ns}, if 'y' terms are missing in std. form, then, multiply initial trial sol^n by x to get final trial sol^n.
* Note :- If, $R(x)$ is a polynomial f^n of degree 'n' (in std. form). Then, INITIAL trial sol^n is also a polynomial f^n (of same degree 'n')

S4) Using ⑤ in ①.

$$\Rightarrow (20d_1x^3 + 12d_2x^2 + 6d_3x + 2d_4) + (5d_1x^4 + 4d_2x^3 + 3d_3x^2 + 2d_4x + d_5) = 10x^4 + 2$$

$$\Rightarrow x^4(5d_1) + x^3(20d_1 + 4d_2) + x^2(12d_2 + 3d_3) + x(6d_3 + 2d_4) + 1.(2d_4 + d_5)$$

Comparing coeff. & solve

$$\Rightarrow \left.\begin{aligned} d_1 &= 2 \\ d_2 &= -10 \\ d_3 &= 40 \\ d_4 &= -120 \\ d_5 &= 242 \end{aligned}\right\} \rightarrow 6$$

S5) Substituting ⑥ in ⑤

$$\Rightarrow y_p(x) = 2x^5 - 10x^4 + 40x^3 - 120x^2 + 242x$$

S6) Complete sol^n, $y(x) = y_g(x) + y_p(x)$

$$\Rightarrow y(x) = (C_1 + C_2e^{-x}) + (2x^5 - 10x^4 + 40x^3 - 120x^2 + 242x)$$

Ans

* If RHS is a constt, particular solⁿ is also that constt.

Q.5) $y'' - y' + y = 2 + e^x + \sin x \longrightarrow (A)$

Comparing with std. form, we find

$R(x) = 2 + e^x + \sin x$

Its of case - IV

$\therefore$ Its as a sum of 3 terms, so, divide the eqⁿ in 3 eqⁿˢ.

Particular solⁿ

So, $y'' - y' + y = 2 + e^x + \sin x \Rightarrow \begin{cases} y'' - y' + y = 2 \Rightarrow y_1 \to (B) \\ y'' - y' + y = e^x \Rightarrow y_2 \to (C) \\ y'' - y' + y = \sin x \Rightarrow y_3 \to (D) \end{cases}$

So, particular solⁿ for $y(x)$,

$y_P = (y_1 + y_2 + y_3) \quad (1)$

S1) homogeneous part is same ∀ eqⁿˢ.

Solving :- $y'' - y' + y = 0 \longrightarrow (2)$

Writing auxiliary eqⁿ

$\Rightarrow m^2 - m + 1 = 0$

$\Rightarrow m = \frac{1 \pm \sqrt{3}i}{2} \;(\Rightarrow \alpha + i\beta)$

Complex root pair

So, $y_g(x) = e^{\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) \longrightarrow (3)$

* Finding particular solⁿ for eqⁿˢ (1).

Particular solⁿ for (B)

S2) Trial solⁿ :- RHS = 2 = h

$h' = 0$

So, $y = d_1$, a constt

Substituting in (B)

$\Rightarrow \underbrace{d_1''}_{0} - \underbrace{d_1'}_{0} + d_1 = 2 \Rightarrow \boxed{d_1 = 2}$

$\therefore$ req^d particular sol^n for eq^n (B) is
$y_1 = 2$.

// Particular sol^n for (C).

S2) Trial sol^n $\therefore$ $RHS = e^x = h$

$h' = e^x$

$h'' = e^x$, remains same.

So, $y_1 = d_2 e^x$
, constt.

Substituting in (C)

$\Rightarrow \cancel{d_2 e^x} - \cancel{d_2 e^x} + d_2 e^x = e^x \Rightarrow d_2 = 1$

So, particular sol^n, $y_2 = 1$

// Particular sol^n for (D).

S2) Trial sol^n: $RHS = \sin x = h$

$h' = \cos x$

$h'' = -\sin x$, repeating terms

So, $y = d_3 \sin x + d_4 \cos x$

$\Rightarrow y' = d_3 \cos x - d_4 \sin x$

$y'' = -d_3 \sin x - d_4 \cos x$

Substituting in (D)

$\Rightarrow -\cancel{d_3 \sin x} - \cancel{d_4 \cos x} - d_3 \cos x + d_4 \sin x + \cancel{d_3 \sin x} + \cancel{d_4 \cos x} = \sin x$

$\Rightarrow d_3 = 0, \; d_4 = 1$

So, particular sol^n, $y_3 = \cos x$.

S5 So, reqd particular sol^n, $y_p = y_1 + y_2 + y_3$

$\Rightarrow y_p = 2 + e^x + \cos x$

S6) So, complete sol^n, $y(x) =$ Ans

$y(x) = y_g(x) + y_p(x) = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right) + \left(2 + e^x + \cos x\right)$

Section - 22

☆ Higher Order Linear D E : Homogeneous

form:- $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \rightarrow (1)$

↳ n^{th} order homogeneous D.E

Way to solve :- (already told in beginning of Sec-18)

✓ Write in terms of :-

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1}D + a_n)y = 0$

↳ $\underbrace{D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, \dots \; D^n = \frac{d^n}{dx^n}.}_{\text{Differential operators}}$

✓ Write auxiliary eq^n $(D = m, D^n = m^n)$

$m^n + a_1 m^{n-1} + \dots + a_{n-1}m + a_n = 0$

Solve auxiliary eq^n & find its n roots.

- Based on roots (nature), write the general sol^n of eq^n (1)
- Nature of roots.

real equal complex.

Problems

Q1) $y''' - 3y'' + 2y' = 0$

2) $y''' - 3y'' + 4y' - 2y = 0$

3) $y''' + y = 0$

4) $y^{(4)} + 5y'' + 4y = 0$

5) $y^{(4)} + 2y''' + 2y'' + 2y' + y = 0$

☆ $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
☆ $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

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① $y''' - 3y'' + 2y' = 0$
$\Rightarrow (D^3 - 3D^2 + 2D)y = 0$

$\therefore$ Auxiliary eq^n:-
$m^3 - 3m^2 + 2m = 0$
$\Rightarrow m(m^2 - 3m + 2) = 0$
$\Rightarrow m(m-1)(m-2) = 0$
$\Rightarrow m = 0, 1, 2$
The roots are real & distinct
So, sol^n (general) :-
$y(t) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
$= c_1 e^0 + c_2 e^x + c_3 e^{2x}$
$\Rightarrow y(t) = c_1 + c_2 e^x + c_3 e^{2x}$

☆ Note :- RHS = 0 in part ①
$\Rightarrow$ Particular sol^n $\cancel{\exists}$

② $y''' - 3y'' + 4y' - 2y = 0 \longrightarrow$ ①
$\Rightarrow (D^3 - 3D^2 + 4D - 2)y = 0$,
Writing Auxiliary eq^n
$y''' = m^3,\ y'' = m^2,\ y' = m,\ y = 1$
$\Rightarrow (m^3 - 3m^2 + 4m - 2) = 0 \longrightarrow$ ②

M1 : to solve cubic eq^n
$\Rightarrow m^3 - 1 - (3m^2 - 4m + 1) = 0$
$\Rightarrow (m-1)(m^2 + 1 + m) - (3m^2 - 3m - m + 1) = 0$
$\Rightarrow (m-1)(m^2 + m + 1) - [(3m)(m-1) - (1)(m-1)] = 0$
$\Rightarrow (m-1)(m^2 + m + 1) - (m-1)(3m-1) = 0$
$\Rightarrow (m-1)[m^2 + m + 1 - 3m + 1] = 0$
$\Rightarrow (m-1)[m^2 - 2m + 1 + 1] = 0$
$\Rightarrow (m-1)[(m-1)^2 + 1] = 0$

$$\Rightarrow \boxed{m = 1,\ 1 \pm i} \rightarrow \text{imaginary root pairs}$$

$\hookrightarrow$ real root

M2: to solve cubic eqⁿ — Using trial & error method & finding one of the roots

Sum of coeff $= 1 - 3 + 4 - 2 = 0$

So, $m = 1$ is a root of eqⁿ (2)

So $(m-1)$ is a factor in eqⁿ (2).

Now, finding remaining roots :-

Method (a) to ~~solve~~ find remaining roots

$$\begin{array}{r|l}
 & m^2 - 2m + 2 \\
m-1 & m^3 - 3m^2 + 4m - 2 \\
 & \underline{m^3 - m^2} \\
 & \quad -2m^2 + 4m - 2 \\
 & \quad \underline{-2m^2 + 2m} \\
 & \qquad 2m - 2 \\
 & \qquad \underline{2m - 2} \\
 & \qquad\quad 0
\end{array}$$

So, $(m^3 - 3m^2 + 4m - 2) = (m-1)(m^2 - 2m + 2)$

Method (b) : Synthetic Division — to find remaining roots

Coeff of eqⁿ (2)

$$\begin{array}{c|cccc}
 & m^3 & m^2 & m & \text{const} \\
1 & 1 & -3 & 4 & -2 \\
 & 0 & 1 & -2 & 2 \\
\hline
+ & 1 & -2 & 2 & \boxed{0}
\end{array}$$

one of the roots we got

always (root × 0) + 1

(root × 1) + (−3)

we got 0 in end ⇒ it's a factorised now.

$\Rightarrow m^2 - 2m + 2 = 0$ is other eqⁿ.

Solving, we get $m = 1 \pm i$

So, both M1 & M2 give same roots

(use any method :-
either M1 (by analysing) OR M2

M2: OR
- Method a (Dividing)
- Method b (Synthetic division))

So, general solⁿ is

$$y_g(x) = y(x) = c_1 e^{m_1 x} + e^{\alpha x}(c_2 \cos\beta x + c_3 \sin\beta x)$$

$$\Rightarrow y(x) = \underbrace{c_1 e^{x}}_{\text{for } m=1} + \underbrace{e^{x}(c_2 \cos x + c_3 \sin x)}_{\text{corresponding to } m = 1 \pm i}$$

---

(3) $y''' + y = 0$

Auxiliary eqⁿ

$m^3 + 1 = 0$

M1 $\Rightarrow (m+1)(m^2 - m + 1) = 0 \quad [\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)]$

M2 Method (b) Synthetic division

One of the roots is -1 (clearly)

$\Rightarrow m+1$ is a factor

$$\Rightarrow \quad -1 \;\; \begin{array}{|cccc} m^3 & m^2 & m & \text{const} \\ 1 & 0 & 0 & 1 \\ \hline 0 & -1 & 1 & -1 \\ \hline 1 & -1 & 1 & \boxed{0} \end{array}$$

$\Rightarrow m^2 - m + 1 = 0$ is the other factor

$\Rightarrow (m+1)(m^2 - m + 1) = 0$ are factors

(3) $m^3 = -1$

$\Rightarrow m$ = cube root of -1

$\Rightarrow -1, \omega, \omega^2$

So, Solving, we get

$$m = -1, \frac{1 \pm i\sqrt{3}}{2}.$$

or $-1, \omega, \omega^2$ → complex root pair.

So, general solⁿ,

$$y_g(x) = y(x) = c_1 e^{m_1 x} + e^{\alpha x}[c_2 \cos\beta x + c_3 \sin\beta x]$$

$$\Rightarrow y(x) = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right]$$

Ans

(4) $y^{(4)} + 5y'' + 4y = 0$

Auxiliary eqⁿ :-

$m^4 + 5m^2 + 4 = 0$

$\Rightarrow m^4 + 4m^2 + m^2 + 4 = 0$

$\Rightarrow m^2(m^2+4) + 1(m^2+4) = 0$

$\Rightarrow (m^2+1)(m^2+4) = 0$

$\Rightarrow m = \pm i, \; m = \pm 2i$

2 imaginary root pairs

So, $y_g(x) = e^{\alpha_1 x}[c_1 \cos\beta_1 x + c_2 \sin\beta_1 x]$

$+ e^{\alpha_2 x}[c_3 \cos\beta_2 x + c_4 \sin\beta_2 x]$

$= e^0[c_1 \cos x + c_2 \sin x]$

$+ e^0[c_3 \cos 2x + c_4 \sin 2x]$

$\Rightarrow y_g(x) = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x = y(x)$

(5) $y^{(4)} + 2y''' + 2y'' + 2y' + y = 0$.

Auxiliary eq^n:-

$$m^4 + 2m^3 + 2m^2 + 2m + 1 = 0$$

(M1) $\Rightarrow m^4 + 2m^3 + m^2 + m^2 + 2m + 1 = 0$

$\Rightarrow m^2[m^2 + 2m + 1] + 1(m^2 + 2m + 1) = 0$

$\Rightarrow (m^2+1)(m^2 + 2m + 1) = 0$

$\Rightarrow (m^2+1)(m+1)^2 = 0$.

$\Rightarrow m = -1, -1, \pm i$

(M2) Sum of even coeff = Sum of odd coeff

even coeff → power is even

$$\Rightarrow \underset{\text{even}}{y^{(4)}} + \underset{\text{odd}}{2y'''} + \underset{\text{even}}{2y''} + \underset{\text{odd}}{2y'} + \underset{\text{even}}{y} = 0$$

$$\Rightarrow 1 + 2 + 1 = 2 + 2$$

Same.

Valid everywhere. ⭑ If sum of even coeff = Sum of odd coeff. $\Rightarrow -1$ is a root to given eq^n.

Now, $m+1$ is a factor

By Synthetic division

$$\begin{array}{r|rrrrr} -1 & 1 & 2 & 2 & 2 & 1 \\ & 0 & -1 & -1 & -1 & -1 \\ \hline & 1 & +1 & 1 & 1 & 0 \end{array}$$

$$\underset{\text{odd}}{m^3} + \underset{\text{even}}{m^2} + \underset{\text{odd}}{m} + \underset{\text{even}}{1}$$

So, $1 + 1 = 1 + 1$ (even coeff = odd coeff sum)

$\Rightarrow -1$ is again a root of cubic eq^n

⇒ Using synthetic division again

$$\begin{array}{r|rrrr|l} -1 & 1 & 1 & 1 & 1 & 0 \\ & 0 & -1 & 0 & -1 & \\ \hline & 1 & 0 & 1 & 1 & \boxed{0} \end{array}$$

$\Rightarrow m^2 + 1 = 0$

Resulting quadratic eq^n

So, the final factors are

$$(m+1)(m+1)(m^2+1) = 0$$

So, roots are

$$m = \underbrace{-1, -1}_{\text{real \& equal}}, \underbrace{\pm i}_{\text{complex pair}}$$

So,

general sol^n

$$y(x) = (c_1 + c_2 x)e^{mx} + e^{\alpha x}[c_3 \cos\beta x + c_4 \sin$$

$$\Rightarrow y(x) = (c_1 + c_2 x)e^{-x} + e^0[c_3 \cos x + c_4 \sin x$$

Ans

Section - 23

Method 2) OPERATION METHODS FOR FINDING PARTICULAR SOLUTION

(Method 1 : Method of Undetermined Coeff.)

Let

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x) \longrightarrow ①$$

↳ non homogeneous, n^{th} order, linear DE

* Note :- here, the RHS $\neq 0$ (i.e., $f(x) \neq 0$).
So, we'll need to find the particular solⁿ, $y_p(x)$.
If $f(x) = 0$, then, $y(x) = y_g(x) + \cancel{y_p(x)} \to 0$
or $y(x) = y_g(x)$

In the operator form:-

$$D^n y + a D^{n-1} y + \cdots + a_{n-1} D y + a_n y = f(x)$$

↳ $\left(D^n = \frac{d^n}{dx^n},\ D^{n-1} = \frac{d^{n-1}}{dx^{n-1}},\ \ldots\ D = \frac{d}{dx} \right)$

∴ eqⁿ ① can be rewritten as:

$$\underbrace{\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n\right)}_{\text{a polynomial in } D} y = f(x)$$

$$\Rightarrow \left(P(D)\right) y = f(x) \longrightarrow ②$$

↳ $P(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n$

↳ a polynomial of n^{th} degree in D.

Now, solving eq^n ②.
Idea, factorize the above (P(D)), s.t :-

$$P(D) = (D-r_1)(D-r_2) - - - - (D-r_n) \longrightarrow ④$$

↳ $r_1, r_2, ..., r_n$: roots of $P(D) = 0$

Now,
Particular sol^n for given DE is :-

$$y_P = \left\{ \frac{1}{P(D)} \right\} f(x) \longrightarrow ③$$

→ called as operator f^n

★ Note ① : We know, $D = \frac{d}{dx}$ & $D^2 = \frac{d^2}{dx^2}$

$$\therefore \frac{1}{D}(\ ) = \int (\ ) dx$$

$$\& \frac{1}{D^2}(\ ) = \iint (\ ) dx$$

→ Consider $P(D) = Dy$
So, eq^n ① becomes $Dy = f(x)$

$$\text{or } y = \frac{1}{D} f(x) \rightarrow ⓐ$$

We can say $\frac{d}{dx}(y) = f(x)$

$$\text{or } y = \int f(x) dx \rightarrow ⓑ$$

from ⓐ & ⓑ,

$$\frac{1}{D} f(x) = \int f(x) dx$$

$$\text{or } \frac{1}{D}(\ ) = \int (\ ) dx$$

i.e, operating $\frac{1}{D}$ on anything ⇒ Integrate that thing

* Note (2) : Now, consider eq^n (1) as

$$(D-r)y = f(x) \longrightarrow (c)$$

r : constt.

$$\Rightarrow y = \frac{1}{(D-r)} f(x) \longrightarrow (d)$$

eq^n (c) can be rewritten as

$$Dy - ry = f(x)$$

or $$\underbrace{\frac{d}{dx}y - ry = f(x)}_{\text{linear D.E 1'st order.}} \longrightarrow (e)$$

~~from (d) & (e)~~

Solving :- $P(x) = -r$, $Q(x) = f(x)$

$$IF = e^{\int P(x)dx} = e^{-rx}$$

$$\Rightarrow y(IF) = \int (IF)Q(x)dx$$

$$\Rightarrow y(e^{-rx}) = \int f(x) e^{-rx} dx$$

Or $$y = e^{rx} \int e^{-rx} f(x) dx \longrightarrow (f)$$

Now, from (d) & (f)

**
$$\boxed{\begin{aligned} \frac{1}{(D-r)} f(x) &= e^{rx} \int e^{-rx} f(x) dx \\ \text{IIIly, } \frac{1}{(D+r)} f(x) &= e^{-rx} \int e^{rx} f(x) dx \end{aligned}} \longrightarrow (A)$$

These formulae can be used $\forall$ problems which have an operator acting.

★ Solving eqⁿ (3):
METHODS to operate $\left\{\frac{1}{P(D)}\right\}$ on $f(x)$.

METHOD 1: Successive Integration.

From (3), $y_p = \left\{\frac{1}{P(D)}\right\} f(x)$

$$= \left[\frac{1}{(D-r_1)(D-r_2)\cdots(D-r_n)}\right] f(x) \quad (\text{from } (4))$$

$$= \left[\frac{1}{(D-r_1)} \cdot \frac{1}{(D-r_2)} \cdots\cdots \frac{1}{(D-r_n)}\right] f(x)$$

1st operⁿ gives a fⁿ

2nd

(n-1)th operⁿ

nth operⁿ

(using the idea :- $\left(\frac{1}{D-r_n}\right) f(x)$ can be solved by previously shown methods) → See eqⁿ (A)

METHOD 2: Partial Fractions Decomposⁿ of Operators:

from (3), $y = \left\{\frac{1}{P(D)}\right\} f(x)$

$$= \left[\frac{1}{(D-r_1)} \cdot \frac{1}{(D-r_2)} \cdot \frac{1}{\cdots} \cdot \frac{1}{(D-r_n)}\right] f(x)$$

(from (4))

Called as Heaviside expansion of $\frac{1}{P(D)}$

$$\Rightarrow y = \left[\frac{A_1}{D-r_1} + \frac{A_2}{D-r_2} + \dots + \frac{A_n}{D-r_n}\right] f(x)$$

$A_1, A_2, \dots A_n$: constts.

$$\Rightarrow y = A_1\left[\frac{1}{(D-r_1)} f(x)\right] + A_2\left[\frac{1}{(D-r_2)} f(x)\right] + \dots A_n\left[\frac{1}{(D-r_n)} f(x)\right]$$

↳ can be solved by previously shown method. (see eqn (A))

METHOD 3: **Series expansions of operators :**

↳ Suitable for polynomial f^{ns}.

From (3), $y = \left\{\frac{1}{P(D)}\right\} f(x)$

$= [P(D)]^{-1} f(x)$

↳ expand using any theorem. & solve (eqn (A))

✱ Note :-

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

METHOD 4: **Exponential shift method**

↳ method used whenever the f^n $f(x)$ (RHS of eqn (1)) has a factor of the form e^{kx}.

Thus, $f(x) = e^{kx} \cdot g(x)$

Now, from (3), $y = \left\{\frac{1}{P(D)}\right\} f(x)$

$$\Rightarrow y = \frac{1}{P(D)}\left(e^{kx} \cdot g(x)\right)$$

or, $y = e^{kx}\left[\frac{1}{P(D+k)}\, g(x)\right]$

→ Consider,

$$(D-r)f(x) = (D-r)e^{kx}g(x)$$

$$= D\left[e^{kx}g(x)\right] - re^{kx}g(x)$$

$$= \left[e^{kx}\left[D\,g(x)\right] + ke^{kx}g(x)\right] - re^{kx}g(x)$$

$$\Rightarrow \boxed{(D-r)\,e^{kx}g(x) = e^{kx}\left[D+k-r\right]g(x)}$$

↳ i.e., when $r=0$

$$D\left[e^{kx}g(x)\right] = e^{kx}\left[D+k\right]g(x)$$

or $$\frac{1}{D}e^{kx}g(x) = e^{kx}\left[\frac{1}{D+k}\,g(x)\right]$$

→ Then, this eq^n can be solved by using Method 1, 2 or 3.

Problems Q.1) Use any method to solve :-

1) $y'' + 4y' + 4y = 10x^3 e^{-2x}$
2) $y'' - y = x^2 e^{2x}$
3) $y'' - 2y' + y = 2x^3 - 3x^2 + 4x + 5$.

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1) (here, apt. method is M4. But, for practice, it has been done by other method)

So, $(D^2+4D+4)y = 10x^3 e^{-2x}$

$$\therefore y = \left(\frac{1}{(D+2)(D+2)}\right) \times 10x^3 e^{-2x}$$

Now, By successive integrⁿ method,

$$y = \frac{1}{D+2}\left[\frac{1}{D+2}\left(10x^3 e^{-2x}\right)\right]$$

$$\left(\frac{1}{D+k} f(x) = e^{-kx}\int e^{kx} f(x)\,dx\right)$$

$$= \frac{1}{D+2}\left[e^{-2x}\int e^{2x}\left(10x^3 e^{-2x}\right)dx\right]$$

$$= \frac{1}{D+2}\left[e^{-2x}\int 10x^3\,dx\right]$$

$$= \frac{1}{D+2}\left[10e^{-2x}\int x^3\,dx\right]$$

$$= \frac{1}{D+2}\left[10e^{-2x}\left(\frac{x^4}{4}\right)\right]$$

$$= \frac{1}{D+2}\left[\frac{5}{2}x^4 e^{-2x}\right]$$

$$= e^{-2x}\int e^{\cancel{2x}}\left(\frac{5}{2}x^4 e^{\cancel{-2x}}\right)dx$$

$$\Rightarrow y = e^{-2x} \times \frac{5}{2}\int x^4\,dx = \frac{e^{-2x} \times x^5}{2}$$

Particular solⁿ Ans

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2) $y'' - y = x^2 e^{2x}$

Using M2: Method of operators

$$(D^2 - 1)y = x^2 e^{2x}$$
$$\Rightarrow y = \frac{1}{(D^2-1)} x^2 e^{2x}$$
$$= \left(\frac{1}{(D+1)(D-1)}\right) x^2 e^{2x}$$

↳ from partial fractions / analysing

$$= \frac{1}{2}\left[\frac{\overbrace{D+1-(D-1)}^{2=}}{(D+1)(D-1)}\right] x^2 e^{2x}$$
$$= \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right] x^2 e^{2x}$$
$$= \frac{1}{2}\left[\left(\frac{1}{D-1}\right)x^2 e^{2x} - \left(\frac{1}{D+1}\right) x^2 e^{2x}\right]$$

From eqn (A),

$$= \frac{1}{2}\left[e^{x}\int e^{-x}(x^2 e^{2x})dx - e^{-x}\int e^{x}(x^2 e^{2x})dx\right]$$
$$= \frac{1}{2}\left[e^{x}\int \underset{u}{x^2}\,\underset{v}{e^{x}}\, dx - e^{-x}\int \underset{u}{x^2}\,\underset{v}{e^{3x}}\, dx\right]$$
$$= \frac{1}{2} \cdot e^{x}\left[x^2 e^{x} - 2x e^{x} + 2e^{x}\right]$$
$$- \frac{1}{2} e^{-x}\left[x^2 \frac{e^{3x}}{3} - (2x)\left(\frac{e^{3x}}{3^2}\right) + 2\cdot\frac{e^{3x}}{3^3}\right]$$

(Bernoulli's formula → $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$)

$$\Rightarrow y = \frac{1}{2}\left[x^2e^{2x} - 2xe^{2x} + 2e^{2x} - \frac{x^2e^{2x}}{3} + \frac{2}{9}xe^{2x} - \frac{2}{27}e^{2x}\right]$$

$$= \frac{1}{2}e^{2x}\left[\left(1-\frac{1}{3}\right)x^2 - \left(2-\frac{2}{9}\right)x + \left(2-\frac{2}{27}\right)\right]$$

$$\Rightarrow y = \frac{1}{2}e^{2x}\left[\frac{2}{3}x^2 - \frac{16}{9}x + \frac{52}{27}\right]$$

$$\Rightarrow y = \frac{e^{2x}}{27}\left[9x^2 - 24x + 26\right]$$

ALITER :- Using M4 : Exponential shift

$$y = \left(\frac{1}{D^2-1}\right)e^{2x}x^2$$

$$= e^{2x}\left[\left(\frac{1}{(D+2)^2-1}\right)x^2\right]$$

(By replacing $D \longrightarrow D+2$)

$$= e^{2x}\left[\frac{1}{[(D+2)+1][(D+2)-1]}x^2\right]$$

$$= e^{2x}\left[\frac{1}{(D+3)(D+1)}x^2\right]$$

$(a^2-b^2 = (a+b)(a-b))$

$\Rightarrow$ By Successive Integrⁿ, $y = e^{2x}\left[\frac{1}{(D+3)} \cdot e^{-x}\int e^x \cdot x^2\,dx\right]$

$$= e^{2x}\cdot\left(\frac{1}{D+3}\right)\left[e^{-x}\left[\underbrace{x^2e^x - 2xe^x + 2e^x}_{\text{Bernoulli's}}\right]\right]$$

$$\Rightarrow y = e^{2x}\left[\frac{1}{D+3}\cdot\left(x^2-2x+2\right)\right]$$

$$= e^{2x}\left[e^{-3x}\int \underbrace{e^{3x}}_{v}\,\underbrace{\left(x^2-2x+2\right)}_{u}\,dx\right]$$

$$= e^{2x}\left[e^{-3x}\left[\left(x^2-2x+2\right)\frac{e^{3x}}{3} - (2x-2)\frac{e^{3x}}{3^2} + 2\,\frac{e^{3x}}{3^3}\right]\right]$$

Bernoulli's

$$= e^{2x}\left[\frac{x^2-2x+2}{3} - \frac{(2x-2)}{9} + \frac{2}{27}\right]$$

$$= \frac{e^{2x}}{27}\left[9x^2-18x+18-6x+6+2\right]$$

$$\Rightarrow y = \frac{e^{2x}}{27}\left[9x^2-24x+26\right]$$

(Same as previous ans.)

—x—

3) $y''' - 2y' + y = 2x^3 - 3x^2 + 4x + 5$

Finding particular solⁿ: using operator method

$$\left(D^3-2D+1\right)y = 2x^3-3x^2+4x+5$$

RHS is polynomial f^n. So, M3 is applicable

$$\Rightarrow y = \left(\frac{1}{D^3-2D+1}\right)\left(2x^3-3x^2+4x+5\right)$$

$$\Rightarrow y = \left[1+(D^3-2D)\right]^{-1}\underbrace{(2x^3-3x^2+4x+5)}_{\text{polynomial of deg. 3}}$$

$$\parallel\!\mid$$

$$(1+x)^{-1}$$

we know

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

Idea: we have a polynomial of deg. 3.
So, all higher powers of D ($D^4, D^5, \ldots$) will make polynomial zero.
So, discard them.

$$\Rightarrow y = \left[1-(D^3-2D)+(D^3-2D)^2-(D^3-2D)^3+\cdots\right]\left[2x^3-3x^2+4x+5\right]$$

$$= \begin{bmatrix} 1 - D^3 + 2D + [\cancel{D^6} - \cancel{4D^4} + 4D^2] \\ -[\cancel{D^9} - \cancel{6D^7} + \cancel{12D^5} - 8D^3] + \\ + \ldots (\text{terms of higher degree}) \end{bmatrix} \times \begin{bmatrix} 2x^3 - 3x^2 \\ + 4x + 5 \end{bmatrix}$$

$$\Rightarrow y = (1 - D^3 + 2D + 4D^2 + 8D^3 + 0)\left[2x^3 - 3x^2 + 4x + 5\right]$$

$$= (7D^3 + 4D^2 + 2D + 1)\left[2x^3 - 3x^2 + 4x + 5\right]$$

$$= 1.(2x^3-3x^2+4x+5) + 2\left[6x^2 - 6x + 4 + 0\right]$$

$\frac{d}{dx}$ of polynomial

$$+ 4\underbrace{\left[12x - 6 + 0\right]}_{\frac{d^2}{dx^2}\text{ of polynomial}} + 7\underbrace{\left[12\right]}_{\frac{d^3}{dx^3}\text{ of polynomial}}.$$

$$\Rightarrow y = 2x^3 - 3x^2 + 4x + 5 + 12x^2 - 12x + 8 + 48x - 24 + 84$$

$$\Rightarrow y = 2x^3 + 9x^2 + 40x + 73$$

$$\Rightarrow y = \underbrace{2x^3 + 9x^2 + 40x + 73}_{\text{particular sol}^n} = y_p(x)$$

Note : for finding complete sol^n,
find sol^n for homogeneous eq^n $\rightarrow$ general sol^n.
then, total sol^n = $y_g(x) + y_p(x)$

done before.

Q 4) $y^{(6)} - y = x^{10}$.
In terms of operator,
$$(D^6 - 1)y = x^{10}.$$
The req^d particular sol^n,
$$y = \left(\frac{1}{D^6 - 1}\right) x^{10}.$$

degree of polynomial = 10.
So, discard higher powers of D, from D^{11}.

$$\Rightarrow y = -\underbrace{[1 - D^6]^{-1}}_{(1-x)^{-1}\,;\, x = D^6.} x^{10}$$

$$= -[1 + x + x^2 + x^3 + \dots]\, x^{10}$$
$$= -[1 + D^6 + D^{12} + \dots]\, x^{10}$$

$$= -[1 + D^6]\, x^{10}$$

$$\Rightarrow y_p = -[x^{10}] - \underbrace{[10.9.8.7.6.5x^4]}_{D^6 . x^{10}.}$$

Ans

Q5) $y'' + y' - y = 3x - x^4$.

By operator method,

$$(D^2 + D - 1)y = 3x - x^4$$

Finding particular sol^n:-

$$y = \frac{1}{\underbrace{(D^2+D-1)}_{\text{not factorizable}}}(3x - x^4)$$

So, cannot be solved as :- $\frac{1}{(D-r_1)(D-r_2)} \cdot P(x)$

$$= \left[\frac{A}{r_1(1-\frac{D}{r_1})} + \frac{B}{r_2(1-\frac{D}{r_2})}\right] \times P(x)$$

$$\Rightarrow y = -[\underbrace{1-(D^2+D)}_{(1-x)^{-1}}]^{-1}\,\underbrace{(3x-x^4)}$$

Polynomial of deg: 4.
So, D^5 & higher powers neglected

$$= -[1+x+x^2+x^3+\dots][x^4-3x]$$

$$= [1+(D^2+D)+(D^2+D)^2+(D^2+D)^3+(D^2+D)^4+\dots+(\text{higher powers})][x^4-3x]$$

$$= [1+(D^2+D)+(D^4+D^2+2D^3)+(\cancel{D^6}+D^3+\cancel{3D^5}+3D^4)+(\cancel{D^8}+\cancel{4D^7}+\cancel{6D^6}+\cancel{4D^5}+D^4)+\dots]$$

$$\times (x^4-3x)$$

Note :- $(a+b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 a b^3 + {}^4C_4 b^4$

* $(a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$

$$\Rightarrow y = \left(1 + 2D^2 + D + 3D^3 + 5D^4\right)(x^4 - 3x)$$

$$= (x^4 - 3x) + [4x^3 - 3] + 2[12x^2 - 0]$$

$$+ 3[24x] + 5[24]$$

$$\Rightarrow y_p = x^4 + 4x^3 + 24x^2 + 69x + 117$$ Ans

Q. 6) $y'' - 7y' + 12y = e^{2x}(x^3 - 5x^2)$

(Idea: we see an exp. f^n. So, use M4: exponential shifting)

By operator method

$\Rightarrow (D^2 - 7D + 12)y = e^{2x}(x^3 - 5x^2)$

So, particular sol^n:-

$$y = \left(\frac{1}{D^2 - 7D + 12}\right) e^{2x}(x^3 - 5x^2)$$

By exponential shift :-

$$y = e^{2x}\left[\frac{1}{(D+2)^2 - 7(D+2) + 12}\right](x^3 - 5x^2)$$

$$= e^{2x}\left[\frac{1}{D^2 + 4D + 4 - 7D - 14 + 12}\right](x^3 - 5x^2)$$

$$= e^{2x}\left[\frac{1}{D^2 - 3D + 2}\right](x^3 - 5x^2)$$

$$= e^{2x}\left[\frac{1}{(D-2)(D-1)}\right][x^3 - 5x^2]$$

$$\Rightarrow y = e^{2x}\left[\frac{(D-1)-(D-2)}{(D-2)(D-1)}\right](x^3-5x^2)$$

$$= e^{2x}\underbrace{\left[\left(\frac{1}{D-2}\right)-\left(\frac{1}{D-1}\right)\right]}_{\text{Aliter : partial fraction (M2)}}(x^3-5x^2)$$

$$= e^{2x}\left\{\left(\frac{1}{D-2}\right)\underbrace{x^3-5x^2}_{\text{Polynomial fn}} - \left(\frac{1}{D-1}\right)\underbrace{x^3-5x^2}\right\}$$

So, dont use eq^n (1) for solving. Solve by M3.

$$= e^{2x}\underbrace{\left[-\frac{1}{2}\left(1-\frac{D}{2}\right)^{-1}(x^3-5x^2)+(1-D)^{-1}(x^3-5x^2)\right]}_{\text{Aliter, solve directly without doing partial.}}$$

Now, its polynomial of deg. 3 ∴ So, discard terms of $D^4, D^5, \ldots$

$$= e^{2x}\left[-\frac{1}{2}\left[1+\frac{D}{2}+\left(\frac{D}{2}\right)^2+\left(\frac{D}{2}\right)^3+\ldots\right](x^3-5x^2) + \left[1+D+D^2+D^3+\cancel{D^4}+\ldots\right](x^3-5x^2)\right]$$

$$= e^{2x}\left[-\frac{1}{2}(x^3-5x^2)-\frac{(3x^2-10x)}{4}-\frac{1}{8}(6x-10) - \frac{1}{16}(6) + (x^3-5x^2)+(3x^2-10x)+(6x-10)+6\right]$$

Note :- $\frac{1}{D^2} f(x) = \iint f(x)\,dx$

$$\Rightarrow y = e^{2x}\left[\begin{array}{llll} -x^3/2 & +10/4\, x^2 & +20/8\, x & 20/16 \\ +2x^3/2 & -3/4\, x^2 & -6/8\, x & -6/16 \\ & -20/4\, x^2 & -80/8\, x & -160/16 \\ & +12/4\, x^2 & +48/8\, x & +96/16 \end{array}\right]$$

$$\Rightarrow y = e^{2x}\left[+\frac{x^3}{2} - \frac{x^2}{4} - \frac{18x}{8} - \frac{50}{16} \right]$$

It is particular solⁿ.

Q.7) $(D-2)^2 y = e^{2x} \sin x$

Particular solⁿ :-

$$y = \frac{1}{(D-2)^2} e^{2x} \sin x$$

$$= e^{2x} \left[\left[\frac{1}{(D+2)-2}\right]^2 \right] \sin x$$

$$= e^{2x} \left[\frac{1}{D^2} \right] \sin x$$

$$= e^{2x} \left(D^{-2}\right) \sin x$$

$$= e^{2x} \iint \sin x\,dx$$

$$\Rightarrow y = -e^{2x} \sin x$$

Ans

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§ Section - 19

METHOD OF VARIATION OF PARAMETERS

→ method to find particular solⁿ of 2nd order, non-homogeneous D.E

★. General 2nd ord eqⁿ :-

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \longrightarrow ①$$

→ Procedure to find particular solⁿ to eqⁿ ①.

S1) Find general solⁿ of corresponding homogeneous eqⁿ i.e,

$$y'' + P(x)y' + Q(x)\,y = 0 \longrightarrow ②$$

(say) $y_g(x) = C_1 y_1(x) + C_2 y_2(x)$

→ $y_1, y_2(x)$: 2 independent solⁿˢ of eqⁿ ②

S2) The reqᵈ particular solⁿ of eqⁿ ① ie,

$$y_p(x) = v_1(x)\,y_1(x) + v_2(x)\,y_2(x)$$

→ $v_1(x) = \int \dfrac{-y_2(x)R(x)}{W(y_1(x), y_2(x))}\,dx$

→ $v_2(x) = \int \dfrac{y_1(x)\,R(x)}{W(y_1(x), y_2(x))}\,dx$

Wronskian $= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

Note :- While Integrating at any step, dont write any integration constt. (∵ particular solⁿ is being found)

Problems

Q1) $y'' + 2y' + y = e^{-x}\log x$

2) $y'' + 4y = \tan 2x$

3) $y'' + y = \sec x$

4) $y'' + y = \cot 2x$

5) $y'' + y = \sec x \tan x$

6) $y'' + y = \cot^2 x$

1) $y'' + 2y' + y = e^{-x}\log x$

$P(x) = 2$

$Q(x) = 1$

$R(x) = e^{-x}\log x$.

S1) Finding $y_g(x)$

$\Rightarrow y'' + 2y' + y = 0$.

Auxiliary eqⁿ :-

$m^2 + 2m + 1 = 0$

$\Rightarrow m = \underbrace{-1, -1}_{\text{Real \& equal roots}}$

$\Rightarrow y_g(x) = (C_1 + C_2 x)e^{mx}$

$\Rightarrow y_g(x) = (C_1 + C_2 x)e^{-x}$

$= C_1 e^{-x} + C_2 x e^{-x}$; $y_1 = e^{-x}$, $y_2 = xe^{-x}$

S2) Finding $y_p(x)$

Now, $V_1(x) = \int \frac{-y_2(x)R(x)}{W(y_1, y_2)}\,dx$

$$W(y_1, y_2) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x}(1-x) \end{vmatrix}$$

$\Rightarrow W = e^{-2x}(1-x) + xe^{-2x} = e^{-2x}$

$$\Rightarrow W(y_1, y_2) = e^{-2x}$$

$$\Rightarrow V_1(x) = \int \frac{-x\,\cancel{e^{-x}}\cdot\left(\cancel{e^{-x}}\log x\right)}{\cancel{e^{-2x}}}\,dx$$

$$\Rightarrow V_1(x) = \int \underset{\text{II}}{-x}\ \underset{\text{I}}{\log x}\,dx$$

$$= -\left[(\log x)\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx\right]$$

$$= -\left[(\log x)\frac{x^2}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right)\right]$$

$$\Rightarrow V_1(x) = -\frac{x^2}{2}\left(\log x - \frac{1}{2}\right)$$

$$\& \quad V_2(x) = \int \frac{y_1(x)\ R(x)}{W(y_1, y_2)}\,dx$$

$$= \int \frac{e^{-x}\cdot\left(e^{-x}\log x\right)}{e^{-2x}}\,dx$$

$$= \int \underset{\text{I}}{\log x}\,dx$$

$$= \log x\,(x) - \int \frac{1}{x}\,x\,dx$$

$$\Rightarrow V_2(x) = x(\log x - 1)$$

So,

$$y_p(x) = V_1\,y_1(x) + V_2(x)\,y_2(x)$$

$$= -\frac{x^2}{2}\left(\log x - \frac{1}{2}\right)e^{-x} + x(\log x - 1)\cdot x e^{-x}$$

$$= \frac{x^2}{2}\left(\frac{1}{2} - \log x\right)e^{-x} + x^2 e^{-x}(\log x - 1)$$

$$= x^2 e^{-x}\left(\frac{1}{4} - \frac{\log x}{2} + \log x - 1\right)$$

$$\Rightarrow y_p(x) = x^2 e^{-x}\left(\frac{\log x}{2} - \frac{3}{4}\right) \quad \text{Ans}$$

2) $y'' + 4y = \tan 2x$.

$\Rightarrow y'' + 0y' + 4y = \tan 2x$

$P(x) = 0$

$Q(x) = 4$

$R(x) = \tan 2x$

S1) Finding $y_g(x)$

$y'' + 4y = 0$.

Auxiliary eq^n :-

$\Rightarrow m^2 + 4 = 0$

$\Rightarrow m = \underbrace{\pm 2i}_{\text{Imaginary pair}} \equiv 0 \pm 2i$; $\alpha = 0, \beta = 2$

So, $y_g(x) = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$\Rightarrow y_g(x) = c_1 \underbrace{\cos 2x}_{y_1(x)} + c_2 \underbrace{\sin 2x}_{y_2(x)}$.

S2) Finding $y_p(x)$.

$$W(y_1(x), y_2(x)) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$\Rightarrow W(y_1, y_2) = 2$$

Now,

$$v_1(x) = \int -\frac{y_2(x) R(x)}{W(y_1, y_2)}\, dx$$

$$= \int -\frac{\sin 2x \cdot \tan 2x}{2}\, dx$$

$$\Rightarrow v_1(x) = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= -\frac{1}{2}\left[\int \frac{1-\cos^2 2x}{\cos 2x}\,dx\right]$$

$$= -\frac{1}{2}\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$= -\frac{1}{2}\left[\frac{\log[\sec 2x+\tan 2x]}{2} - \frac{\sin 2x}{2}\right]$$

chain rule

$$\Rightarrow v_1(x) = -\frac{1}{4}\left[\log(\sec 2x+\tan 2x) - \sin 2x\right].$$

Now,

$$v_2(x) = \int \frac{y_1(x)\,R(x)}{W(y_1, y_2)}\,dx$$

$$= \int \frac{\cos 2x \tan 2x}{2}\,dx$$

$$= \frac{1}{2}\int \sin 2x\,dx$$

$$\Rightarrow v_2(x) = -\frac{1}{2}\left(\frac{\cos 2x}{2}\right)$$

$$\Rightarrow v_2(x) = -\frac{\cos 2x}{4}$$

So, $y_p(x) = v_1(x)\,y_1(x) + v_2(x)\,y_2(x)$

$$= \frac{1}{4}\left[\left[\sin 2x - \log(\sec 2x+\tan 2x)\right]\cdot\cos 2x\right] - \frac{1}{4}\sin 2x\cos 2x$$

$$\Rightarrow y_p(x) = -\frac{1}{4}\cos 2x\,\log(\sec 2x+\tan 2x)$$

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3) $y'' + y = \sec x$

$P(x) = 0$

$Q(x) = 1$

$R(x) = \sec x$

S1) $y_g(x)$

$y'' + y = 0$

$\Rightarrow$ Auxiliary eq^n :-

$m^2 + 1 = 0$

$\Rightarrow m = \underbrace{\pm i}_{\text{Imaginary pair}} = 0 \pm i \; ; \alpha = 0, \beta = 1$

$\Rightarrow y_g(x) = c_1 \underbrace{\cos x}_{y_1(x)} + c_2 \underbrace{\sin x}_{y_2(x)}$

S2) $y_p(x)$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$\Rightarrow W(y_1, y_2) = 1$

Now, $V_1(x) = \int \frac{-y_2(x) R(x)}{W(y_1, y_2)} dx$

$= \int \frac{-\sin x \cdot \sec x}{1} dx$

$= \int -\tan x \, dx$

$\Rightarrow V_1(x) = \log(\cos x)$

$$V_2(x) = \int \frac{y_1(x) R(x)}{W} dx$$

$$= \int \frac{\cos x \cdot \sec x}{1} dx$$

$$\Rightarrow V_2(x) = x$$

So,

$$y_p(x) = V_1(x)\, y_1(x) + V_2(x)\, y_2(x)$$

$$\Rightarrow y_p(x) = \log(\cos x) \cdot \cos x + x \sin x$$

Ans

4) $y'' + y = \cot 2x$

S1) $y_g(x) \rightarrow$ same as in (3)

$$\Rightarrow y_g(x) = c_1 \underbrace{\cos x}_{y_1(x)} + c_2 \underbrace{\sin x}_{y_2(x)}$$

S2) $W(y_1(x), y_2(x)) = 1$

$$V_1(x) = \int \frac{-(\sin x)(\cot 2x)}{1} dx$$

$$= \int - \frac{\sin x \cos 2x}{\sin 2x} dx$$

$$= \int - \frac{\cancel{\sin x} \cos 2x}{2 \cancel{\sin x} \cos x} dx$$

$$= -\frac{1}{2} \int \frac{2\cos^2 x - 1}{\cos x} dx$$

$$= -\int \cos x \, dx + \frac{1}{2} \int \sec x \, dx$$

$$\Rightarrow V_1(x) = -\sin x + \frac{1}{2} \log(\sec x + \tan x)$$

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Now, $V_2(x) = \int \frac{\cos x \cdot \cot 2x}{1} dx$

$= \int \frac{\cos x \cdot \cos 2x}{\sin 2x} dx$

$= \int \frac{\cos x \cdot \cos 2x}{2 \sin x \cos x} dx$

$= \frac{1}{2} \int \frac{1 - 2\sin^2 x}{\sin x} dx$

$= \frac{1}{2} \int \operatorname{cosec} x \, dx - \int \sin x \, dx$

$\Rightarrow V_2(x) = -\frac{1}{2} \log(\operatorname{cosec} x - \cot x) + \cos x$

So, $y_p(x) = V_1(x) y_1(x) + V_2(x) y_2(x)$

$= \cancel{-\sin x \cos x} + \frac{\cos x \log(\sec x + \tan x)}{2}$

$\cancel{+ \sin x \cos x} - \frac{\sin x \log(\operatorname{cosec} x - \cot x)}{2}$

$\Rightarrow y_p(x) = \frac{1}{2}\left[\cos x \log(\sec x + \tan x) - \sin x \log(\operatorname{cosec} x - \cot x)\right]$

Ans

5) $y'' + y = \sec x \tan x$.

S1) Same as before

$\Rightarrow y_g(x) = c_1 \underbrace{\cos x}_{y_1(x)} + c_2 \underbrace{\sin x}_{y_2(x)}$

$W(y_1, y_2) = 1$.

S2) $v_1(x) = \int \frac{(-\sin x)(\sec x \tan x)}{1} \, dx$

$= \int -\frac{\sin^2 x}{\cos^2 x} \, dx$

$= \int \frac{\cos^2 x - 1}{\cos^2 x} \, dx$

$= \int dx - \int \sec^2 x \, dx$

$\Rightarrow v_1(x) = x - \tan x$

$v_2(x) = \int (\cancel{\cos x})(\cancel{\sec x} \tan x) \, dx$

$\Rightarrow v_2(x) = \log(\sec x)$

So, $y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$

$\Rightarrow y_p(x) = (x - \tan x)\cos x + (\log \sec x) \sin x$

Ans

6) $y'' + y = \cot^2 x$

S1) $y_g(x) = c_1 \underbrace{\cos x}_{y_1(x)} + c_2 \underbrace{\sin x}_{y_2(x)}$

$W = 1$

S2) $y_p(x)$

$v_1(x) = \int (-\sin x) \cot^2 x \, dx$

$\star \int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$

$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x} = -\log(\operatorname{cosec} x - \cot x)$

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$$\Rightarrow V_1(x) = -\int \frac{\cos^2 x}{\sin x} dx$$

$$= \int \frac{\sin^2 x - 1}{\sin x} dx$$

$$= \int dx - \int \operatorname{cosec} x \, dx$$

$$\Rightarrow V_1(x) = x + \log(\operatorname{cosec} x - \cot x)$$

$$V_2(x) = \int (\cos x)(\cot^2 x) dx$$

$$= \int \frac{\cos^3 x}{\sin^2 x} dx$$

$$= \int \frac{\cos x(1 - \sin^2 x)}{\sin^2 x} dx$$

$$= \int \frac{\cos x \, dx}{\sin^2 x} - \int \cos x \, dx$$

$$= \int \operatorname{cosec} x \cot x \, dx - \int \cos x \, dx$$

$$\Rightarrow V_2(x) = -\operatorname{cosec} x - \sin x$$

Aliter

$\int \frac{(1 - \sin^2 x) \, d(\sin x)}{\sin^2 x}$

$= \int \left(\frac{1}{\sin^2 x} - 1\right)$

$= \int \left(\frac{1}{t^2} - 1\right)$

So, $y_p(x) = V_1(x) y_1(x) + V_2(x) y_2(x)$

$$\Rightarrow y_p(x) = (x + \log(\operatorname{cosec} x - \cot x))\cos x - (\operatorname{cosec} x + \sin x)\sin x$$

Ans

* Closed form solⁿ to a DE ⇒ a solⁿ satisfying DE
* Not a closed form solⁿ ⇒ a solⁿ which itself is a fn.

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SERIES SOLUTION METHOD

IDEA * Consider D.E : $y'' + 5y' + 6 = 0 \rightarrow (1)$

$\Rightarrow m^2 + 5m + 6 = 0$

$\Rightarrow m = -2, -3$

$\Rightarrow y(x) = c_1 e^{-2x} + c_2 e^{-3x}$

$= c_1\left(1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} \ldots\right) +$

$c_2\left(1 - \frac{3x}{1!} + \frac{(3x)^2}{2!} - \ldots\right)$

So,

the solⁿ can be alternatively written in terms of power series. (||ly, solⁿ of $y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$ can also be written in terms of power series).

So, considering a solⁿ of D.E (1), as $\sum_{n=0}^{\infty} a_n x^n$. Then, bringing it to the form of power series, we'll solve the D.E.

* POINTS

* (1) An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

is called a power series in x

(2) $$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n$$

is called a power series in $(x - x_0)$.

→ series made by shifting origin to x_0.

Looking into convergence of f^n :-

eg $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

It doesnt converge unless $|x| < 1$

$|x|<1$ is radius of convergence.

eg ② $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

doesnt converge unless $|x|>1$

eg ③ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

It's convergent $\forall x$

eg ④ $f(x) = 1 + x + (2!)x^2 + (3!)x^3 + \dots$

It's always diverging $\forall x$
($\because$ coeff$_s$ are large values)

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SERIES SOL^n : for 1st order D.E

Q1) Solve $y' = 2xy$ ① using power series method.

Sol^n:- Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the power series sol^n for ①

S1) Find y'

$y' = \sum_{n=1}^{\infty} a_n (n x^{n-1})$ $\rightarrow$ $\because$ If we start from $n=0$, we'll get 1st term as x^{-1} i.e. a -ve power $\rightarrow$ not desired

$y = \sum_{n=1}^{\infty} a_n x^n = a_0 + a_1 x + \dots + a_r x^r + \dots$

$\therefore y' = 0 + a_1 + a_2(2x) + \dots + a_r(rx^{r-1}) + \dots$

or $y' = \sum_{n=1}^{\infty} a_n (n x^{n-1})$

→ will be needed in later sections

$y'' =$

$y'' = a_2(2) + a_3(2(3))x + \dots + a_k(k\cdot(k-1))x^{k-2} + \dots$

or $y'' = \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}$.

S2) Substitute y & y' in eq^n ①.

$$\Rightarrow \sum_{n=1}^{\infty} a_n n \cdot x^{n-1} = 2x \sum_{n=0}^{\infty} a_n x^n$$

x^{n-1} x^n
diff^t form of power.

S3) Make powers of x same on both sides (if not same).
So, put $n-1 = k$
$\Rightarrow n = k+1$

$$\Rightarrow \sum_{k+1=1}^{\infty} a_{k+1}(k+1)x^k = 2x \sum_{n=0}^{\infty} a_n x^n$$

same form of power ✓

$$\Rightarrow \sum_{k=0}^{\infty} a_{k+1}(k+1)x^k = 2x \sum_{n=0}^{\infty} a_n x^n.$$

S4) Change symbol $k \to n$ again
So that both LHS & RHS have same variable.

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n = 2x \sum_{n=0}^{\infty} a_n x^n.$$

↳ This process in all gives same form of power on both sides.

S5) Remove all terms multiplied by Σ

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n = 2\sum_{n=0}^{\infty} a_n x^{n+1}$$ (bringing x inside)

diff^t power again

So, put $n+1 = k$
$\Rightarrow n = k-1$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n = 2\sum_{k-1=0}^{\infty} a_{k-1}x^k$$

$$= 2\sum_{k=1}^{\infty} a_{k-1}x^k$$

Changing $k \to n$.

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n = 2\sum_{n=1}^{\infty} a_{n-1}x^n.$$

finally, same power, & nothing multiplied to Σ terms.

S6) Make starting terms of Σ as same.

$\Rightarrow$ Removing 1st term from LHS.

$$\Rightarrow a_1(0+1)x^0 + \sum_{n=1}^{\infty} a_{n+1}(n+1)x^n = 2\sum_{n=1}^{\infty} a_{n-1}x^n$$

same starting term ✓

$$\Rightarrow a_1 + \sum_{n=1}^{\infty} a_{n+1}(n+1)x^n = 2\sum_{n=1}^{\infty} a_{n-1}x^n.$$

57) Comparing coeff. on both sides

LHS RHS

Constt. term : $a_1 = 0$

coeff. of x^n : $a_{n+1}(n+1) = 2a_{n-1}$

(∵ Starting term & x^n power in Σ is same on both sides &, compare coeff. of x^n directly)

$$\Rightarrow \boxed{a_{n+1} = \frac{2a_{n-1}}{(n+1)}}$$ ☆→ Recurrence Relⁿ.

↳ valid from $n = 1, 2, \ldots \infty$

At $\underline{n=1}$

$\Rightarrow a_2 = \frac{2a_0}{2} \Rightarrow a_2 = a_0$

$\underline{n=2}$ ∵ $a_1 = 0$

$\Rightarrow a_3 = \frac{2a_1}{3} = \frac{2(0)}{3} = 0$.

$\underline{n=3}$

$\Rightarrow a_4 = \frac{2a_2}{4} = \frac{2(a_0)}{2} = \frac{a_0}{2}$

$\underline{n=4}$

$\Rightarrow a_5 = \frac{2a_3}{5} = \frac{2(0)}{5} = 0$

$\underline{n=5}$

$\Rightarrow a_6 = \frac{2a_4}{6} = \frac{1}{3} \cdot \frac{1}{2} a_0 = \frac{a_0}{6}$

$\underline{n=6}$

$\Rightarrow a_7 = \frac{5}{7} a_5 = \frac{5}{7}(0) = 0$.

$n = 7$

$\Rightarrow a_8 = \frac{2a_6}{8} = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot a_0$

Seeing the pattern,

$a_1, a_3, a_5, a_7, \ldots a_{2n-1} = 0$,

So, the starting power series

$y = a_0 + \cancel{a_1 x} + a_2 x^2 + \cancel{a_3 x^3} + a_4 x^4 + \cancel{a_5 x^5} + \ldots$

$\Rightarrow y = a_0 + a_0 x^2 + \frac{1}{2} a_0 x^4 + \frac{1}{3}\frac{1}{2} a_0 x^6 + \ldots$

$\Rightarrow y = a_0 \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{2 \cdot 3} + \ldots \right)$

$\Rightarrow y = a_0 \left[\frac{x^0}{0!} + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots \right]$

$= a_0 \left[\frac{(x^2)^0}{0!} + \frac{(x^2)^1}{1!} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \ldots \right]$

$\Rightarrow y = a_0 e^{(x^2)}$

Aliter :- Direct method

$\frac{dy}{dx} = 2xy$

$\Rightarrow \frac{dy}{y} = 2x\,dx$

any constt

$\Rightarrow \log y - \log c = x^2$.

$\Rightarrow y = c e^{x^2}$

(same ans. as before)

* Power series method, done for eq^{ns} where direct method $\nexists$.

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Q2) Solve $y' + y = 1$ — ①
by power series method & compare power series sol^n with direct analytical sol^n

S1) Let $y = \sum_{n=0}^{\infty} a_n x^n$ be structure of unknown sol^n of ①

$$\Rightarrow y' = \sum_{n=1}^{\infty} a_n (n x^{n-1})$$

Substituting in ①

$$\Rightarrow \sum_{n=1}^{\infty} a_n (n x^{n-1}) + \sum_{n=0}^{\infty} a_n x^n = 1$$

S2) Powers matching for x

Put $n-1 = k$ in first term
$\Rightarrow n = k+1$

$$\Rightarrow \sum_{k+1=1}^{\infty} a_{k+1} (k+1)(x^k) + \sum_{n=0}^{\infty} a_n x^n = 1$$

$$\Rightarrow \sum_{k=0}^{\infty} a_{k+1} (k+1)(x^k) + \sum_{n=0}^{\infty} a_n x^n = 1$$

Replace $k \to n$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+1} (n+1)(x^n) + \sum_{n=0}^{\infty} a_n x^n = 1$$

So, x^n & Σ terms are same now.

S3) Comparing RHS & LHS.

$\underline{n=0}$: Const. term

$\Rightarrow a_1(1) + a_0 = 1 \Rightarrow a_1 = (1 - a_0)$

For $n \geq 1$, coeff of x^n

$$a_{n+1}(n+1) + a_n = 0. \quad (\forall\ n \geq 1)$$

$$\Rightarrow a_{n+1}(n+1) = -a_n$$

$$\Rightarrow a_{n+1} = -\frac{a_n}{n+1}$$

$\to n = 1$

$$\Rightarrow a_2 = -\frac{a_1}{2} = -\frac{1}{2}(1-a_0)$$

$\to n = 2$

$$a_3 = -\frac{a_2}{3} = -\frac{1}{3}\cdot\left(\frac{-1}{2}\right)(1-a_0)$$

$$= \frac{1}{2\cdot 3}(1-a_0).$$

$$= (-1)^2\left(\frac{1}{2\cdot 3}\right)(1-a_0)$$

$\to n = 3$

$$\Rightarrow a_4 = \frac{-a_3}{4} = -\frac{1}{4}\left(\frac{1}{2}\cdot\frac{1}{3}\right)(1-a_0)$$

$$\Rightarrow a_4 = (-1)^3\left(\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\right)(1-a_0)$$

So, from pattern, the req^d sol^n is

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + (1-a_0)x + (-1)\left(\frac{1}{2}\right)(1-a_0)x^2 +$$

$$(-1)^2\left(\frac{1}{2}\cdot\frac{1}{3}\right)(1-a_0)x^3 + (-1)^3\left(\frac{1}{2\cdot 3\cdot 4}\right)(1-a_0)x^4$$

$$+ \cdots$$

$\Rightarrow y = 1 + (-1)^1(1-a_0) + (-1)^2(1-a_0)x + (-1)^3\left(\frac{1}{2}\right)(1-a_0)x^2$
$+ (-1)^4\left(\frac{1}{2}\cdot\frac{1}{3}\right)(1-a_0)x^3 + \ldots$

or

$y = 1 - (1-a_0)\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right]$

$\Rightarrow y = 1 - (1-a_0)e^{-x}$

Alter: Using Std. method :-

M1

$y' + y = 1$

$\Rightarrow \frac{dy}{dx} + \underbrace{1}_{P(x)} \cdot y = \underbrace{1}_{Q(x)}$

$IF = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$

$\Rightarrow y(IF) = \int Q(x)(IF)\,dx$

$\Rightarrow ye^x = \int e^x\,dx$

$\Rightarrow ye^x = e^x + c$

$\Rightarrow y = 1 + ce^{-x}$

If $c = -(1-a_0)$

$\Rightarrow -c = 1 - a_0$

$\Rightarrow y = 1 - (1-a_0)e^{-x}$

(Same as before) Ans

M2

$y' = 1 - y$

$\Rightarrow \frac{dy}{dx} = 1 - y$

$\Rightarrow \frac{dy}{1-y} = dx$

$\Rightarrow -\log(1-y) + \log c = x$

$\Rightarrow \log\left(\frac{c}{1-y}\right) = x$

$\Rightarrow \frac{c}{1-y} = e^x$

$\Rightarrow c = (1-y)e^x$

$\Rightarrow y = 1 - ce^{-x}$

Same form

* Analytic f^{ns} at $x = x_0$ $\equiv$ f^{ns} that are differentiable at $x = x_0$

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Section - 28

Second Order L.E.

Name of a model (a type in 2nd ord. LE)

* ORDINARY PTS

$\Rightarrow$ General form $\boxed{y'' + P(x)y' + Q(x)y = 0} \longrightarrow (1)$

If

Defin^n $x = x_0$ is said to be an ordinary pt. of a f^n, $f(x)$, then $f(x)$ should be ANALYTIC at $x = x_0$ (which means that $f(x)$ has a ordinary power series expansion in some neighbourhood of x_0)

* Analytic f^{ns}: f^{ns} that are diff (roughly)

eg x^2 : analytic $\forall\ x \in \mathbb{R}$

$\frac{1}{x}$: " $\forall\ x \in \mathbb{R} - \{0\}$

$\frac{1}{x-a}$: not analytic at $x = a$

$(x-a)^3$ = analytic everywhere.

Note Here, if $P(x)$ & $Q(x)$ are analytic at $x = x_0$ i.e, $x = x_0$ is an ordinary pt., then, every sol^n of (1) is also analytic at this pt.

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both $P(x)$ & $Q(x)$ are analytic at $x = x_0$

* Only in this case, power series method for finding solⁿ for eqⁿ ① becomes feasible.
* Any pt, not an ordinary pt. of ① is called a SINGULAR PT.

eg: for $\frac{1}{x-a}$; $x = a$: singular pt

; $x \ \forall \ \mathbb{R} - \{a\}$: Ordinary pts.

eg: for $(x-a)^3$: doesn't have any singular pt.

* Singular pt. of a fⁿ:
  The pts. where the fⁿ ceases to be analytic are called singular pts. of the fⁿ.

eg: $\frac{1}{x^2 - 5x + 6}$; $x = 2, 3$: Singular pts

; $x \in \mathbb{R} - \{2, 3\}$: ordinary pts.

* (Steps done in detail)

Q. Solve the Legendre's DE by power series method.

Legendre's D.E : $(1-x^2)y'' - 2xy' + p(p+1)y = 0 \longrightarrow ①$

p : any constt.

From ①

$$y'' - \left(\frac{2x}{1-x^2}\right)y' + \frac{p(p+1)}{(1-x^2)} y = 0$$

$$P(x) = \frac{-2x}{1-x^2}, \quad Q(x) = \frac{p(p+1)}{1-x^2}$$

Both $P(x)$ & $Q(x)$ are analytic at $x = 0$.
(not analytic at $x = \pm 1$ only)

only for idea

hence, power series sol^n for eq^n ① is possible around $x=0$

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be power series sol^n of ①

$\hookrightarrow$ general eq^n of power series,

$$y(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$a=0$, here

$$\Rightarrow y' = \sum_{n=1}^{\infty} a_n n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n \cdot n(n-1)\, x^{n-2}$$

found in previous sections.

Now, ① $\Rightarrow (1-x^2)\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - 2x \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$

$$+ p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^n - 2\sum_{n=1}^{\infty} a_n \cdot n \cdot x^n$$

$$+ p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

(first term should be changed to x^n)

in first Σ, put $n-2 = k \Rightarrow n = k+2$

$$\Rightarrow \sum_{k+2=2}^{\infty} a_{k+2}(k+2)(k+1) x^k - \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^n$$

$$- 2\sum_{n=1}^{\infty} a_n \cdot n \cdot x^n + p(p+1)\sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

Changing symbol $\Rightarrow k \to n$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} a_n(n)(n-1)x^n$$

$$-2\sum_{n=1}^{\infty} a_n \cdot n \cdot x^n + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Now, Starting term should start with $n=2$, so separate terms. $\to$ ∴ to highest

$$\Rightarrow \left[a_2(0+2)(0+1)x^0 + a_3(1+2)(1+1)x^1 + \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)x^n\right.$$

$$+\left[-\sum_{n=2}^{\infty} a_n(n)(n-1)x^n\right]$$

$$-\left[2a_1(1)x^1 - 2\sum_{n=2}^{\infty} a_n n x^n\right]$$

$$+\left[p(p+1)a_0x^0 + p(p+1)a_1 \cdot x^1 + p(p+1)\sum_{n=2}^{\infty} a_n x^n\right] = 0$$

$$\Rightarrow \left[2a_2 + 6a_3x - 2a_1x + p(p+1)a_0 + p(p+1)a_1x\right]$$

$$+\left[\sum_{n=2}^{\infty} (a_{n+2})(n+2)(n+1)x^n - (a_n)(n)(n-1)x^n - 2a_n(n)x^n + p(p+1)a_n x^n\right] = 0$$

$$\Rightarrow \underbrace{2a_2 + p(p+1)a_0} + \underbrace{\left[p(p+1)a_1 - 2a_1 + 6a_3\right]}x$$

$$+\left[\sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1) - a_n(n)(n-1) - 2a_n(n) + p(p+1)a_n\right]x^n = 0.$$

Comparing coeff. on both sides

(1) $a_{n+2}(n+2)(n+1) - a_n(n)(n-1) - 2a_n(n) + p(p+1)a_n = 0$ } valid for $n \geq 2$

(2) $2a_2 + p(p+1)a_0 = 0$

(3) $2 \cdot 3\, a_3 - 2a_1 + p(p+1)a_1 = 0$

From (2) & (3)

$\Rightarrow a_2 = -\dfrac{p(p+1)a_0}{2}$

$a_3 = \dfrac{1}{2\cdot 3}\left[2 - p(p+1)\right]a_1$

$= \dfrac{1}{2\cdot 3}\left[-p^2 - p + 2\right]a_1 = \dfrac{-1}{6}(p^2+p-2)a_1$

$\Rightarrow a_3 = -\dfrac{(p+2)(p-1)}{2\cdot 3}a_1$

From (1)

$a_{n+2} = \dfrac{1}{(n+1)(n+2)}\left[n(n-1) + 2n - p(p+1)\right]a_n$

$\Rightarrow a_{n+2} = \left[\dfrac{n^2 - n + 2n - p^2 - p}{(n+1)(n+2)}\right]a_n$

$\Rightarrow a_{n+2} = -\left(\dfrac{p^2 - n^2 + p - n}{(n+1)(n+2)}\right)a_n$

$\Rightarrow a_{n+2} = -\left(\dfrac{(p-n)(p+n) + (p-n)}{(n+1)(n+2)}\right)a_n$

$$\Rightarrow a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)} \cdot a_n \quad \text{valid for } n \geq 2$$

→ for $n = 2$

$$a_4 = -\frac{(p-2)(p+3)}{(3)(4)} a_2$$

$$= -\frac{(p-2)(p+3)}{(3)(4)} \times \left(\frac{-p(p+1)}{2} \right) a_0$$

$$\Rightarrow a_5 = \frac{(-1)^2 \, p(p-2)(p+1)(p+3)}{4!} a_0$$

→ $n = 3$

$$\Rightarrow a_5 = -\frac{(p-3)(p+4)}{(4)(5)} a_3$$

$$= -\frac{(p-3)(p+4)}{(4)(5)} \times \left(-\frac{(p+2)(p-1)}{2 \cdot 3} a_1 \right)$$

$$\Rightarrow a_5 = \frac{(-1)^2 (p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

→ $n = 4$, solving

$$a_6 = -\frac{(p-4)(p+5)}{(5)(6)} a_4$$

$$= \frac{(-1)^3 \, p(p-2)(p+1)(p+3)(p-4)(p+5)}{6!} a_0$$

→ $n = 5$

$$a_7 = -\frac{(p-5)(p+6)}{(6)(7)} a_5$$

$$\Rightarrow a_7 = \frac{(-1)^3 (p-1)(p-3)(p+2)(p+4)(p-5)(p+6)}{7!} a_1$$

Now,
By substituting values in the power series:-

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots$$

a_0, a_1 : constt.
others $(a_2, a_3 \ldots)$: substitute in terms of a_0 & a_1

So,
$$y = a_0 + a_1 x - \frac{p(p+1)}{2} a_0 x^2 - \frac{(p-1)(p-2)}{2\cdot 3} a_1 x^3$$
$$+ \left[\frac{p(p-2)(p+1)(p+3)}{2\cdot 3\cdot 4} a_0\right] x^4$$
$$+ \left[\frac{(p-3)(p-1)(p+2)(p+4)}{2\cdot 3\cdot 4\cdot 5} a_1\right] x^5$$
$$- \left[\frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{2\cdot 3\cdot 4\cdot 5\cdot 6} a_0\right] x^6$$
$$- \left[\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} a_1\right] x^7 \cdots$$

$$\Rightarrow y = a_0 \underbrace{\left[1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} x^6 + \cdots\right]}_{y_0(x)}$$

$$+ a_1 \underbrace{\left[x - \frac{(p-1)(p-2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 - \frac{(p-5)(p-3)(p-1)(p+2)(p+4)(p+6)}{7!}x^7 + \dots \right]}_{y_1(x)}$$

$$\Rightarrow y = a_0\, y_0(x) + a_1\, y_1(x)$$

→ here, $y_0(x)$ & $y_1(x)$ are 2 independent sol^ns for Legendre's eq^n

→ $y_0(x)$ & $y_1(x)$: Legendre f^ns

Seeing $y_0(x)$, the sequence terminates when p is even, otherwise its infinite series.
||ly, $y_1(x)$ terminates when p is odd integer.
Otherwise, they are infinite series

If series $y_0(x)$ or $y_1(x)$ terminates for specific values of p, then, they are called Legendre's polynomial of degree p.

★ The f^ns, $y_0(x)$ & $y_1(x)$ solved above are named as

Legendre's f^ns	Legendre's polynomials
If series is infinite.	If series is finite (or terminating)

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Q) Find general solⁿ of :-

$$(1+x^2)y'' + 2xy' - 2y = 0 \longrightarrow (1)$$

in terms of power series of x.
Can you express solⁿ by means of an already known std. fⁿ.

Now,
let $y = \sum_{n=0}^{\infty} a_n (x-0)^n$ be
the power series solⁿ of (1)

$$\Rightarrow y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n \cdot n \cdot n-1 \cdot x^{n-2}$$

Using them in eqⁿ (1).

Checking Analyticity

$$y'' + \left(\frac{2x}{1-x^2}\right)y' - \left(\frac{2}{1-x^2}\right)y = 0$$

$$P(x) = \frac{2x}{1-x^2}, \quad Q(x) = \frac{-2}{1-x^2}$$

both are analytic at $x=0$
Hence, $x=0$ is an ordinary pt for given D.E.
Hence, power series solⁿ exists for given D E

$$\Rightarrow (1+x^2)\sum_{n=2}^{\infty} a_n \cdot n(n-1)x^{n-2} + 2x\left(\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}\right)$$

$$-2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

$n-2=k$
$\Rightarrow n = k+2$

* Making x^n terms in every term :

$$(1) \quad \sum_{k+2=2}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{n=2}^{\infty} a_n \cdot n(n-1)x^n$$

$$+ 2\sum_{n=1}^{\infty} a_n \cdot n \cdot x^n - 2\sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$k \to n$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)x^n + \sum_{n=2}^{\infty} a_n \cdot n(n-1)x^n$$

$$+2\sum_{n=1}^{\infty} a_n \cdot n \cdot x^n - 2\sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

* Now starting term should start with $n=2$

$$\Rightarrow \underbrace{a_2(2)x^0}_{n=0} + \underbrace{a_3(2)(3)x^1}_{n=1} + \sum_{n=2}^{\infty} a_{n+2}(n+1)(n+2)x^n$$

$$+ \sum_{n=2}^{\infty} a_n(n)(n-1)x^n$$

$$+ \underbrace{2(a_1)x}_{n=1} \qquad + 2\sum_{n=2}^{\infty} a_n \cdot n\, x^n$$

$$- \underbrace{2a_0x^0}_{n=0} - \underbrace{2a_1x}_{n=1} \qquad - 2\sum_{n=2}^{\infty} a_n x^n = 0$$

$$\Rightarrow (2a_2 - 2a_0) + (2\cdot 3a_3 + 2(a_1 - a_1))x$$

$$+ \sum_{n=2}^{\infty}\left[(a_{n+2})(n+1)(n+2) + a_n(n)(n-1) + (2n-2)a_n\right]x^n = 0$$

$$\Rightarrow 2(a_2 - a_0) + (2\cdot 3)a_3 x$$

$$+ \sum_{n=2}^{\infty}\left[(a_{n+2})(n+1)(n+2) + a_n(n(n-1)+2(n-1))\right]x^n = 0$$

Comparing coeff on both sides

$\Rightarrow 2(a_2 - a_0) = 0 \Rightarrow \boxed{a_2 = a_0}$

$(2 \cdot 3)a_3 + 2(a_1 - a_1) = 0$

$\underbrace{\boxed{a_3 = 0 \cdot (a_1)}}_{\text{arbitrary}} = 0$

&

$(a_{n+2})(n+1)(n+2) + a_n(n+2)(n-1) = 0$ condt

$$\Rightarrow a_{n+2} = \left[\frac{-\cancel{(n+2)}(n-1)}{(n+1)\cancel{(n+2)}}\right] a_n$$

$$\Rightarrow a_{n+2} = -\frac{(n-1)}{(n+1)} a_n \quad ; \text{ for all } n \geq 2$$

→ for $n=2$.

$\Rightarrow a_4 = \frac{-1}{3} a_2 = \frac{-1}{3} \cdot \frac{1}{1} \cdot a_0 = \frac{-1}{1.3} a_0.$

→ for $n=3$

$a_5 = -\frac{2}{4} a_3 = 0$

→ for $n=4$

$a_6 = -\frac{3}{5} a_4 = -\frac{3}{5}\left(\frac{-1}{3} \cdot \frac{1}{1}\right) a_0 = \frac{1}{5} a_0$

→ for $n=5$

$a_7 = -\frac{4}{6} a_5 = 0.$

→ $n=6$,

$\Rightarrow a_8 = -\frac{5}{7} a_6 = -\frac{5}{7}\left(\frac{1}{5}\right) a_0 = -\frac{1}{7} a_0$

Substituting these values in power series

$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + \cancel{a_3 x^3} + a_4 x^4 + \cancel{a_5 x^5} + a_6 x^6 + \cancel{a_7 x^7} + a_8 x^8 + \dots$

$\Rightarrow y = (a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \dots) + a_1 x$

$= (a_0 + a_0 x^2 - \frac{1}{3} a_0 x^4 + \frac{1}{5} a_0 x^6 - \frac{1}{7} a_0 x^8 + \dots) + a_1 x$

$\Rightarrow y = a_0 \left[1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \dots\right] + a_1 x$

Ans

optional: solve further

$\rightarrow$ We know, $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$\Rightarrow x \tan^{-1}(x) = x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \dots$

Same as above series

$\Rightarrow y = a_0 [1 + x \tan^{-1}(x)] + a_1 x$

(general solⁿ in terms of known fⁿ.

Solve $y'' + xy' + y = 0 \longrightarrow (1)$

here, $\underbrace{P(x) = x \;\&\; Q(x) = 1}_{\text{Analytic everywhere}}$

All pts on real plane are ordinary pts. So, power series solⁿ is possible

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the sol^n of ①.

$$y' = \sum_{n=1}^{\infty} a_n . n . x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n . n.(n-1) x^{n-2}$$

① ⇒

$$\underbrace{\sum_{n=2}^{\infty} a_n \cdot n \cdot x^{n-2}}_{\text{Make it } x^n} + \sum_{n=1}^{\infty} a_n n . x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$\Rightarrow n - 2 = k$

$\Rightarrow n = k+2$

$k \to n$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} a_{n+2} (n+1)(n+2) x^n + \sum_{n=1}^{\infty} a_n \cdot n \cdot x^n}_{\text{Make starting term from } n=1} + \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\Rightarrow a_2 \cdot 2 x^0 + \quad + \sum_{n=1}^{\infty} a_{n+2} (n+2) x^n$$

$$+ \quad + \sum_{n=1}^{\infty} a_n \cdot n \cdot x^n$$

$$+ a_0 x^0 \quad + \sum_{n=1}^{\infty} a_n \cdot x^n = 0$$

$$\Rightarrow (2a_2 + a_0) +$$

$$\sum_{n=1}^{\infty} \left[(a_{n+2})(n+1)(n+2) + a_n (n) + a_n \right] x^n = 0$$

Comparing coeff :-

$$2a_2 + a_0 = 0 \Rightarrow a_0 = -2a_2 \longrightarrow (2)$$

& $a_{n+2}(n+1)(n+2) + a_n(n+1) = 0$

$$\Rightarrow \boxed{a_{n+2} = -\left(\frac{1}{n+2}\right) a_n} \longrightarrow (3)$$

Recurrence Relation

valid $\forall\ n \geq 1$

$\rightarrow n=1$

$$\Rightarrow a_3 = -\frac{1}{3} a_1$$

$\rightarrow n=2$

$$a_4 = -\frac{1}{4} a_2 = \frac{(-1)^2}{4 \cdot 2} \cdot a_0$$

$\rightarrow n=3$

$$\Rightarrow a_5 = -\frac{1}{5} a_3 = \frac{(-1)^2}{5 \cdot 3} a_1$$

$\rightarrow n=4$

$$a_6 = -\frac{1}{6} a_4 = \frac{(-1)^3}{6 \cdot 4 \cdot 2} a_0$$

$\rightarrow n=5$

$$a_7 = -\frac{1}{7} a_5 = \frac{(-1)^3}{7 \cdot 5 \cdot 3} a_1$$

So, the seq^d sol^n is :-

$$y = [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots]$$

$$\Rightarrow y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{a_1}{3} x^3 + (-1)^2 \frac{a_0}{2 \cdot 4} x^4 + (-1)^2 \frac{a_1 x^5}{3 \cdot 5} + \cdots$$

$$\Rightarrow y = a_0\underbrace{\left[1 - \frac{x^2}{2} + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + \cdots\right]}_{y_1(x)}$$

$$+ a_1\underbrace{\left[x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + \cdots\right]}_{y_2(x)}$$

$$\Rightarrow y = a_0 y_1(x) + a_1 y_2(x) \quad \text{Ans}$$

Q. Solve Chebyshev's eq^n:-

$$(1-x^2)y'' - xy' + p^2 y = 0 \quad ; \; p : \text{constt.} \quad \to (1)$$

by finding 2 linearly independent series sol^n valid for $|x|<1$.

Also, show that, if $p = n$, where $n \in \mathbb{Z}, n \geq 0$, then, $\exists$ a polynomial sol^n of degree n.

From (1)

$$y'' - \underbrace{\frac{x}{1-x^2}}_{P(x)} y' + \underbrace{\frac{p^2}{1-x^2}}_{Q(x)} y = 0$$

It is not analytic at $x = \pm 1$. So, for $x<1$ it is analytic.

Hence, power series sol^n for eq^n (1) in domain of $x<1$ is possible.

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be power series solⁿ of eqⁿ ①.

$\Rightarrow y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$

$y'' = \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2}$

Using above in ①

$$\Rightarrow (1-x^2) \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^{n-2} - x \sum_{n=1} a_n \cdot n \cdot x^{n-1}$$

$$+ p^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \underbrace{\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2}} - \sum_{n=2}^{\infty} a_n (n)(n-1) x^n$$

$$- \sum_{n=1}^{\infty} a_n \cdot (n) x^n + p^2 \sum_{n=0}^{\infty} a_n \cdot x^n = 0.$$

[Making power of x as x^n.
Put $n-2 = k \Rightarrow n = k+2$
$k \to n$ replaced]

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2} (n+1)(n+2) x^n - \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^n$$

$$- \sum_{n=1}^{\infty} a_n \cdot n \cdot x^n$$

$$+ p^2 \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

Making the starting term from $n=2$ to ∞.

$$\Rightarrow 2a_2x^0 + 2\cdot3\cdot a_3x + \sum_{n=2}^{\infty} a_{n+2}(n+1)(n+2)x^n$$

$$+ \sum_{n=2}^{\infty} a_n\, n\,(n-1)\,x^n$$

$$- a_1x - \sum_{n=2}^{\infty} a_n\cdot n\cdot x^n$$

$$+ p^2a_0 + p^2a_1x + p^2\sum_{n=2}^{\infty} a_n x^n = 0$$

$$\Rightarrow (2a_2 + p^2a_0) + [2\cdot3\, a_3 - a_1 + p^2a_1]\,x$$

$$+ \sum_{n=2}^{\infty}\begin{bmatrix} a_{n+2}(n+1)(n+2) - a_n(n)(n-1) \\ - a_n\cdot n + p^2a_n \end{bmatrix} x^n = 0$$

Comparing coeff. on both sides.

$$\Rightarrow 2a_2 + p^2a_0 = 0 \Rightarrow a_2 = -\frac{p^2a_0}{2} \rightarrow (2)$$

$$6a_3 - (1-p^2)a_1 = 0$$

$$\Rightarrow a_3 = \frac{(1-p^2)}{2\cdot3}a_1 \rightarrow (3)$$

$$a_{n+2}(n+1)(n+2) - a_n[n(n-1) + n - p^2] = 0$$

$$\Rightarrow a_{n+2} = \frac{n^2-p^2}{(n+1)(n+2)}a_n \rightarrow (4)$$

$\hookrightarrow$ valid for $n \geq 2$.

for $n = 2$

$$\hookrightarrow a_4 = \frac{4-p^2}{(3)(4)} a_2 = \frac{p^2(p^2-2^2)}{2\cdot3\cdot4} a_0 .$$

for $n = 3$

$$\hookrightarrow a_5 = \frac{9-p^2}{(4)(5)} a_3 = \frac{(3^2-p^2)(1-p^2)}{2\cdot3\cdot4\cdot5} a_1$$

$n = 4.$

$$\hookrightarrow a_6 = \frac{4^2-p^2}{(5)(6)} a_4$$

$$\Rightarrow a_6 = (-1)\frac{(p^2)(p^2-2^2)(p^2-4^2)}{2\cdot3\cdot4\cdot5\cdot6} a_0 .$$

$n = 5.$

$$\hookrightarrow a_7 = \frac{5^2-p^2}{(6)(7)} a_5$$

$$= \frac{(1^2-p^2)(3^2-p^2)(5^2-p^2)}{2\cdot3\cdot4\cdot5\cdot6\cdot7}$$

$$- - - - - - -$$

So, req^d sol^n, from (1)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x - \frac{p^2}{2} a_0 x^2 + \left(\frac{1-p^2}{2\cdot3}\right) a_1 x^3 + \frac{p^2(p^2-2^2)}{2\cdot3\cdot4} a_0 x^4$$

$$+ \frac{(3^2-p^2)(1-p^2)}{2\cdot3\cdot4\cdot5} a_1 x^5 - \frac{p^2(p^2-2^2)(p^2-4^2)}{6!} a_0 x^6$$

$$+ - - - - -$$

$$= \underbrace{a_0 \left[1 - \frac{p^2 x^2}{2!} + \frac{p^2(p^2-2^2)x^4}{4!} - \frac{p^2(p^2-2^2)(p^2-4^2)}{6!} x^6 + - - - - \right]}_{y_1(x)}$$

$$\underbrace{+ a_1\left[x - \frac{(p^2-1^2)}{3!}x^3 + \frac{(p^2-1^2)(p^2-3^2)}{5!}x^5 - \frac{(p^2-1^2)(p^2-3^2)(p^2-5^2)}{7!}x^7 + \cdots\right]}_{y_2(x)}$$

$$\Rightarrow y = a_0 y_1(x) + a_1 y_2(x)$$

Ans

Part-(b)

If p takes a finite value, one of the infinite series ($y_1(x)$ or $y_2(x)$) becomes a polynomial.

$\hookrightarrow$ When p = even, infinite series $y_1(x)$ becomes polynomial. ($y_2(x)$ is still infinite)

Reverse happens when p = odd ($y_2(x)$ series becomes polynomial)

HW

Q Solve Airy's eq^n:-

$y'' + xy = 0$ ... (1) using power series method.

$P(x) = 0$, $Q(x) = x$

It is analytic $\forall\ x \in R$.

So, power series solution exists.

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be power series sol^n of eq^n (1)

$\Rightarrow y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$, $y'' = \sum_{n=2}^{\infty} a_n \cdot n \cdot n-1 \cdot x^{n-2}$

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Using in eq^n ①

$$\Rightarrow \sum_{n=2}^{\infty} a_n \cdot n \cdot n-1 \cdot x^{n-2} + \sum_{n=0}^{\infty} a_n \cdot x^{n+1} = 0$$

Matching powers of x.

$n-2 = k$ $\quad\quad\quad$ $n+1 = k$

$\Rightarrow n = k+2$ $\quad\quad\quad$ $\Rightarrow n = k-1$

$k \to n.$ $\quad\quad\quad$ $k \to n$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)x^n + \sum_{n=1}^{\infty} a_{n-1} \cdot x^n = 0$$

Matching starting Σ

$$\Rightarrow a_2(2)x^0 + \sum_{n=1}^{\infty} a_{n+2}(n+1)(n+2)x^n + \sum_{n=1}^{\infty} a_{n-1} \cdot x^n = 0$$

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty}\left[(a_{n+2})(n+1)(n+2) + a_{n-1}\right] x^n = 0$$

Comparing coeff. on both sides

$\Rightarrow 2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

& $a_{n+2}(n+1)(n+2) + a_{n-1} = 0 \Rightarrow a_{n+2} = -\dfrac{1}{(n+1)(n+2)} a_{n-1}$

valid for $n \geq 1$

• for $n=1$

$a_3 = \dfrac{-1}{3!} a_0 = \dfrac{-1}{6} a_0$

• for $n=2$

$a_4 = \dfrac{-1}{3.4} a_1 = \dfrac{-1}{12} a_1$

• for $n=3$

$a_5 = \dfrac{-1}{4.5} a_2 = 0$

• for $n=4$

$a_6 = \dfrac{-1}{5.6} a_3 = \dfrac{1}{6.5.6} a_0 = \dfrac{1}{180} a_0$

• for $n=5$

$a_7 = \dfrac{-1}{6.7} a_4 = \dfrac{1}{3.4.6.7} a_1 = \dfrac{1}{504} a_1$

So, the required solution is :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

$$\Rightarrow y = a_0 + a_1 x + 0 \cdot x^2 - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + 0 \cdot x^5$$

$$+ \frac{1}{180} a_0 x^6 + \frac{1}{504} a_1 x^7 + \ldots$$

$$\Rightarrow y = a_0 \left[1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{(2 \cdot 3)(5 \cdot 6)} x^6 - \ldots \right]$$

$$+ a_1 \left[x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{(3 \cdot 4)(6 \cdot 7)} x^7 - \ldots \right]$$

$$\Rightarrow y = a_0 y_1(x) + a_1 y_2(x)$$

Ans

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Section - 29

REGULAR SINGULAR POINTS & Solving of D.E. with regular singular pts.

* Defin^n: Regular singular pt: D.E, $y'' + P(x)y + Q(x)y = 0$

A singular pt, x_0, of a ~~f^n $f(x)$~~ is said to be regular if the f^{ns}

$(x - x_0)P(x)$ & $(x - x_0)^2 Q(x)$

are analytic f^{ns}.

$P(x)$ & $Q(x)$ taken alone are NOT analytic

Else, x_0 is said to be IRREGULAR Singular pt.

* Idea: for a D.E $y' + P(x)y + Q(x)y = 0$

If $x = x_0$ is an ordinary pt. of this D.E, then $P(x)$ & $Q(x)$ are analytic f^{ns} at $x = x_0$.

With this being valid, power series sol^n exists.

* $x_0 = 0$ (mostly)

Power series sol^n.

$x_0 = 0$: $y = \sum_{n=0}^{\infty} a_n x^n$

$x_0 = a$: $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

ex for regular singular pts.

ex ① Legendre's D.E:-

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$

$$\Rightarrow y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

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$$P(x) = \frac{-2x}{1-x^2}, \quad Q(x) = \frac{p(p+1)}{1-x^2}$$

So, $P(x)$ & $Q(x)$ are not analytic at $x = \pm 1$.
In other words, $x = \pm 1$ are singular pts. for $P(x)$ & $Q(x)$

Now, checking whether, singular pts. $x=1$ & $x=-1$ are regular or not.

For $x=1$

$$(x-1)P(x) = (x-1)\frac{(-2x)}{1-x^2} = \frac{2x}{1+x}$$

differentiable

It is dfb at $x=1$

$$\& \quad (x-1)^2 Q(x) = \frac{p(p+1)\cancel{(x-1)}(x-1)}{(-1)\cancel{(x-1)}(x+1)}$$

$$= \frac{(1-x)\,p(p+1)}{(1+x)}$$

It is dfb at $x=1$.

Hence, $x=1$ is a regular singular pt.

• For $x=-1$

$$\Rightarrow [x-(-1)]P(x) = \frac{\cancel{(x+1)}(-2x)}{(1-x)\cancel{(1+x)}} = \frac{2x}{x-1}$$

It is dfb at $x=-1$.

$$\& \quad [x-(-1)]^2 Q(x) = \frac{(x+1)\cancel{(x+1)}\,p(p+1)}{(1-x)\cancel{(1+x)}}$$

$$= \frac{(x+1)\,p(p+1)}{1-x}$$

It is dfb at $x=-1$.

So, IIly, $x=-1$ is also a regular singular pt.

eg (2) : Bessel's eq^n of order, p

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

$$\Rightarrow y'' + \frac{1}{x}y' + \left[1 - \left(\frac{p}{x}\right)^2\right]y = 0$$

$$P(x) = \frac{1}{x}, \quad Q(x) = 1 - \left(\frac{p}{x}\right)^2.$$

The f^{ns} $P(x)$ & $Q(x)$ are not analytic at $x = 0$. So, $x = 0$ is a singular pt. for the given eq^n.

Checking for regular pt. at $x = 0$

$$(x - x_0)P(x) = (x - 0)P(x) = x\left(\frac{1}{x}\right) = 1$$

It is d/b & hence analytic at $x = 0$.

$$(x - x_0)^2 Q(x) = x^2\left(1 - \left(\frac{p}{x}\right)^2\right)$$

$$= x^2 - p^2$$

It is d/b and analytic at $x = 0$

Hence, $x = 0$ is a regular singular pt.

eg (3) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$.

$$\Rightarrow y'' - \frac{2}{x^3}y' + \frac{3}{x^2(x-1)}y = 0,$$

$$P(x) = -\frac{2}{x^3}, \quad Q(x) = \frac{3}{x^2(x-1)}$$

$\to x = 0$ $\quad \to x = 0$ & $x = 1$

$P(x)$ & $Q(x)$ are non analytic at $x = 0, 1$
hence, $x = 0, 1$ are singular pts.
checking for regular pts.

$\boxed{x = 0}$

$\Rightarrow$ $x P(x) = \dfrac{-2}{x^2}$

It is not dfb at $x = 0$
So, its not a regular singular pt. for $P(x)$

$x^2 Q(x) = \dfrac{3}{x-1}$

It is dfb at $x = 0$
So, it is a regular singular pt. for $Q(x)$

$\because$ Both cond^ns are not satisfied, $x = 0$ is not a regular singular pt.

$\boxed{x = 1}$

$(x-1) P(x) = \dfrac{-2(x-1)}{x^3}$

It is dfb at $x = 1$
So, it is a regular singular pt. for $P(x)$

$(x-1)^2 Q(x) = \dfrac{3(x-1)}{x^2}$.

It is dfb at $x = 1$

$\because$ both cond^ns are satisfied. So, $x = 1$ is a regular singular pt. at $x = 1$.

Q. Locate & classify the singular pts. on x-axis for the following D.Es. :-

(i) $x^2 (x^2 - 1)^2 y'' - x(1-x) y' + 2y = 0$

(ii) $(3x+1)xy'' - (x+1)y' + 2y = 0$

(iii) $x^2 y'' + (2-x)y' = 0$

(i) $y'' + \dfrac{1}{x(x^2-1)(x+1)} y' + \dfrac{2}{x^2(x^2-1)^2} y = 0$

$$P(x) = \frac{1}{x(x+1)^2(x-1)}, \quad Q(x) = \frac{2}{x^2(x^2-1)^2}$$

The f^{ns} $P(x)$ & $Q(x)$ are non analytic at $x = 0, +1, -1$.

Hence, the singular pts. are $0, 1, -1$.

Checking for regular singular pts.

$\boxed{x=0}$

$$x P(x) = \frac{1}{(x-1)(x+1)^2}$$

It is dfb at $x=0$

$$x^2 Q(x) = \frac{2}{(x^2-1)^2}$$

It is also dfb at $x=0$.

So, $x=0$ is a regular singular pt.

$\boxed{x=1}$

$$(x-1)P(x) = \frac{1}{x(x+1)^2}$$

It is dfb at $x=1$.

$$(x-1)^2 Q(x) = \frac{2}{x^2(x+1)^2}$$

It is dfb at $x=1$

$\therefore$ $x=1$ is a regular singular pt.

$\boxed{x=-1}$

$\Rightarrow (x+1)\,P(x) = \dfrac{1}{x(x-1)(x+1)}$

It is not dfb at $x=-1$

$(x+1)^2 Q(x) = \dfrac{2}{x^2(x-1)^2}$

It is dfb at $x=-1$

$\therefore$ both $P(x)$ & $Q(x)$ are not satisfied, $x=-1$ is an irregular singular pt

(ii) $(3x+1)xy'' - (x+1)y' + 2y = 0$

$\Rightarrow y'' - \dfrac{(x+1)}{x(3x+1)} + \dfrac{2}{(3x+1)x}\,y = 0$

$P(x) = -\dfrac{(x+1)}{x(3x+1)}, \quad Q(x) = \dfrac{2}{x(3x+1)}$

The f^{ns} $P(x)$ & $Q(x)$ are non analytic at $x = 0, -\dfrac{1}{3}$

So, singular pts are $0, -\dfrac{1}{3}$

Now, checking for regular singular pts. :-

$\boxed{x=0}$

$x(P(x)) = -\dfrac{(x+1)}{(3x+1)}$, dfb at $x=0$

$x^2 Q(x) = \dfrac{2x}{3x+1}$, dfb at $x=0$

So, $x=0$ is a regular singular pt

$\boxed{x = -\frac{1}{3}}$

$$\Rightarrow \left(x+\frac{1}{3}\right)P(x) = \frac{(3x+1)P(x)}{3}$$

$$= \frac{(3x+1)}{3} \times \frac{(-(x+1))}{x(3x+1)}$$

$$= -\frac{(x+1)}{3x} \text{ ; dfb at } x = -\frac{1}{3}$$

$$\left(x+\frac{1}{3}\right)^2 Q(x) = \frac{(3x+1)^2}{9} Q(x)$$

$$= \frac{2(3x+1)}{9x} \text{ , dfb at } x = \frac{-1}{3}$$

So, $x = \frac{-1}{3}$ is also a regular singular pt.

(iii) $x^2 y'' + (2-x)y' = 0$

$$\Rightarrow y'' + \left(\frac{2-x}{x^2}\right)y' + 0\,y = 0$$

$$\Rightarrow P(x) = \frac{2-x}{x^2}, \quad Q(x) = 0$$

(= analytic $\forall x$)

$Q(x)$, being 0 is analytic $\forall x$.

& $P(x)$ is non analytic at $x = 0$ = singular pt.

Now, checking for regular singular pt :-

$\boxed{x = 0}$

$x\,P(x) = \frac{2-x}{x}$, which is non dfb at $x = 0$.

$x^2 Q(x) = 0$, dfb at $x = 0$

$\therefore$ both $P(x)$ & $Q(x)$ are not satisfied so, its an irregular singular pt.

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(iv) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$

Ans (v) (a) $y'' + \sin x\, y = 0$

(b) $x y'' + \sin x\, y = 0$

(c) $x^2 y'' + \sin x\, y = 0$

(d) $x^3 y'' + \sin x\, y = 0$

(e) $x^4 y'' + \sin x\, y = 0$

(iv) $$y'' - \frac{2}{x^3}y' + \frac{3}{x^2(x-1)}y = 0$$

$$P(x) = -\frac{2}{x^3}, \quad Q(x) = \frac{3}{x^2(x-1)}$$

The non analytic pts. for both $P(x)$ & $Q(x)$ are :- $x = 0$, $x = 1$ → only for $Q(x)$

Checking for regular singular pt

$\boxed{x = 0}$

$x\,P(x) = \frac{-2}{x^2}$ = non analytic at $x = 0$ (or singular at $x = 0$)

$x^2 Q(x) = \frac{3}{(x-1)}$ = analytic at $x = 0$ (or $x = 0$ is not a singular pt.)

∴ Both are not satisfied, so, $x = 0$ is an irregular singular pt.

$\boxed{x = 1}$

$(x-1)P(x) = \frac{-2(x-1)}{x^3}$ = analytic at $x = 1$.

$(x-1)^2 Q(x) = \frac{3(x-1)}{x^2}$ = analytic at $x = 1$.

So, $x = 1$ is a regular singular pt.

(v) (a) $y'' + \sin x \, y = 0$

$\equiv y'' + 0\, y' + \sin x \cdot y = 0$

$P(x) = 0, \quad Q(x) = \sin x$

Clearly, both $P(x)$ & $Q(x)$ are analytic $\forall x$

So, $\exists$ no singular pts.

(b) $y'' + 0\, y' + \frac{1}{x} \sin x \, y = 0$

$P(x) = 0, \quad Q(x) = \frac{\sin x}{x}$

~~$Q(x)$ is non analytic at $x = 0$ = singular pt~~

~~$P(x)$ is analytic $\forall x$~~

~~Checking for regular singular pt~~

$\lim_{x \to 0} \frac{\sin x}{x} = 1$. So, $x = 0$ is not a singular pt

Hence, the given DE is analytic $\forall x$

(c) $y'' + 0\, y' + \frac{1}{x^2} \sin x \, y = 0$

The singular pts or non analytic pt is $\underline{\underline{x = 0}}$

Regular singular pt. check:

$\boxed{x = 0}$

$$\left.\begin{aligned} x\, P(x) &= 0 \\ x^2 Q(x) &= \sin x \end{aligned}\right\} \text{analytic at } x = 0$$

So, $x = 0$ is a regular singular pt

(d) $y'' + 0y' + \frac{1}{x^3}\sin x \, y = 0$

$x = 0$ is the only singular pt
So, Checking for regular singular pt.

$x P(x) = 0$: analytic at $x = 0$
$x^2 Q(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$: analytic at $x = 0$.

Hence, $x = 0$ is a regular singular pt. for the given D.E

(e) $y'' + 0y' + \frac{1}{x^4}\sin x \, y = 0$

$P(x) = 0$ $\quad\quad$ $Q(x) = \frac{\sin x}{x^4}$

The only singular pt. is $x = 0$.
Checking for regular singular pt.

$x P(x) = 0$: analytic at $x = 0$
$x^2 Q(x) = \frac{\sin x}{x^2} = \frac{1}{x}\left(\lim_{x \to 0} \frac{\sin x}{x}\right) = \frac{1}{x}$

$=$ non analytic at $x = 0$.

So, $x = 0$ is an irregular singular pt.

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★ NOTE : FROBENIUS SERIES METHOD

* Method to solve a D.E. with $x=0$ as a regular singular point :

S1) Let the power series solⁿ of the D.E at $x=0$ be of the form:

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$; where m is a fixed constt (+ve or −ve) ≠ 0 or fraction

i.e, $\boxed{y = \sum_{n=0}^{\infty} a_n x^{m+n}}$ → FROBENIUS SERIES

S2) Substitute above series in the D.E & proceed as in the previous model.

→ for $x = a$, $y = \sum_{n=0}^{\infty} a_n (x-a)^{m+n}$

★ NOTE ② :

* Method to find INDICIAL EQUATION

S1) Substituting the above Frobenius series in the given DE & equating the lowest power of x to 0 will give us an algebraic eqⁿ in 'm' called as Indicial eqⁿ & whose roots are the values of the constt 'm'.

IDEA

Indirect method

Find indicial eqⁿ & its roots for

① $x^2 y'' + p x y' + q y = 0$ → given D E .

Std. form:

$$y'' + \frac{p}{x} y' + \frac{q}{x^2} y = 0$$

Indirect method

The req^d indicial eq^n :-

$$\boxed{m(m-1) + pm + q = 0}$$

→ p : coeff of $\frac{1}{x}$ in y' term

→ q : coeff of $\frac{1}{x^2}$ in y term.

(2) $y'' + \left(\frac{p_0 + p_1 x + p_2 x^2 + \cdots}{x}\right)y' + \left(\frac{q_0 + q_1 x + q_2 x^2 \cdots}{x^2}\right)y = 0$

Finding indicial eq^n :-

$$m(m-1) + \underbrace{p_0}_{\text{coeff. of } \frac{1}{x}} m + \overbrace{q_0}^{\text{coeff. of } \frac{1}{x^2}} = 0$$

Q Find indicial eq^n & roots.

(a) $4x^2 y'' + (2x^4 - 5x)y' + (3x^2+2)y = 0$

(b) $2x^2 y'' + x(2x+1)y' - y = 0$

(c) $x^3 y'' + (\cos 2x - 1)y' + 2xy = 0$

(a) $y'' + \left(\frac{2x^4 - 5x}{4x^2}\right)y' + \left(\frac{3x^2+2}{4x^2}\right)y = 0$

$\Rightarrow y'' + \left[\frac{1}{2}(x^2) - \underbrace{\frac{5}{4}}_{p}\left(\frac{1}{x}\right)\right]y' + \left(\frac{3}{4} + \underbrace{\frac{1}{2}}_{q}\frac{1}{x^2}\right)y = 0$

So, By Indirect method, the indicial eq^n :-

$m(m-1) + pm + q = 0$

$\Rightarrow m(m-1) - \frac{5}{4}m + \frac{1}{2} = 0$

$\Rightarrow (4m-1)(m-2) = 0 \Rightarrow m = \underbrace{1/4, 2}_{\text{Indicial roots.}}$

$$\cos x = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right]$$

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(b) $2x^2 y'' + x(2x+1)y' - y = 0$

$$\Rightarrow y'' + \frac{(2x+1)}{2x} y' - \left(\frac{1}{2x^2}\right) y = 0$$

$$\Rightarrow y'' + \left[1 + \underbrace{\left(\frac{1}{2}\right)}_{p}\left(\frac{1}{x}\right)\right] y' + \left[\underbrace{\left(\frac{-1}{2}\right)}_{q}\left(\frac{1}{x^2}\right)\right] y = 0$$

Indicial eq^n:-

$$m(m-1) + pm + q = 0$$

$$\Rightarrow m(m-1) + \frac{m}{2} - \frac{1}{2} = 0$$

$$\Rightarrow 2m^2 - 2m + m - 1 = 0$$

$$\Rightarrow 2m^2 - m - 1 = 0$$

$$\Rightarrow m = \frac{1 \pm \sqrt{1+8}}{4} = 1, \frac{-1}{2} \rightarrow \text{Indicial roots}$$

a series

(c) $y'' + \left(\frac{\cos 2x - 1}{x^3}\right) y' + \left(\frac{2}{x^2}\right) y = 0$

$$\Rightarrow y'' + \frac{\left\{\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots\right] - 1\right\}}{x^3} y' + 2\left(\frac{1}{x^2}\right) y = 0$$

$$\Rightarrow y'' \left\{\left(\frac{1}{x^3} - \underbrace{(2)}_{p}\left(\frac{1}{x}\right) + \frac{2^4 x}{4!} - \dots\right) - 1\right\} y' + \underbrace{2}_{q}\left(\frac{1}{x^2}\right) y = 0$$

Indicial eq^n:-

$$m(m-1) + pm + q = 0$$

$$\Rightarrow m(m-1) - 2m + 2 = 0$$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-2)(m-1) = 0 \Rightarrow m = 1, 2 = \text{Indicial roots}$$

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Q. Verify that the origin is a regular singular pt. & calculate 2 Independent Frobenius series solⁿ for the DE:

$$4xy'' + 2y' + y = 0$$

$$\Rightarrow y'' + \left(\frac{1}{2x}\right)y' + \left(\frac{1}{4x}\right)y = 0 \quad \text{--- (1)}$$

Now, check whether $x=0$ is a regular singular pt.

In DE (1), $p(x) = \frac{1}{2x}$, $q(x) = \frac{1}{4x}$

In both of them, $x=0$ is a singular pt

Now, $x\,p(x) = \frac{1}{2}$

$x^2 q(x) = \frac{x}{4}$ } analytic at $x=0$.

So, $x=0$ is a regular singular pt

$\Rightarrow$ Frobenius series method is applicable to solve DE (1).

S1) Let $y = \sum_{n=0}^{\infty} a_n x^{m+n}$ be the trial solⁿ of eqⁿ (1).

So, $y' = \sum_{n=0} a_n (m+n) x^{m+n-1}$

& $y'' = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$

we have an added x^m in our term. So, -ve power of x is not possible. $\Rightarrow$, use from $n=0$

Now, $\frac{y}{x^2} = \sum_{n=0}^{\infty} a_n x^{m+n-2}$

$\frac{y'}{x} = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-2}$

$\times\ \lambda\ \div$ by x so that we get y/x^2 term

from ①,

$$y'' + \frac{1}{2}\left(\frac{y'}{x}\right) + \frac{1}{4}x\left(\frac{y}{x^2}\right) = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{m+n-2} + \frac{1}{2}\sum_{n=0}^{\infty} a_n(m+n)x^{m+n-2} + \frac{1}{4}x\sum_{n=0}^{\infty} a_n x^{m+n-2} = 0$$

Power of x is same & starting term is also same. So, that's why take terms of $\frac{y'}{x}$ & $\frac{y}{x^2}$

$\div$ both sides by x^{m-2} (to remove equal powers)

$$\Rightarrow \sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^n + \frac{1}{2}\sum_{n=0}^{\infty} a_n(m+n)x^n + \frac{x}{4}\sum_{n=0}^{\infty} a_n x^n = 0$$

Taking it inside will change power of x

(Powers are reduced now)

$$\Rightarrow \sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^n + \frac{1}{2}\sum_{n=0}^{\infty} a_n(m+n)x^n + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now, make powers of x & starting term same

So, in 3rd term: $n+1 = k$

$n = k-1$

$k \to n.$

$\Rightarrow n \to n-1$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^n + \frac{1}{2}\sum_{n=0}^{\infty} a_n(m+n)x^n + \frac{1}{4}\sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

$$\Rightarrow a_0(m-1)(m)x^0 + \sum_{n=1}^{\infty} a_n(m+n)(m+n-1)x^n$$

$$+ \frac{1}{2}a_0(m)x^0 + \sum_{n=1}^{\infty} a_n(m+n)x^n$$

$$+ \frac{1}{4}\left(\sum_{n=1}^{\infty} a_{n-1}x^n\right) = 0$$

$$\Rightarrow \left[(m)(m-1) + \left(\frac{m}{2}\right)\right]a_0x^0 + \sum_{n=1}^{\infty}\left\{\left[(m+n)(m+n-1) + \frac{(m+n)}{2}\right]a_nx^n + \frac{1}{4}a_{n-1}x^n\right\} = 0$$

lowest power of x

Comparing constant term (x^0 term) to zero

$$\Rightarrow (m)\left[m - \frac{1}{2}\right]a_0 = 0$$

∵ $a_0 \neq 0$, so, Indicial eq^n is $m(m - 1/2) = 0$

& Indicial roots are: $\boxed{m = 0, 1/2}$

Equating coeff. of x^n to zero

$$\Rightarrow a_n(m+n)(m+n-1) + \frac{1}{2}a_n(m+n) + \frac{1}{4}a_{n-1} = 0$$

for $n = 1, 2, 3, \ldots$

$$\Rightarrow a_n\left[(m+n)(m+n-1/2)\right] = -\frac{1}{4}a_{n-1}$$

$$\Rightarrow a_n\left[(m+n)\left(\frac{2(m+n)-1}{\cancel{2}}\right)\right] = \frac{-1}{\cancel{4}\,2}a_{n-1}$$

$$\Rightarrow a_n (m+n)(2(m+n)-1) = \frac{-1}{2} a_{n-1}$$

$$\Rightarrow a_n = -\frac{1}{2(m+n)(2(m+n)-1)} a_{n-1}$$

→ recurrence eq^n

→ valid for $n = 1, 2, \ldots$

Now,

- CASE - I : $m = 0$

$$\Rightarrow a_n = \frac{-1}{2(0+n)(2(0+n)-1)} a_{n-1}$$

$$\Rightarrow a_n = \frac{-1}{2n(2n-1)} a_{n-1}$$

→ recurrence rel^n when $m=0$

→ $n \in 1, 2, 3 \ldots$

→ for $n = 1$

$$\Rightarrow a_1 = -\frac{a_0}{2}$$

→ for $n = 2$

$$a_2 = -\frac{a_1}{4.3} = \frac{(-1)^2 a_0}{1.2.3.4}$$

→ for $n = 3$

$$a_3 = -\frac{a_2}{6.5} = \frac{(-1)^3 a_0}{1.2.3.4.5.6}$$

$\vdots$

So, the Frobenius series is

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} = x^0 \sum_{n=0}^{\infty} a_n x^n.$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= \frac{a_0}{0!} - \frac{1}{2!} a_0 x + \frac{a_0}{4!} x^2 - \frac{a_0}{6!} x^3 + \ldots$$

$\Rightarrow y = a_0\left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \dots\right]$

$\Rightarrow y = a_0\left[\cos\sqrt{x}\right]$

→ one of the independent solutions

→ a_0 : any constt

• CASE - II : $m = 1/2$

$\Rightarrow a_n = \frac{-1}{2(1/2+n)[2(1/2+n)-1]} a_{n-1}$

$a_n = -\frac{a_{n-1}}{2n(2n+1)}$

→ recurrence relⁿ for $m = 1/2$

→ valid for $n \in \{1, 2, 3, \dots\}$

→ for $n=1$

$a_1 = \frac{-a_0}{2\cdot 3}$

→ for $n=2$

$a_2 = -\frac{a_1}{4\cdot 5} = (-1)^2 \frac{a_0}{2\cdot 3\cdot 4\cdot 5}$

→ for $n=3$

$a_3 = -\frac{a_2}{6\cdot 7} = (-1)^3 \frac{a_0}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$

⋮

So, the Frobenius series is

$y = x^m \sum_{n=0}^{\infty} a_n x^n = x^{1/2} \sum_{n=0}^{\infty} a_n x^n$

$= x^{1/2}\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\right]$

$\Rightarrow y = x^{1/2}\left[a_0 - \frac{a_0}{3!} x + \frac{a_0}{5!} - \frac{a_0}{7!} + \dots\right]$

$$\Rightarrow y = a_0\left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \cdots\right]$$

$$\Rightarrow y = a_0\left[\cos\sqrt{x}\,\right]$$

→ one of the independent solutions

→ a_0 : any constt.

- **CASE - II : $m = 1/2$**

$$\Rightarrow a_n = \frac{-1}{2(1/2+n)\left[2(1/2+n)-1\right]}\, a_{n-1}$$

$$a_n = -\frac{a_{n-1}}{2n(2n+1)}$$

→ recurrence relⁿ for $m = 1/2$

→ valid for $n \in \{1, 2, 3, \ldots\}$

→ for $n = 1$

$$a_1 = \frac{-a_0}{2\cdot 3}$$

→ for $n = 2$

$$a_2 = -\frac{a_1}{4.5} = (-1)^2\frac{a_0}{2\cdot3\cdot4\cdot5}$$

→ for $n = 3$

$$a_3 = -\frac{a_2}{6\cdot 7} = (-1)^3\frac{a_0}{2\cdot3\cdot4\cdot5\cdot6\cdot7}$$

⋮

So, the Frobenius series is

$$y = x^m\sum_{n=0}^{\infty} a_n x^n = x^{1/2}\sum_{n=0}^{\infty} a_n x^n$$

$$= x^{1/2}\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right.$$

$$\Rightarrow y = x^{1/2}\left[a_0 - \frac{a_0}{3!}x + \frac{a_0}{5!} - \frac{a_0}{7!} + \cdots\right.$$

$$\Rightarrow y = a_0\left[\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} - \frac{(\sqrt{x})^7}{7!} + \dots\right]$$

$$\Rightarrow y = a_0 \sin\sqrt{x}$$

$\hookrightarrow$ the 2nd independent sol^n

Hence, the general sol^n,

$$y(x) = c_1 \cos\sqrt{x} + c_2 \sin\sqrt{x} \quad \left(\equiv c_1 y_1(x) + c_2 y_2(x)\right)$$

$\hookrightarrow$ a_0 is merging with constts c_1 & c_2.

Q. Verify if $x=0$ is a regular singular pt. & find Frobenius series sol^n for DE:

$$2x^2 y'' + xy' - (x+1)y = 0 \rightarrow (1)$$

$$\Rightarrow y'' + \frac{1}{2x}y' - \frac{(x+1)}{2x^2}y = 0 \rightarrow (2)$$

Si) Check whether $x=0$ is regular singular pt.

$$P(x) = \frac{1}{2x}, \quad Q(x) = -\frac{(x+1)}{2x^2}$$

$x=0$ is a singular pt.

Now, $\left.\begin{array}{l} xP(x) = 1/2 \\ x^2 Q(x) = -\frac{(x+1)}{2} \end{array}\right\}$ non singular pt.

$\Rightarrow x=0$ is a regular singular pt.

S2) Writing initial trial solⁿ for Frobenius series :-

Let

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} \text{ be the trial sol}^n$$

$$y' = \sum_{n=0}^{\infty} a_n (m+n) x^{(m+n-1)}$$

$$y'' = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{(m+n-2)}$$

Now, $$\frac{y}{x^2} = \sum_{n=0}^{\infty} a_n x^{m+n-2}$$

$$\frac{y'}{x} = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-2}$$

from (2)

$$y'' + 2\left(\frac{y'}{x}\right) - \left(\frac{x+1}{2}\right)\left(\frac{y}{x^2}\right) = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$$

$$+ \frac{1}{2}\sum_{n=0}^{\infty} a_n (m+n) x^{m+n-2}$$

$$-\left(\frac{x+1}{2}\right)\sum_{n=0}^{\infty} a_n x^{m+n-2} = 0.$$

Cancelling x^{m-2}

$$\Rightarrow \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^n$$

change required

$$+ \frac{1}{2}\sum_{n=0}^{\infty} a_n (m+n) x^n - \frac{1}{2}\sum_{n=0}^{\infty} a_n x^{n+1} - \frac{1}{2}\sum_{n=0}^{\infty} a_n x^n$$

$$= 0$$

So, for 3rd term :- $m+1 = k$

$$n = k-1$$

$$k \to n$$

So, basically, $n \to n-1$

$$\Rightarrow \sum_{n=0}^{\infty} \left\{ \left[(m+n)(m+n-1) \right] a_n + \tfrac{1}{2}(m+n) a_n \right\} x^n$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} a_{n-1} x^n - \frac{1}{2} \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \left[(m+0)(m+0-1) a_0 + \tfrac{1}{2}(m+0) a_0 \right] x^0 + \sum_{n=1}^{\infty} \left[\begin{matrix} (m+n)(m+n-1) \\ + \tfrac{1}{2}(m+n) \end{matrix} \right] a_n x^n$$

$$- \frac{1}{2} a_0 x^0 \qquad - \frac{1}{2} \sum_{n=1}^{\infty} a_n x^n$$

$$- \frac{1}{2} \sum_{n=1}^{\infty} a_{n-1} x^n .$$

$$= \left[m(m-1) a_0 + \tfrac{1}{2} m a_0 - \frac{a_0}{2} \right]$$

$$+ \sum_{n=1}^{\infty} \left(\left[(m+n)(m+n-1) + \tfrac{1}{2}(m+n) - \frac{1}{2} \right] a_n - \frac{1}{2} a_{n-1} \right) x^n = 0 .$$

Comparing constt. term

$$\Rightarrow m^2 a_0 - \frac{m a_0}{2} - \frac{a_0}{2} = 0$$

$$\Rightarrow a_0 \left[2m^2 - m - 1 \right] = 0$$

$\because a_0 \neq 0$

$\Rightarrow 2m^2 - m - 1 = 0$: Indicial eq^n

$\Rightarrow m = m^2 - m + m^2 - 1$

$= m(m-1) + (m-1)(m+1)$

$\Rightarrow (2m+1)(m-1) = 0$

$\Rightarrow m = -\frac{1}{2}, 1$: Indicial roots

Now, equating coeff of x^n

$$\Rightarrow \left[(m+n)(m+n-1) + \tfrac{1}{2}(m+n)\right] a_n - \frac{a_{n-1}}{2} = 0$$

$$\Rightarrow a_n \left[(m+n)\left(m+n-\frac{1}{2}\right) \right] = \frac{1}{2} a_{n-1}$$

$$\Rightarrow a_n = \frac{a_{n-1}}{\left[\left(2(m+n)+1\right)(m+n-1)\right]}$$

→ recurrence eq^n

→ valid for $n = 1, 2, ---$

Case I : $m = -\frac{1}{2}$

$$a_n = \frac{a_{n-1}}{\left[\left(2\left(n+\frac{1}{2}\right)+1\right)\left(n+\frac{1}{2}-1\right)\right]}$$

$$\Rightarrow a_n = \frac{a_{n-1}}{n(2n-3)}$$

→ for $n=1$

$$\Rightarrow a_1 = \frac{a_0}{1\cdot(-1)} = -a_0$$

→ for $n=2$

$$a_2 = \frac{a_1}{2\cdot 1} = -\frac{a_0}{1\cdot 2}$$

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↳ for $n = 3$

$$a_3 = \frac{a_2}{3(3)} = -\frac{a_0}{1.2.3.3} = -\frac{a_0}{18}$$

Hence, the Frobenius series :-

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$

$$= x^{-1/2}\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\right]$$

$$\Rightarrow y = x^{-1/2}\left[\underbrace{a_0}_{\to 1,\text{ say}} - a_0 x - \frac{a_0}{2}x^2 - \frac{a_0}{18}x^3 - \dots\right]$$

$$= x^{-1/2}\left[1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \dots\right]$$

Case II : $m = 1$

$$\Rightarrow a_n = \frac{a_{n-1}}{(2(n+1)+1)(1+n-1)}$$

$$= \frac{a_{n-1}}{n(2n+3)}$$

Solution :

$$y = x^m(a_0 + a_1 x + a_2 x^2 + \dots)$$

$$\hookrightarrow a_1 = \frac{a_0}{1.5}, \dots a_2 = \frac{a_0}{70} \dots a_3 = \frac{a_0}{1890}, \dots$$

$$\Rightarrow y = a_0 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \dots\right)$$

Q. Solve the Bessel's eq^n, when $p = 0$

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

$\hookrightarrow p = 0$

$$\Rightarrow x^2y'' + xy' + x^2y = 0.$$

Sol^n S1) Check for regular singular pt.

$$y'' + \frac{1}{x}y' + y = 0 \longrightarrow (1)$$

$P(x) = \frac{1}{x}$ } $x = 0$ is a singular pt.

$Q(x) = 1$

Now, $x\,P(x) = 1$
$x^2 Q(x) = x^2$ } both are analytic

So, $x = 0$ is not a singular pt or, its analytic.

So, $x = 0$ is a regular singular pt.

S2) Solving by Frobenius series method,

Let the trial sol^n be $y = x^m \sum_{n=0}^{\infty} a_n x^n$.

$$\Rightarrow y' = \sum_{n=0}^{\infty} a_n \cdot (m+n) x^{m+n-1}$$

$$\& \ y'' = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$$

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Now, $\frac{y}{x^2} = \sum_{n=0}^{\infty} a_n x^{m+n-2}$

$\frac{y'}{x} = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-2}$

So, from ①

$$y'' + \frac{y'}{x} + x^2\left(\frac{y}{x^2}\right) = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{m+n-2}$$

$$+ \sum_{n=0}^{\infty} a_n(m+n)x^{m+n-2}$$

$$+ x^2\sum_{n=0}^{\infty} a_n x^{m+n-2} = 0$$

$\div x^{m-2}$ both sides

$$\Rightarrow \sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^n$$

$$+ \sum_{n=0}^{\infty} a_n(m+n)x^n$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Make x^n terms same

$\Rightarrow n+2 = k$

& $n = k-2$

& $k \to n$

$\Rightarrow n \to n-2$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^n + \sum_{n=0}^{\infty} a_n(m+n)x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\Rightarrow a_0(m)(m-1)x^0 + a_1(m+1)(m)x + \sum_{n=2}^{\infty} a_n(m+n)(m+n-1)x^n$$

$$+ a_0(m)x^0 + a_1(m+1)x + \sum_{n=2}^{\infty} a_n(m+n)x^n$$

$$+ \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\Rightarrow a_0[m + m(m-1)]$$
$$+ a_1[m(m+1) + (m+1)]x$$
$$+ \sum_{n=2}^{\infty} [(m+n)(m+n-1) + (m+n)] a_n x^n + a_{n-2}x^n = 0$$

$$\Rightarrow a_0 m^2 + a_1(m+1)^2 x$$
$$+ \sum_{n=2}^{\infty} \left[(m+n)^2 a_n + a_{n-2}\right] x^n = 0$$

Equating constt. term

$\Rightarrow a_0 m^2 = 0 \Rightarrow m^2 = 0$ Indicial eq^n

$\Rightarrow m = 0, 0 \quad (\because a_0 \neq 0)$

Equating coeff of x

$\Rightarrow a_1(m+1)^2 = 0 \Rightarrow a_1(0+1)^2 = 0$

$\Rightarrow a_1 = 0$

So, the indicial root is $m = 0$

Now, equating coeff of x^n

$\Rightarrow (m+n)^2 a_n + a_{n-2} = 0$

$\Rightarrow \boxed{a_n = \dfrac{-1}{(m+n)^2} a_{n-2}}$ → recurrence rel^n → valid for $n \geq 2$

Case I : for $m = 0$

$$\Rightarrow a_n = \frac{-1}{n^2} a_{n-2}$$

$\to n = 2$

$$a_2 = \frac{-1}{4} a_0$$

$\to n = 3$

$$a_3 = \frac{-1}{9} a_1 = 0$$

$\to n = 4$

$$a_4 = \frac{-1}{4^2} a_2 = \frac{+1}{4^3} a_0$$

$\to n = 5$

$$a_5 = \frac{-1}{5^2} a_3 = \frac{+1}{5^2 3^2} a_1 = 0$$

$\to n = 6$

$$a_6 = \frac{-1}{6^2} a_4$$

$$\Rightarrow a_6 = \frac{-1}{2^2 4^2 6^2} a_0$$

$$\Rightarrow a_{2k} = (-1)^k \frac{a_0}{2^2 4^2 \dots (2k)^2} ; n \in \mathbb{Z}$$

$\Rightarrow$ Frobenius series solⁿ is

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= x^0 \left(a_0 - \frac{a_0}{2^2} x^2 + \frac{a_0}{2^2 4^2} x^4 - \frac{a_0}{2^2 4^2 6^2} x^6 + \dots \right)$$

$$\Rightarrow y = a_0 \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots \right)$$

Now, $a_{2K} = (-1)^k \dfrac{a_0}{2^2 \cdot 4^2 \dots (2K)^2} = \dfrac{(-1)^k a_0}{(2^2)^k [1^2 \cdot 2^2 \cdot 3^2 \dots k^2]}$

Taking 2^2 common from everyterm of denominator

$$\Rightarrow a_{2k} = (-1)^k \frac{a_0}{2^{2k}(k!)^2}$$

So, $$y = \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

$$\Rightarrow y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} a_0 x^{2n}.$$

Section - 30

REGULAR SINGULAR POINTS
(Special Cases)

for the eq^n, $y'' + P(x)y' + Q(x)y = 0$.

CASE - I :

If roots of indicial eq^n are equal, i.e, $m_1 = m_2$.

→ Second Frobenius series solution $[(\nexists)]$.
So, in this case, only one sol^n $\exists$.

CASE - II :

If roots of indicial eq^n are m_1 & m_2 & $m_1 = m_2 + p$, where p is a +ve integer.
or $(m_1 - m_2) = p$.

→ In this case we may/may not have a Second independent sol^n.

→ If, second independent sol^n $\exists$, then, its given by

$$y_2(x) = x^{m_2} \sum_{n=0}^{\infty} b_n x^n + a \cdot \log x \; y_1(x)$$

→ a : a constt (may be zero)
(in this case, it can't be a log term).

CASE - III

If roots of indicial eq^n are distinct. Then, the given DE possess 2 independent sol^n.

$(m_1 \neq m_2)$

& $m_1 - m_2 \neq$ an integer.

($\because$ otherwise, it'll be Case - II)

Problem 1) $x^2 y'' - 3xy' + (4x+4)y = 0$] equal

Hw 2) $4x^2y'' - 8x^2y' + (4x^2+1)y = 0$] root

Find the Frobenius series sol^n for 1) & 2)

3) $x^2y'' - x^2y' + (x^2-2)y = 0 \rightarrow$ distinct root differing by integer

3) $y'' - x\left(\frac{y'}{x}\right) + (x^2-2)\left(\frac{y}{x^2}\right) = 0$.

Write after checking: Here, $x = 0$ is a regular singular point. So, Frobenius series sol^n is possible.

Let $y = x^m \sum_{n=0}^{\infty} a_n x^n$ be sol^n of (1)

$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^{m+n}$. m : constt.

Solving in the same way as before, we get an eq^n with constt term, x^n terms ...

Now,

equating coeff. of x^0 to zero (constt term)

$\Rightarrow a_0 [m(m-1) - 2] = 0 \longrightarrow (4)$

Coeff of x^n to zero, ; $n \in 2, 3, 4 \ldots$

$$\Rightarrow a_n\left[(m+n)(m+n-1)-2\right] - a_{n-1}(m+n-1) + a_{n-2} = 0 \longrightarrow (2)$$

is called as 3 term formula ($\because$ 3 terms $\to a_n, a_{n-1}, a_{n-2}$ are related)

Coeff. of x to zero.

$$\Rightarrow a_1\left[(m+1)(m)\right] - a_0 m - 2a_1 = 0 \longrightarrow (3)$$

Using value of a_0 & a_1 in eq^n (2), we get
$\to$ from (4) & (3)

$$a_n = \frac{a_{n-1}(m+n-1) - a_{n-2}}{\left[(m+n)(m+n-1)-2\right]}$$ Three term Recurrence eq^n

$\hookrightarrow n = 2, 3, \ldots$

Indicial eq^n : coeff. of $x^0 = 0$

$\Rightarrow$ from (4)

$$m(m-1) - 2 = 0 \qquad (\because a_0 \neq 0)$$

$$\Rightarrow m^2 - m - 2 = 0$$

$$\Rightarrow m = \frac{1 \pm 3}{2} = 2, -1 = m_1, m_2$$

It comes under case II ($\because m_1 - m_2 = 2-(-1) = 3$)

Case (a) :- when $m = 2$

$$a_n = \frac{a_{n-1}(n+1) - a_{n-2}}{\left[(n+2)(n+1)-2\right]}$$

$\hookrightarrow n = 2, 3, 4, \ldots$

From (3),

$$a_1(2+1)(2) - a_0(2) - 2a_1 = 0$$

$$\Rightarrow 6a_1 - 2a_1 - 2a_0 = 0$$

$$\Rightarrow \boxed{a_1 = \frac{1}{2}a_0}$$

for $n = 2$

$$\Rightarrow a_2 = \frac{a_1(3) - a_0}{4(3) - 2}$$

$$\Rightarrow a_2 = \frac{\frac{a_0}{2}(3) - a_0}{10} = \frac{a_0}{20}$$

for $n = 3$

$$a_3 = -\frac{a_0}{60}$$

Substitute these values in Frobenius series,

$$\Rightarrow y = x^2\left[a_0 + \frac{a_0}{2}x + \frac{a_0}{20}x^2 - \frac{a_0}{60}x^3 + \dots\right]$$

$$= a_0 x^2\left[1 + \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{60} + \dots\right]$$

for $a_0 = 1$

$$\Rightarrow y = x^2\left[1 + \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{60} + \dots\right]$$

Ily, case (b) :- $m = -1$

Solving in same way, we get

$$y = x^{-1}\left[1 + \frac{x}{2} + \frac{x^2}{2} - \frac{x^4}{8} + \dots\right]$$

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1) $y'' - 3\left(\frac{y'}{x}\right) + (4x+4)\left(\frac{y}{x^2}\right) = 0$

here, after solving,

$x=0$ is a regular singular point

So, Frobenius series solⁿ is valid.

Let

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}.$$

Substituting Frobenius series & equating coeff. to zero, we get

coeff. of x^n: $a_n\left((m+n)(m+n-4)+4\right) + 4a_{n-1} = 0 \rightarrow$ (2)

$\rightarrow n = 1, 2, 3, \ldots$

coeff. of x^0: $a_0\left(m(m-1) - 3m + 4\right) = 0 \longrightarrow$ (3)

Indicial eqⁿ :- (3) = 0

$\Rightarrow m(m-1) - 3m + 4 = 0$

$\Rightarrow m^2 - 4m + 4 = 0$

$\Rightarrow m = \underbrace{2, 2}_{\text{equal roots} \Rightarrow \text{Case I}}$

From (2)

$$a_n = \frac{-4a_{n-1}}{[(m+n)(m+n-4)+4]}$$

When $m=2$ (only one independent solⁿ)

$$a_n = \frac{-4a_{n-1}}{(n+2)(n-2)+4} = \frac{-4a_{n-1}}{n^2-4+4}$$

$\Rightarrow a_n = \frac{-4a_{n-1}}{n^2}$, recurrence relⁿ, valid from $n = 1, 2, \ldots$

for $n=1$

$$a_1 = -\frac{4a_0}{1^2} = -4a_0$$

$$a_2 = -\frac{4a_1}{2^2} = -a_1 = +4a_0.$$

$$a_3 = -\frac{4a_2}{3^2} = -\frac{4(4a_0)}{9}$$

$$= -\frac{16}{9}a_0$$

$$a_4 = -\frac{4a_3}{4^2}$$

$$= -\frac{4}{4^2}\left(-\frac{16}{9}a_0\right)$$

$$= +\frac{4}{9}a_0$$

The req^d sol^n :-

$$y = x^m\left[a_0 + a_1x + a_2x^2 + \ldots\right]$$

$$\Rightarrow y = x^2\left[a_0 - 4a_0x + 4a_0x^2 - \frac{16}{9}a_0x^3 + \ldots\right]$$

Ans

HW

Problem :- Solve Bessel's eq^n of order $p = 1/2$:

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Show that $m_1 - m_2 = 1$

But, nevertheless, the eq^n has 2 independent Frobenius series sol^n. Find the sol^{ns}.

Section – 31

HYPERGEOMETRIC EQ^n.
↳ by GAUSS.

Std. form :-

$$\boxed{x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0} \longrightarrow (1)$$

↳ a, b, c : constts.

So, by Frobenius series method,

$$P(x) = \frac{c-(a+b+1)x}{x(1-x)} \; ; \; Q(x) = \frac{-ab}{x(1-x)}$$

& $x=0$ & $x=1$ are singular pts. of (1).

Now,

Finding the sol^n of eq^n (1) around the regular singular point $x=0$, using Frobenius series method gives the following results :

Indicial eq^n :- $m(m-(1-c)) = 0$

$\Rightarrow m = 0, 1-c$: Indicial roots

Finding the sol^n of (1) when

Case (1) : $m = 0$

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$

Remember

The result after doing Frobenius will be used directly

$$\Rightarrow y = 1 + \frac{ab}{1 \cdot c} + \frac{a(a+1) \cdot b(b+1)}{1 \cdot 2 \cdot c(c+1)} x^2 + \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} x^3 + \cdots$$

↳ when $m = 0$ ☆☆

$$\Rightarrow \boxed{y = F(a, b, c, x)}$$

$\rightarrow F(a,b,c,x) = 1 + \frac{a\,b}{1\cdot c}x + \frac{a(a+1)b(b+1)}{1\cdot 2\, c(c+1)}x^2 + \dots$

$\rightarrow$ Given an eq^n:-

s1) Identify if its hypergeometric.

s2) Find values of a, b & c

s3) Use the std. result to answer.

Case 2 :- $m = 1 - c$

$$\Rightarrow \boxed{y = x^{1-c} F(\underbrace{a-c+1}_{k_1}, \underbrace{b-c+1}_{k_2}, \underbrace{2-c}_{k_3}, x)}$$

$$\Rightarrow \boxed{y = x^{1-c} F(k_1, k_2, k_3, x)}$$

☆ $\rightarrow F(a,b,c,d) \rightarrow$ changed to this.

So, the general sol^n of eq^n 1 around $x = 0$ is :-

$$\boxed{y = c_1 F(a, b, c, x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c, x)} \rightarrow (2)$$

Remember

Now,

General sol^n of 1 around the regular singular point $x = 1$

The initial trial sol^n:- $y = x^m \sum_{x=0}^{\infty} a_n (x-1)^n$

M1) $\rightarrow$ Using it, find Indicial eq^n, indicial roots, solve and get the sol^n.

M2) Alternative

When sol^n of one of the singular pts is known,

Put $t = 1-x$ in ①

↳ when $x = 1$, $t = 0$

$$\Rightarrow \boxed{t(1-t)\, y'' + [(a+b-c+1) - (a+b+1)t]\, y' - aby = 0} \quad \text{③}$$

$y'' = \frac{d^2y}{dt^2}$, $y' = \frac{dy}{dt}$

→ Now, this is our hypergeometric eq^n.
We have to find its sol^n for $t = 0$

Note: General sol^n of ③ at $t = 0$ is same as general sol^n of ① at $x = 1$

Comparing eq^{ns} ① & ③ we find, the change is there in value of c,
Clearly, $c = a+b-c+1$

$$\Rightarrow y = c_1 F(a, b, a+b-c+1, t) + c_2 t^{c-a-b} F(c-b, c-a, c-a-b+1, t)$$

$t \to 1-x$

Remember

$$\Rightarrow \boxed{y = c_1 F(a, b, a+b-c+1, 1-x) + c_2 (1-x)^{c-a-b} F(c-b, c-a, c-a-b+1, (1-x))} \quad \text{④}$$

*

↳ Req^d general sol^n of ① around the regular singular pt, $x = 1$.

* More general form of Hypergeometric eq^n :-

$$(x-A)(x-B)y'' + (C+Dx)y' + Ey = 0 \longrightarrow (A)$$

$\hookrightarrow$ A, B, C, D, E : const's

To find the sol^n of general Hypergeometric eq^n :

Let $t = \dfrac{x-A}{B-A}$

- In eq^n (A), $x = A$, $x = B$ are singular pts. (by finding $P(x)$ & $Q(x)$)
- Check for regular singular pt.

Point $x = A$ & put $t = \dfrac{x-A}{B-A}$

Note :- If we want to find sol^n of eq^n (A) at $x = B$, then, $t = \dfrac{x-B}{A-B}$

Substituting the value of t in (A)

$$\Rightarrow t(1-t)y'' + (F+Gt)y' + Hy \longrightarrow (4)$$

, where $F = C$

$G = -(A+B+1)$

$H = -AB$

Also, $y'' = \dfrac{d^2y}{dt^2}$, $y' = \dfrac{dy}{dt}$

eq^n (4) can be solved at $t = 0$.

$\ast\ \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\ast\ -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

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Solⁿ of (4) at $t=0$ is same as solⁿ of (A) at $x = A$.

Q. Verify the following results :-

(1) $x\,F(1,1,2,-x) = \log(1+x)$

(2) $\lim\limits_{b\to\infty} F\left(a,b,a,\frac{x}{b}\right) = e^x$.

(3) $x\left[\lim\limits_{a\to\infty} F\left(a,a,\frac{3}{2},\frac{-x^2}{4a^2}\right)\right] = \sin x$

(1) Now, $x\,F(1,1,2,-x)$

$$\equiv$$

$$x\,F(a,b,c,x)$$

So, Here $a=1,\ b=1,\ c=2,\ x \longrightarrow -x$

So, solⁿ :-

$$y = x\left[1 + \frac{ab}{1\cdot c}(x) + \frac{a(a+1)\,b(b+1)}{1\cdot 2\ c(c+1)}(x)^2 + \cdots\right]$$

$$= x\left[1 + \frac{1\cdot 1}{1\cdot 2}(-x) + \frac{1\cdot(1+1)\,1(1+1)}{1\cdot 2\ \ 2\cdot(2+1)}(-x)^2 + \frac{1\cdot(1+1)(1+2)\ 1(1+1)(1+2)}{1\cdot 2\cdot 3\ (2)(2+1)(2+2)}(-x)^3 + \cdots\right]$$

$$= x\left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right]$$

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \log(1+x) \quad \text{(Direct formula)}$$

(2) We know,

$$y_1(x) = F(a,b,c,x) = 1 + \frac{ab}{1\cdot c}(x) + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}(x)^2 + \dots$$

Now

$$\text{LHS} :- \lim_{b\to\infty}\left[1 + \frac{ab}{1\cdot a}\left(\frac{x}{b}\right) + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot a(a+1)}\left(\frac{x}{b}\right)^2 + \dots\right]$$

$$= \lim_{b\to\infty}\left[1 + x + \frac{(1+1/b)}{2!}x^2 + \frac{(1+1/b)(1+2/b)}{3!}x^3 \dots\right]$$

$$\left(\lim_{b\to\infty}\frac{k}{b} = 0\right)$$

$$\Rightarrow \text{LHS} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= e^x.$$

$$= \text{RHS}.$$

So, LHS = RHS

(3) $$x\left[\lim_{a\to\infty} F\left(a, a, \frac{3}{2}, \frac{-x^2}{4a^2}\right)\right] = \sin x$$

(a, b, c, x)

$$\text{LHS} = x\lim_{a\to\infty}\left[1 + \frac{a\,a}{1\cdot\left(\frac{3}{2}\right)}\left[\frac{-x^2}{4a^2}\right] + \frac{a(1+a)(a)(1+a)}{1\cdot 2\cdot\left(\frac{3}{2}\right)\left(\frac{3}{2}+1\right)}\left[\frac{-x^2}{4a^2}\right]^2\right.$$
$$\left. + \frac{a(1+a)(2+a)(a)(1+a)(2+a)}{1\cdot 2\cdot 3\left(\frac{3}{2}\right)\left(\frac{3}{2}+1\right)\left(\frac{3}{2}+2\right)}\left(\frac{-x^2}{4a^2}\right)^3 \dots\right]$$

$$= x\lim_{a\to\infty}\left[1 - \frac{x^2}{3!} + \frac{(1+1/a)^2}{5!}x^4 - \frac{(1+1/a)^2(1+2/a)^2}{7!}x^6 \dots\right]$$

$$= x\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right]$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sin x = \text{RHS}$$

H.P.

Note:- $\underbrace{F(a,b,c,x) = F(b,a,c,x)}$

valid $\forall$ questions

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① Using hypergeometric results, find general solⁿ:-

$$x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0$$

Given:- Singular point $= x = 0$

General form of hypergeometric eqⁿ:-

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby = 0$$

↳ matches the given eqⁿ.

Now, by comparing,

$$c = 3/2$$
$$ab = -2$$
$$a+b+1 = 2$$
$$\Rightarrow a+b = 1$$
$$\Rightarrow a = 1-b$$
$$\Rightarrow b(1-b) = -2$$
$$\Rightarrow b^2 - b - 2 = 0$$
$$\Rightarrow b = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = 2, -1$$

$$\Rightarrow \underline{a = -1, b = 2} \quad \text{or} \quad \underline{a = 2, b = -1}$$

∴, the solⁿ of given eqⁿ:-

$$y = c_1 F(a,b,c,x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c, x)$$
$$= c_1 F\left(-1, 2, \frac{3}{2}, x\right) + c_2 x^{-1/2} F\left(-\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, x\right)$$

$$\text{or } y = c_1 F\left(-1, 2, \frac{3}{2}, x\right) + c_2 x^{-1/2} F\left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, x\right)$$

Using the known expansion, result

* Note:- If $F(a,b,c,x)$ has a or b as zero or -ve integers, then, infinite series breaks off into finite polynomial.
If c is zero or -ve integers, then, the solⁿ doesn't exist.

(2) $(x^2-1)y'' + (5x+4)y' + 4y = 0$ at $x = -1$

General form of hypergeometric eq^n :-

$$(x-A)(x-B)y'' + (C+Dx)y' + Ey = 0$$

→ Convert to regular form of hypergeometric eq^n :- $t = \dfrac{x-A}{B-A}$

↳ If singular pt. is reg^d at $x = A$

$$\Rightarrow (x+1)(x-1)y'' + (4+5x)y' + 4y = 0$$

$A = -1, \ B = 1$

$$\Rightarrow t = \frac{x+1}{1-(-1)} = \frac{x+1}{2} \Rightarrow x = 2t-1$$

Now, $x+1 = 2t$

$x-1 = 2(t-2)$

$$\Rightarrow (2t)(2t-2)\,y'' \rightarrow \frac{d^2y}{dt^2}$$

Now, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{dy}{dt} \times \dfrac{1}{2}$

$$\Rightarrow \boxed{\frac{dy}{dt} = 2\frac{dy}{dx}}$$

Now, $\dfrac{d^2y}{dx^2} = \dfrac{1}{2}\dfrac{d}{dx}\left(\dfrac{dy}{dt}\right)$

$$= \frac{1}{2}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

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$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}\frac{d^2y}{dt^2}\cdot\frac{1}{2} = \frac{1}{4}\frac{d^2y}{dt^2}$$

Now, given the eqⁿ

$$(x+1)(x-1)y'' + (5x+4)y' + 4y = 0$$

$$(2t)(2t-2)\left(\frac{1}{4}\frac{d^2y}{dt^2}\right) + [5(2t-1)+4]\left(\frac{1}{2}\frac{dy}{dt}\right) + 4y = 0$$

$$\Rightarrow t(t-1)\frac{d^2y}{dt^2} + \frac{(10t)-1}{2}\frac{dy}{dt} + 4y = 0$$

$$\Rightarrow t(1-t)\left(\frac{d^2y}{dt^2}\right) + \left(\frac{1}{2}-5t\right)dy/dt - 4y = 0$$

$$x(x-1)y'' + [c-(a+b+1)]y' - ab\,y = 0$$

Now, comparing

$$\Rightarrow c = +1/2$$

$$a+b+1 = +5, \quad ab = 4$$

$$\Rightarrow a = 2, \quad b = 2$$

So, the solⁿ is given as :-

$$y = c_1 F(a,b,c,t) + c_2 t^{1-c} F(a-c+1, b-c+1, 2-c, t)$$

$$\Rightarrow y = c_1 F(2,2,1/2,t) + c_2 t^{1/2} F\left(\frac{5}{2}, \frac{5}{2}, \frac{3}{2}, t\right)$$

$$\Rightarrow y = c_1 F\left(2,2,\frac{1}{2},\frac{1+x}{2}\right) + c_2\left(\frac{1+x}{2}\right)^{1/2} F\left(\frac{5}{2},\frac{5}{2},\frac{3}{2},\frac{1+x}{2}\right)$$

Ans

Chapter -5

APPENDIX-D

(Pg-231)

Chebyshev's Polynomials

* CHEBYSHEV'S D.E

$$(1-x^2)y'' - xy' + n^2 y = 0 \longrightarrow (1)$$

$\hookrightarrow$ n : non-ve integer.

The sol^{ns} of (1) are called Chebyshev's Polynomials & they are divided by $T_n(x)$.

$\hookrightarrow T_{(n)}(x)$

$n = \sqrt{n^2}$ in the Chebyshev's D.E

Problem :- Prove that

$$T_n(x) = F\left(n, -n, \frac{1}{2}, \frac{1-x}{2}\right)$$

Proof :-

$$(1-x^2)y'' - xy' + n^2 y = 0 \longrightarrow (1)$$

$$\Rightarrow (x^2-1)y'' + xy' - n^2 y = 0$$

$$\Rightarrow (x+1)(x-1)y'' + xy' - n^2 y = 0.$$

Seeing coeff. of y'', we get singular pts. (Regular : to be proved)

So, singular pts are :- $x = 1, x = -1$.

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Now, finding sol^n for ① at $x = 1 = A$

$$t = \frac{x-A}{B-A} = \frac{x-1}{-1-1} = \frac{1-x}{2}.$$

Now, $y \xrightarrow{fn\ of} t \xrightarrow{fn\ of} x$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{2}\frac{dy}{dt} \ \& \ \frac{d^2y}{dx^2} = \frac{1}{4}\frac{d^2y}{dt^2}$$

So, eq^n ① becomes.

$$\Rightarrow (-2t)(2-2t)\left(\frac{1}{4}\frac{d^2y}{dt^2}\right) + (1-2t)\left(\frac{-1}{2}\right)\frac{dy}{dt} - n^2y = 0$$

$$\Rightarrow -t(1-t)y'' + \left(\frac{-1}{2}+t\right)y' - n^2y = 0$$

$$\Rightarrow t(1-t)y'' + \left(\frac{1}{2}-t\right)y' + n^2y = 0 \longrightarrow ②$$

$$x(1-x)y'' + (c-(a+b+1)x)y' - ab\,y = 0.$$

$$\Rightarrow c = \frac{1}{2},\ a+b+1 = 1,\ ab = -n^2$$
$$\Rightarrow a+b = 0 \qquad ab = -n^2$$
$$\Rightarrow a = n,\ b = -n \quad \text{or} \quad a = -n,\ b = n.$$

Now, sol^n of eq^n ②.

$$y = F\left(n, -n, \frac{1}{2}, t\right)$$

The sol^n of eq^n ① is:-

$$y = F\left(n, -n, \tfrac{1}{2}, \frac{1-x}{2}\right)$$ (≡ a polynomial: 'a' or 'b' is -ve.)

THIS SOL^n IS REFERRED AS :-

$$\boxed{T_n(x) = F\left(n, -n, \frac{1}{2}, \frac{1-x}{2}\right)}$$

Note :- $\because b = -n$ (-ve integer),

① : the above sol^n is a finite polynomial ~~refers~~ referred by $T_n(x)$.

Q TPT :- $\boxed{T_n(x) = \cos n\theta, \text{ where } x = \cos\theta}$

or $\theta = \cos^{-1}(x)$

→ another form of sol^n for Chebyshev's D.E

→ another way to represent Chebyshev's polynomial

So, solve :-

$$(1-x^2)y'' - xy' + n^2 y = 0$$

Let $y = \cos n\theta$ be a sol^n, $x = \cos\theta$

So, $y \xrightarrow{f^n \text{ of}} \theta \xrightarrow{f^n \text{ of}} x$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= (-n \sin n\theta)\left(\frac{1}{dx/d\theta}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin n\theta\,(n)}{(-\sin\theta)}$$

$$\& \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{n\sin n\theta}{(\sin\theta)}\right)$$

$$= \frac{d}{d\theta}\left(\frac{n\sin n\theta}{\sin\theta}\right) \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left[\frac{\sin\theta\left(-n^2\cos n\theta\right) - n\sin n\theta\,(\cos\theta)}{\sin^2\theta}\right]\left[\frac{1}{-\sin\theta}\right]$$

$$\Rightarrow \sin^2\theta\,\frac{d^2y}{dx^2} = \left[-n^2\underbrace{(\cos n\theta)}_{y} + \underbrace{\left(\frac{n\sin n\theta}{\sin\theta}\right)}_{\frac{dy}{dx}}\underbrace{\cos\theta}_{x}\right]$$

$$\Rightarrow \underbrace{(1-\cos^2\theta)}_{x^2}\frac{d^2y}{dx^2} = -n^2y + \frac{dy}{dx}x$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$$

$$\Downarrow$$

$y = \cos n\theta$ satisfies the Chebyshev's eqⁿ.

So, $T_n(x) = \cos n\theta$, $\theta = \cos^{-1}(x)$.

↳ an alternate form of representing.

RESULT:-

(1) $$T_n(x) = \frac{1}{2}\left\{\left[x+\sqrt{x^2-1}\right]^n + \left[x-\sqrt{x^2-1}\right]^n\right\}$$

↳ Proof in textbook (self)

(2) $T_n(x) = \cos n\theta$, $\theta = \cos^{-1}(x)$

$T_0(x) = \cos 0.x = \cos 0 = 1$.

$T_1(x) = \cos 1\,x = \cos x = x$

So, we can say, the D.E having x as result is $(1-x^2)y'' - xy' + 1^2.y = 0$.

Q Prove :- $T_m(T_n(x)) = T_{mn}(x)$

$\Rightarrow T_m(\cos n\theta)$

Prove :- $T_n(x) = \frac{1}{2}\left(\left\{x+\sqrt{x^2-1}\right\}^n + \left\{x-\sqrt{x^2-1}\right\}^n\right) \rightarrow (1)$

Sol^n :- $T_n(x) = \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right)$

Put $x = \cos\theta$, by defin^n (as assumed before)

$\Rightarrow T_n(\cos\theta) = \frac{1}{2}\left((\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\right)$

By de-Moivre's thm,

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

$\Rightarrow T_n(\cos\theta) = \frac{1}{2}\left[\cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}\right]$

$= \frac{1}{2}\left[2\cos n\theta\right]$.

$\Rightarrow T_n(x) = \cos n\theta$

which is the sol^n for Chebyshev's eq^n.

So, eq^n (1) is also a sol^n.

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Note:- $T_n(x) = F\left(n, -n, \frac{1}{2}, \frac{1-x}{2}\right) = \cos n\theta$

↳ Proved before

① → $T_0(x) = 1 \quad (\cos 0)$

$T_1(x) = x \quad (\cos\theta)$

② → $\cos n\theta = \cos[(n-1)\theta + \theta]$

$= \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta$

$\cos(n-2)\theta = \cos[(n-1)\theta - \theta]$

$= \cos(n-1)\theta\cos\theta + \sin(n-1)\theta\sin\theta$

Add

$\Rightarrow \cos n\theta + \cos(n-2)\theta = 2\cos(n-1)\theta\cos\theta$

$\Rightarrow T_n(x) + T_{n-2}(x) = 2T_{n-1}(x) \cdot x$

$\Rightarrow \boxed{T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)}$

↳ Recurrence relⁿ.

③ → for $n = 2$

$T_2(x) = 2x\,T_1(x) - T_0(x)$

$= 2x(x) - 1 = 2x^2 - 1$

for $n = 3$

$T_3(x) = 2x\,T_2(x) - T_1(x)$

$= 2x(2x^2 - 1) - x$

$= 4x^3 - 3x$

for $n = 4$

$T_4(x) = 2x\,T_3(x) - T_2(x)$

$= 2x(4x^3 - 3x) - (2x^2 - 1)$

$= 8x^4 - 8x^2 + 1$

⋮

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⟹ WAY OF ASKING QUESTIONS

Q. ✯ We know, Chebyshev's d.E is
$(1-x^2)y'' - xy' + n^2 y = 0$
& $T_n(x)$ is a solⁿ to it.

→ So, for $T_4(x)$ as solⁿ,
Chebyshev's eqⁿ is
$(1-x^2)y'' - xy' + 4^2 y = 0$

→ for $T_n(x) = \cos x$ as solⁿ,
Chebyshev's eqⁿ is:
$(1-x^2)y'' - xy' + 1^2 y = 0$.

✯ Express x^4 in terms of Chebyshev's polynomials.

We know,
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$\Rightarrow x^4 = \frac{1}{8}\left[T_4(x) + 8x^2 - 1\right]$$
$$= \frac{1}{8}\left(T_4(x) + 8\left(\frac{T_2(x)+1}{2}\right) - 1\right)$$
$$= \frac{1}{8}\left(T_4(x) + 4T_2(x) + 3\right)$$
$$= \frac{1}{8}\left(T_4(x) + 4T_2(x) + 3T_0(x)\right).$$

✯ express $6x^4 - 2x^2$ in terms of Chebyshev's polynomials

Q. State & prove the orthogonal property of Chebyshev's D.E

Orthogonal property

$$\int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0 & ; \quad m \neq n \\ \pi/2 & ; \quad m = n \neq 0 \\ \pi & ; \quad m = n = 0 \end{cases}$$

Solⁿ :- $T_m(x) = \cos m\theta$.

$T_n(x) = \cos n\theta$. , $x = \cos\theta$.

$$\Rightarrow I = \int_{-1}^{1} \frac{\cos m\theta \cdot \cos n\theta}{\sqrt{1-\cos^2\theta}}\, d(\cos\theta)$$

$\hookrightarrow$ when $x = -1$, $\theta = \pi$

$x = 1$, $\theta = 0$

$$\Rightarrow I = \int_{\pi}^{0} \frac{(\cos m\theta)(\cos n\theta)}{\sin\theta}(-\sin\theta\, d\theta)$$

$$= -\int_{\pi}^{0} \cos m\theta \cdot \cos n\theta\, d\theta$$

$$\Rightarrow I = \int_{0}^{\pi} \cos m\theta\, \cos n\theta\, d\theta \qquad \left[\because \int_a^b (\,)dx = -\int_b^a (\,)dx\right]$$

Now, $\boxed{2\cos A \cos B = \cos(A+B) + \cos(A-B)}$

$$\Rightarrow \cos m\theta \cos n\theta = \frac{1}{2}\cos(m+n)\theta + \cos(m-n)\theta$$

Case (i) : for $m \neq n$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi}\left[\cos(m+n)\theta + \cos(m-n)\theta\right] d\theta$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n}\right]_0^{\pi}$$

$$= \frac{1}{2}(0-0) = 0.$$

$\rightarrow$ for $m = n$, it becomes $0/0$ form.

Case(ii) $m = n \neq 0$

$$\Rightarrow I = \int_0^{\pi} \cos^2 n\theta \, d\theta$$

$$= \int_0^{\pi} \left(\frac{1 + \cos 2n\theta}{2} \right) d\theta$$

$$= \left[\frac{\theta}{2} + \frac{\sin 2n\theta}{4n} \right]_0^{\pi}$$

$$= \left(\frac{\pi}{2} + 0 \right) - (0 + 0)$$

$$\Rightarrow I = \frac{\pi}{2}$$

Case(iii) : $m = n = 0$.

$$I = \int_0^{\pi} \cos 0 \cdot \cos 0 \cdot d\theta$$

$$= \int_0^{\pi} d\theta = \pi - 0 = \pi$$

Hence, proved

Q. Show :- $\underset{①}{T_m(T_n(x))} = \underset{②}{T_n(T_m(x))} = \underset{③}{T_{mn}(x)}$.

Proving ① = ③.

IIly, ② = ③ (self)

we know, for Chebyshev's D.E. solⁿ.

$$T_n(x) = \cos n\theta, \quad \theta = \cos^{-1}(x)$$

$$\Rightarrow T_m(T_n(x)) = T_m(\cos n\theta)$$

$$= T_m(\underbrace{\cos(n\cos^{-1}(x))}_{x'''})$$

c) $T_m(T_n(x)) = \cos[m\theta]$

$= \cos[m\cos^{-1}(x)]$

$= \cos[m\cos^{-1}(\cos(n\cos^{-1}(x)))]$

$= \cos[m(n\cos^{-1}(x))]$

$(\because \cos^{-1}\cos(\alpha) = \alpha)$

$= \cos(mn\theta)$

$= \cos(mn)\theta$

$\Rightarrow T_m(T_n(x)) = T_{mn}(x)$

H.P

Q. Solve $\int_0^1 \cos(3\theta)\cos^3(3\theta)\,d\theta$

We know, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$

& $\cos^3(3\theta) = \frac{1}{4}[\cos 9\theta + 3\cos 3\theta]$

$= \int_0^1 (\cos 3\theta)\left[\frac{1}{4}(\cos 9\theta + 3\cos 3\theta)\right] d\theta$

$= \int_0^1 \left[\frac{1}{4}(\cos \underbrace{3\theta}_{m} \cos \underbrace{9\theta}_{n}) + \frac{3}{4}(\cos 3\theta)(\cos 3\theta)\right] d\theta$

$m \neq n$ $\quad m = n \neq 0$

$= \frac{1}{4}(0) + \frac{3}{4}\left(\frac{\pi}{2}\right)$ (By orthogonal property)

$= \frac{3\pi}{8}$ Ans

Section - 44

LEGENDRE'S POLYNOMIALS

* Legendre's D.E:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \longrightarrow (1)$$

$\hookrightarrow n$: non -ve integer.

$\hookrightarrow x = 1$: one regular singular pt.
$x = -1$: other regular singular pt.

Sol^n of (1) near $x = 1$

$\hookrightarrow$ replace $t = \frac{1-x}{2}$

$\hookrightarrow$ when $x = 1$, $t = 0$.

Substitute in eq^n (1)

$$\Rightarrow t(1-t)\frac{d^2y}{dt^2} + (1-2t)\frac{dy}{dt} + n(n+1)y = 0 \rightarrow (2)$$

(hypergeometric eq^n in t)

$$x(1-x)y'' + (c-(a+b+1)x)y' - ab\,y = 0$$

$$\Rightarrow a = -n,\ b = n,\ c = 1.$$

$\therefore$ sol^n of (2) at $t = 0$

$$y = F(a, b, c, t) = F(-n, n, 1, t) \rightarrow (3)$$

The solⁿ of ① is at $x = 1$

$$\Rightarrow \boxed{y = F\left(-n, n, 1, 1-\frac{x}{2}\right)}$$

(one of the 2 solⁿˢ)

here $a = -n$ (a -ve integer), this solⁿ is a polynomial called as LEGENDRE'S polynomial denoted by $P_n(x)$.

i.e, $$\boxed{P_n(x) = F\left(-n, n, 1, 1-\frac{x}{2}\right)}$$

where $F(a, b, c, x)$
$$= 1 + \frac{ab}{1 \cdot c} x + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} x^2 + \ldots$$

So, $$P_n(x) = 1 + \frac{n(n+1)}{(1!)^2 \cdot 2}(x-1) + \frac{n(n-1)(n+1)(n+2)}{(2!)^2 \cdot 2^2}(x-1)^2$$

$$+ \frac{(2n)!}{n!\,(2^n)}(x-1)^n.$$

alternative way to write solⁿ

M2 RODRIGUE'S FORMULA:-

$$\boxed{P_n(x) = \frac{1}{2^n \cdot n!}\left(\frac{d^n (x^2-1)^n}{dx^n}\right)}$$

Putting values $n = 0 \Rightarrow P_0(x) = 1$

$n = 1 \Rightarrow P_1(x) = x$

$n = 2 \Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)$

$n = 3 \Rightarrow P_3(x) = \frac{1}{2}(5x^3 - 3x)$

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☆ GENERATING f^n for LEGENDRE'S POLYNOMIALS:

$$\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots + P_n(x)t^n + \cdots$$

; if $t \leqslant 1$ or $|x| \leqslant 1$

or $(1-2xt+t^2)^{-1/2} = P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots$

or $[1-t(2x+t)]^{-1/2} = P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots$

$$\left(= \sum_{n=0}^{\infty} P_n(x)t^n \right)$$

eg (1) Show : $P_n(1) = 1$

& $P_n(-1) = (-1)^n$

(Use generating f^n)

We need $P_n(1)$ $(\equiv P_n(x))$

So, put $x=1$ in generating f^n

$\Rightarrow \dfrac{1}{\sqrt{1-2t+t^2}} = P_0(1) + P_1(1)t + \cdots$

$\Rightarrow (1-2t+t^2)^{-1/2} = \sum\limits_{n=0}^{\infty} P_n(1)\, t^n$

or

$$\sum_{n=0}^{\infty} P_n(1)t^n = (1-2t+t^2)^{-1/2}$$

$$= [(1-t)^{\cancel{2}}]^{-\cancel{1/2}}$$

$\Rightarrow \sum\limits_{n=0}^{\infty} P_n(1)t^n = (1-t)^{-1}$

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$\Rightarrow (1+t+t^2+t^3+\ldots) = P_0 + P_1(1)t + \ldots$

Comparing coeff. of t^n

$\Rightarrow 1 = P_n(1)$

H.P

Now,

$(1-2(-1)t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-1)t^n$

$\Rightarrow (1+2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-1)t^n$

$\Rightarrow (1+t)^{-1} = \sum_{n=0}^{\infty} P_n(-1)t^n$.

$\Rightarrow 1-t+t^2-t^3 \ldots + (-1)^n t^n + \ldots = \sum_{n=0}^{\infty} P_n(-1)t^n$

Comparing coeff. of t^n

$\Rightarrow (-1)^n = P_n(-1)$

H.P

Q Show : (1) $P_{2n+1}(0) = 0$

(2) $P_{2n}(0) = (-1)^n \dfrac{1.3.5.\ldots(2n-1)}{2^n\ n!}$

Put $n=0$ in Generating f^n.

$\Rightarrow (1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)t^n$.

We know

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r}$$

$$= 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \dots$$

$$+ \frac{n(n-1)\dots(n-(r-1))}{r!}x^r + \dots \infty$$

Now,

$$\sum_{n=0}^{\infty} P_n(0)t^n = (1+t^2)^{-1/2}.$$

$$= 1 + \frac{(-1/2)}{1!}(t^2) + \frac{(-1/2)(-1/2-1)}{2!}(t^2)^2$$

$$+ \dots + \frac{(-1/2)(-1/2-1)\dots(-1/2-r+1)}{r!}(t^2)^r + \dots$$

Comparing the coeff. of t^{2r}

So, $n = 2r$

$$\& \; P_{2r}(0) = \frac{(-1/2)(-1/2-1)\dots(-1/2-r+1)}{r!}$$

$$\Rightarrow P_{2r}(0) = \frac{(-1)^r}{2^r} \frac{1.3.5.\dots(2r-1)}{r!}$$

i.e $r \to n \Rightarrow P_{2n}(0) = \frac{(-1)^n}{2^n}\left[\frac{1.3.5\dots(2n-1)}{n!}\right]$

Now,

$P_{2r+1}(0)$ i.e coeff. of an odd power of t.

on RHS $\exists$ no such term. (odd power)

$\Rightarrow P_{2r+1}(0)$ or $P_{(2n+1)}(0) = 0$ H.P.

Q. Consider the generating f^n :-

$$(1-2x+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

This part might not be given in exams. Ques. can be asked directly from ↓

& prove the following:

(a) $(x-t)\sum_{n=0}^{\infty} P_n(x)\, t^n = (1-2xt+t^2)\sum_{n=1}^{\infty} n P_n(x)\, t^{n-1}$

(b) $(n+1)\, P_{n+1}(x) = (2n+1)\, x P_n(x) - n P_{n-1}(x)$

(c) Deduce $P_2(x)$, $P_3(x)$, $P_4(x)$ & $P_5(x)$

Solⁿ:- Legendre's D.E :-
We know, it is :

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

↳ solⁿ :- $y = F\left(-n, n+1, 1, \frac{1-x}{2}\right)$

$$= P_n(x)$$

Now, we have

$$(1-2x+t^2)^{-1/2} = \sum_{n=0} P_n(x)\, t^n$$

Differentiate w.r.t, both sides

$$\Rightarrow -\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=1}^{\infty} P_n(x)(n t^{n-1})$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=1} n P_n(x)\, t^{n-1}$$

× both sides by $(1-2xt+t^2)$

$$\Rightarrow (x-t)\underbrace{(1-2xt+t^2)^{-1/2}}_{\text{the generating } f^n \text{, as given}} = (1-2xt+t^2)\sum_{n=1}^{\infty} n P_n(x)\, t^{n-1}$$

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$$\Rightarrow (x-t)\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} \rightarrow (1)$$

$\hookrightarrow$ Proof of part (a)

Now, from (1), expanding,

$$\Rightarrow (x-t)\left[P_0(x)+P_1(x)t+P_2(x)t^2+\dots+P_{n-1}(x)t^{n-1} + P_n(x)t^n+\dots\right]$$

$$= (1-2xt+t^2)\left[P_1(x)+2P_2(x)t + \dots (n-1)P_{n-1}(x)t^{n-2} + nP_n(x)t^{n-1} + (n+1)P_{n+1}(x)t^n+\dots\right]$$

Now, seeing for the term of $\underline{t^n}$

$$\Rightarrow t^n\left[-P_{n-1}(x)+xP_n(x)\right] = \left[(n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)\right]t^n$$

$$\Rightarrow xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2nxP_n(x) + (n-1)P_{n-1}(x)$$

$$(n+1)P_{n+1}(x) = (x+2nx)P_n(x) - (1+n-1)P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \rightarrow (2)$$

a recurrence relation; $n=1,2,\dots$

$\hookrightarrow$ Proof of part (b)

By Rodrigues formula

$P_0(x) = 1 \; , \; P_1(x) = x$

Now from (2)

$\rightarrow n = 1$

$\Rightarrow 2P_2(x) = 3x P_1(x) - P_0(x)$

$\Rightarrow 2P_2(x) = 3x^2 - 1$

$\Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\rightarrow n = 2$

$\Rightarrow 3P_3(x) = 5x P_2(x) - 2P_1(x)$

$= 5x\left(\frac{1}{2}(3x^2 - 1)\right) - 2x$

$= \frac{1}{2}\left(15x^3 - 5x\right) - 2x$

$\Rightarrow P_3 = \frac{1}{2}(5x^3 - 3x)$

$\rightarrow n = 3$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$\rightarrow n = 4$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^2 - 15x)$

(8) ORTHOGONAL PROPERTY :-

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & ; m \neq n \\ \dfrac{2}{2n+1} & ; m = n \end{cases}$$

Proof :- ① $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \; ; \; m \neq n.$

Sol^n :- $P_m(x)$ & $P_n(x)$ are sol^{ns} of Legendre's eq^n

i.e $(1-x^2) P_m'' - 2x P_m' + m(m+1) P_m = 0 \rightarrow (i)$

|||ly, $(1-x^2) P_n'' - 2x P_n' + n(n+1) P_n = 0 \rightarrow (ii)$

Now, (i) $\times P_n$ & (ii) $\times P_m$ & subtract them

$$\Rightarrow (1-x^2) P_m'' P_n - 2x P_m' P_n + m(m+1) P_m P_n = 0$$

$$(1-x^2) P_n'' P_m - 2x P_n' P_m + n(n+1) P_m P_n = 0$$

$-$

$$(1-x^2)(P_m'' P_n - P_n'' P_m) - 2x (P_m' P_n - P_n' P_m)$$

$$+ P_m P_n (m(m+1) - n(n+1)) = 0$$

$$\Rightarrow \overset{u}{(1-x^2)} \overset{dv}{\frac{d}{dx}(P_m' P_n - P_n' P_m)} + \overset{du}{\frac{d}{dx}(1-x^2)} \overset{v}{(P_m' P_n - P_n' P_m)}$$

$$+ P_m P_n (m^2 + m - n^2 - n) = 0$$

$$\left(\because \frac{d}{dx}(P_m' P_n - P_n' P_m) = P_m'' P_n + \cancel{P_m' P_n'} - P_n'' P_m - \cancel{P_n' P_m'} \right.$$

$$\left. = P_m'' P_n - P_n'' P_m \right)$$

$d(uv) = u\,dv + v\,du$

$\Rightarrow \frac{d}{dx}\left((1-x^2)(P_m' P_n - P_n' P_m)\right)$

$= (n^2+n-m^2-m) P_m P_n$

Integrate both sides b/w -1 to 1 (w.r.t x)

$\Rightarrow \int_{-1}^{1} \frac{d}{dx}\left((1-x^2)(P_m' P_n - P_n' P_m)\right) dx$

$= \int_{-1}^{1} (n-m)(m+n+1) P_m P_n \, dx$

$\Rightarrow \left[(1-x^2)(P_m' P_n - P_n' P_m)\right]_{-1}^{1} = (n-m)(n+m+1)\int_{-1}^{1} P_m P_n \, dx$

$\Rightarrow 0 = (n-m)(n+m+1) \int_{-1}^{1} P_m(x) P_n(x) dx$

$\Rightarrow \int_{-1}^{1} P_m(x) P_n(x) dx = 0$

$\rightarrow$ valid only if $n \neq m$

$\because$ If $n = m$, we have $0 = 0$ or '0' error.

(2) $\int_{-1}^{1} P_m(P_n(x))\, dx = \frac{2}{2n+1}$; $m = n$

The generating f^n:- $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n \rightarrow$ (a)

Also, $(1-2xt+t^2)^{-1/2} = \sum_{m=0}^{\infty} P_m(x) t^m \rightarrow$ (b)

Multiplying (a) & (b)

$\Rightarrow (1-2xt+t^2)^{-1} = \left[\sum_{m=0}^{\infty} P_m(x) t^m\right]\left[\sum_{n=0}^{\infty} P_n(x) t^n\right]$

Now, integrate both sides b/w limit (-1, 1) w.r.t x.

Now, $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} P_m(x) P_n(x) t^{m+n}\, dx$

$$= \int_{-1}^{1} \frac{1}{1-2xt+t^2}\, dx$$

from previous result,

$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$, $m \neq n$

So, for $m=n$

$\Rightarrow \sum_{n=0}^{\infty}\int_{-1}^{1} P_n^2(x)\, t^{2n}\, dx = \int_{-1}^{1} \frac{1}{1-2xt+t^2}\, dx$

$$= \left[\frac{\log(1+t^2-2xt)}{-2t}\right]_{-1}^{1}$$

(Chain rule)

$$\Rightarrow \sum_{n=0}^{\infty} \int_{-1}^{1} P_n^2(x)\, t^{2n}\, dx = \frac{\log(1-t)^2}{-2t} - \frac{\log(1+t)^2}{-2t}$$

$$\Rightarrow \sum_{n=0}^{\infty} \int_{-1}^{1} P_n^2(x)\, t^{2n}\, dx = \frac{1}{2t}\left[\log\left(\frac{1+t}{1-t}\right)^2\right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \int_{-1}^{1} P_n^2(x)\, t^{2n}\, dx = \frac{1}{t}\left(\log(1+t) - \log(1-t)\right)$$

$$= \frac{1}{t}\left[\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots\right) + \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \dots\right)\right]$$

$$\left(\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \& \quad -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots\right)$$

$$= \frac{2}{t}\left[t + \frac{t^3}{3} + \frac{t^5}{5} + \dots\right]$$

$$= \frac{2}{t}\left(\sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1}\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} \int_{-1}^{1} P_n^2(x)\, t^{2n}\, dx = \sum_{n=0}^{\infty} \left(\frac{2}{2n+1}\right) t^{2n}$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\int_{-1}^{1} P_n^2(x)\, dx\right) t^{2n} = \sum_{n=0}^{\infty} \left(\frac{2}{2n+1}\right) t^{2n}$$

Comparing coeff of t^{2n}, both sides

$$\Rightarrow \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1} \qquad \text{H.P}$$

Combining results from parts (1) & (2),
we prove orthogonal property :-

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & ; m \neq n \\ \frac{2}{2n+1} & ; m = n. \end{cases}$$

Note :-
Any polynomial of degree k can be expressed as a linear combin^n of Legendre polynomials of degree 0 to k.

i.e., $$P(x) = \sum_{n=0}^{k} a_n P_n(x)$$

a polynomial of degree 'k'

a constt

Legendre's polynomials.

∴ Generalising the above note for any arbitrary f^n, $f(x)$:-
We get an infinite series called Legendre's series.

So, we have
$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad : \text{Legendre's series}.$$

Here, to get the constants values $-(a_0, a_1, a_2 \ldots)$
Integrate both sides & $\times$ by $P_K(x)$

$$\Rightarrow \int_{-1}^{1} f(x) P_K(x)\, dx = \sum_{n=0}^{\infty} \int_{-1}^{1} a_n P_n(x) P_K(x)\, dx$$

$$\Rightarrow \int_{-1}^{1} f(x) P_k(x)\,dx = \sum_{n=0}^{\infty} a_n \underbrace{\int_{-1}^{1} P_n(x) P_k(x)\,dx}$$

→ by orthogonal property

$$\begin{cases} = 0 & n \neq k \\ = \dfrac{2}{2k+1} & n = k \end{cases}$$

So, value of Σ exists only when $n = k$

$$\Rightarrow \int_{-1}^{1} f(x) P_k(x)\,dx = a_k \int_{-1}^{1} P_k^2(x)\,dx$$

$$= a_k \left(\frac{2}{2k+1}\right) \quad \because k = n$$

$\Rightarrow$

$$a_k = \left(\frac{2k+1}{2}\right) \int_{-1}^{1} f(x) P_k(x)\,dx$$

→ $k = 0, 1, 2, \ldots$

Changing $k \to n$.

$$\Rightarrow \boxed{a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\,dx}$$

→ $n = 0, 1, 2, 3, \ldots$

Q.1) Given $f(x) = \begin{cases} 0 & ; \ -1 \leq x \leq 0 \\ x & ; \ 0 \leq x \leq 1 \end{cases}$

Find Legendre's series

HW

Q.2) $f(x) = e^x ; \; -1 \leq x \leq 1$

↳ Find Legendre's series.

Q.1) We know

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

where

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)\,P_n(x)\,dx$$

for $n=0$.

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,P_0(x)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} f(x)\cdot 1\,dx$$

$$= \frac{1}{2}\left[\int_{-1}^{0} 0\cdot dx + \int_{0}^{1} x\,dx\right].$$

$$= \frac{1}{4}.$$

for $n=1$

$$\Rightarrow a_1 = \frac{3}{2}\int_{-1}^{1} f(x)\,P_1(x)\,dx$$

$$= \frac{3}{2}\left[\int_{-1}^{0} 0\cdot P_1(x)\,dx + \int_{0}^{1} x P_1(x)\,dx\right]$$

$$= \frac{3}{2}\left[\int_{0}^{1} x\cdot x\,dx\right]$$

$$= \frac{3}{2}\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2}$$

for $n=2$

$$\Rightarrow a_2 = \frac{5}{2}\int_{-1}^{1} f(x) P_2(x)dx$$

$$= \frac{5}{2}\left[\int_{-1}^{0} 0 . P_2(x) + \int_0^1 x P_2(x) dx\right]$$

($P_n(x)$ is found using Rodrigue's formula as

$$P_n(x) = \frac{1}{2^n . n!}\left(\frac{d^n}{dx^n}(x^2-1)^n\right)$$)

$$= \frac{5}{2}\left[\int_0^1 x \frac{(3x^2-1)}{2} dx\right]$$

$$= \frac{5}{4}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_0^1$$

$$= \frac{5}{4}\left[\frac{3}{4} - \frac{1}{2}\right]$$

$$\Rightarrow a_2 = \frac{5}{16}.$$

$\therefore$ Req^d Legendre's series for $f(x)$

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \dots$$

$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) + \dots$$

$\to$ (not necessary to put values of $P_0, P_1, P_2, \dots$)

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Q.2) We know

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\, dx$$

for $n = 0$

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(x) P_0(x)\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^x (1)\, dx$$

$$a_0 = \frac{1}{2}(e - e^{-1})$$

for $n = 1$

$$a_1 = \frac{3}{e}$$

for $n = 2$

$$a_2 = \frac{5}{2}(e - 7e^{-1})$$

$\vdots$

So, Legendre's D.E is

$$f(x) = \frac{1}{2}(e - e^{-1}) P_0(x) + \frac{3}{e} P_1(x) + \frac{5}{2}(e - 7e^{-1}) P_2(x) + \cdots$$

BESSEL'S FUNCTION

$$x^2y'' + xy' + (x^2-p^2)y = 0 \longrightarrow (1)$$

→ Bessel's D.E

→ p: any non -ve constt.

→ for $p = 1/2$,

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

(Solved before).

Solving (1) at regular singular pt $x=0$

We get the solⁿ :- (by applying Frobenius series method at indicial root $m=p$.)

$$y = a_0 x^p \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{[n!\, 2^{2n}(p+1)(p+2)\dots(p+n)]}$$

→ other solⁿ for Frobenius series is possible with indicial root $m=-p$.

→ Taking $a_0 = \frac{1}{2^p \cdot p!}$ & calling solⁿ as Bessel's fⁿ, $J_p(x)$.

$$y = \frac{x^p}{2^p \cdot p!} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}\, n!\,(p+1)(p+2)\dots(p+n)}$$

$$\Rightarrow y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n+p}\, n!\,(p+n)!}$$

$$\Rightarrow y = \boxed{J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,(p+n)!}}$$

Imp

↳ valid ~~only if p~~ $\forall p$

for $m = -p$ → another indicial root

Illy, we get, $$\boxed{J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-p}}{n!\,(-p+n)!}}$$

↳ valid only if p is not Integer

$\therefore$ The general sol^n for Bessel's eq^n

$$y = \underbrace{C_1 J_p(x) + C_2 J_{-p}(x)}$$

↳ valid only if p is not Integer

($\because$ $J_{-p}(x)$ is valid only at that pt., as $(-p+n)!$ gives a $-$ve factorial if $p > n$)

s.t $J_p(x)$ & $J_{-p}(x)$ are LI

Note:- Prove:- $J_{-m}(x) = (-1)^m J_m(x)$

↳ $\boxed{m \in \text{integer}}$

↳ i.e, $J_{-m}(x)$ is dependent on $J_m(x)$.

So they are not LI.

Basically, prove that above sol^n can be got only if $p \notin$ integer.

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Proof :- $$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-m}}{n!\,(-m+n)!}$$

We have taken m as integer.

So, $$\left(\frac{1}{(-m+n)!}\right) = \text{zero}$$

↳ for $n = 0, 1, 2, \ldots (m-1)$

So, $$J_{-m}(x) = \sum_{n=m}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n-m}}{n!\,(-m+n)!}$$

Make starting term of $\sum$ as $n=0$.

So, $n \to n+m$

$$\Rightarrow J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+m} \left(\frac{x}{2}\right)^{n+m}}{(n+m)!\,\underbrace{0!}_{\to 1}}$$

$$= (-1)^m \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{n+m}}{(n+m)!}$$

$$\Rightarrow J_{-m}(x) = (-1)^m J_m(x)$$

↳ Hence, Proved

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Q. Find $J_0(x)$ & $J_1(x)$

// Put $p=0$ in $J_p(x)$

$$\boxed{J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{(n!)(n!)}}$$

$$= 1 - \frac{\left(\frac{x}{2}\right)^2}{(1!)^2} + \frac{\left(\frac{x}{2}\right)^4}{(2!)^2} - $$

$$\Rightarrow J_0(x) = 1 - \frac{x^2}{1^2 \cdot 2^2} + \frac{x^4}{1^2 \cdot 2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

// Put $p=1$ in $J_p(x)$, we get

$$\Rightarrow \boxed{J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{(n!)(n+1)!}}$$

$$\Rightarrow J_1(x) = 1 - \frac{\left(\frac{x}{2}\right)^3}{1!\,2!} + \frac{\left(\frac{x}{2}\right)^5}{2!\,3!} - - - -$$

Q. Prove that

(a) $\frac{d}{dx}[J_0(x)] = -J_1(x)$

(b) $\frac{d}{dx}[x J_1(x)] = x J_0(x)$

where $J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,(n+p)!}$

(a) $\frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{(n!)^2}\right]$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \times \cancel{2}n\left(\frac{x}{2}\right)^{2n-1}\left(\frac{1}{\cancel{2}}\right)$$

Changed from $n=0$ to 1 to avoid −ve power of x.

Note:- $(n+1)! = (n+1) \times n!$
$= (n+1)(n)(n-1)!$
$= (n+1)(n)(n-1)(n-2)!$
- - - - - - - -

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)(\cancel{n})(n-1)!} \times \cancel{n} \left(\frac{x}{2}\right)^{2n-1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(n-1)!} \left(\frac{x}{2}\right)^{2n-1}$$

Put $n \to n+1$

$$\Rightarrow LHS = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!\,(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$

$$= (-1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$

$$= (-1)\ J_1(x)$$
$$= -J_1(x)$$
$$= RHS$$

H.P

(b) $\frac{d}{dx}[x J_1(x)]$

$$= \frac{d}{dx}\left[x \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{(n!)(n+1)!}\right]$$

$$= \frac{d}{dx}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+2}}{(n!)(n+1)! \times (2)^{2n+1}}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2)(x)^{2n+1}}{n!\,(n+1)!\,(2^{2n+1})}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{2}\cancel{(n+1)}\, x^{2n+1}}{n!\,\cancel{(n+1)}(n)!\, 2^{2n} \cdot \cancel{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 \cdot 2^{2n}} \cdot x$$

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$$= x \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{(n!)^2}$$

$$= x J_0(x)$$

$$= RHS$$

H.P

Theory : GAMMA FUNCTIONS.

→ Defin^n :-

$$\boxed{\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} \, dt}$$

→ only when $p > 0$

→ Properties

→ Recurrence rel^n :- $\Gamma(p+1) = p\,\Gamma(p)$

- $\Gamma(1) = 1$
- $\Gamma(p+1) = p!$ (If p is a +ve integer)
- $\Gamma(1/2) = \sqrt{\pi}$

→ $\Gamma(p) = \frac{\Gamma(p+1)}{p}$

→ can be used to compute Gamma f^n values for -ve value of p. (without using the defin^n)

Q TPT :- $\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma(1/2) = \int_0^\infty e^{1/2-1} e^{-t}\, dt$$

Let $t = s^2$

$\Rightarrow dt = 2s\,ds$

$\Rightarrow \dfrac{dt}{s} = 2ds$

$$= \int_0^\infty e^{-1/2} e^{-t}\, dt$$

$$= \int_0^\infty (s^{-1}) \cdot e^{-s^2}\, dt$$

$$= \int_0^\infty e^{-s^2} \left(\frac{dt}{s}\right)$$

$$= \int_0^\infty e^{-s^2} (2ds)$$

$$= 2\int_0^\infty e^{-s^2} ds$$

; s : a dummy var.
The value of this integral by using a different var.

Now,

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx \quad (1) \qquad \& \qquad \Gamma(1/2) = 2e^{-y^2} dy \quad (2)$$

So, $$[\Gamma(1/2)]^2 = 2\left(\int_0^\infty e^{-x^2} dx\right)\left(2\int_0^\infty e^{-y^2} dy\right)$$

$$= 4\int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dx\, dy$$

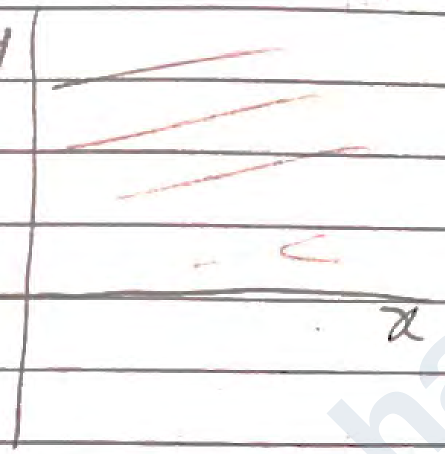
$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

Now, idea :-

$x \to 0$ to ∞

&

$y \to 0$ to ∞



So, in total, the shaded region is the region of integration

Now, changing it to polar coordinates

$$x = r\cos\theta,\quad y = r\sin\theta$$

$$\text{So, } \Gamma(1/2)^2 = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\underbrace{(r\,dr\,d\theta)}_{dx\,dy}$$

Put $t = r^2$

$\Rightarrow dt = 2r\,dr$

$\Rightarrow \frac{dt}{2} = r\,dr.$

$$= 4\int_0^{\pi/2}\int_{t=0}^{\infty}\left(e^{-t}\frac{dt}{2}\right)d\theta \quad (\text{into } d\theta)$$

$$= 2\int_{\theta=0}^{\pi/2}\left(\frac{e^{-t}}{-1}\right)_0^\infty d\theta$$

$$= 2\int_0^{\pi/2}(0-(-1))\,d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = \pi$$

$\Rightarrow \left[\Gamma(1/2)\right]^2 = \pi$

$\Rightarrow \Gamma(1/2) = \sqrt{\pi}$

Ans

☆ NOTE :- In the previously written Bessel's f^n, new denotions:-

$$n! = \Gamma(1/2)$$

↳ when n is a fraction.

eg: $(1/2)! = \Gamma(1/2+1)$

Q Find (a) $(-1/2)!$

$$(-1/2)! = \Gamma(-1/2+1) = \Gamma(1/2) = \sqrt{\pi}$$

(b) $(1/2)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

(c) $\left(\frac{3}{2}\right)!$

$$\left(\frac{3}{2}\right)! = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\left(\frac{1}{2}\sqrt{\pi}\right)$$

$$\Rightarrow \left(\frac{3}{2}\right)! = \frac{3}{4}\sqrt{\pi}$$

(d) $(n+1/2)! = \Gamma\left(n+\frac{3}{2}\right) = \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)$

$$= \frac{2n+1}{2}\Gamma\left(\left(n-\frac{1}{2}\right)+1\right)$$

$$= \left(\frac{2n+1}{2}\right)\left(n-\frac{1}{2}\right)\left[\Gamma\left(n-\frac{1}{2}\right)\right]$$

$$= \left(\frac{2n+1}{2}\right)\left(\frac{2n-1}{2}\right)\Gamma\left(\left(n-\tfrac{3}{2}\right)+1\right)$$

$$= \frac{(2n+1)(2n-1)}{4}\left[\left(\frac{2n-3}{2}\right)\Gamma\left(n-\tfrac{3}{2}\right)\right]$$

$$= \left(\frac{2n+1}{2}\right)\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\tfrac{1}{2}$$

× numerator & den. by even terms of n

$$= \frac{(2n+1)!}{2^{n+1}(2\cdot 4\cdot 6\cdots 2n)} \times \sqrt{\pi}$$

$$= \frac{(2n+1)!\,\sqrt{\pi}}{(2^{n+1})(2^{n})(1\cdot 2\cdot 3\cdots n)}$$

$$= \frac{(2n+1)!\,\sqrt{\pi}}{2^{2n+1}\cdot n!}$$

$$\Rightarrow \left(n+\frac{1}{2}\right)! = \frac{(2n+1)!\,\sqrt{\pi}}{n! \times 2^{2n+1}}$$

Q Find: $\left(n-\frac{1}{2}\right)! = \Gamma\left(n-\tfrac{1}{2}\right)+1 \quad \left(\because p! = \Gamma(p+1)\right)$

$$= \Gamma\left(n+\tfrac{1}{2}\right)$$

Also, $\Gamma\left(n+\frac{3}{2}\right) = \left(n+\frac{1}{2}\right)\Gamma\left(n+\tfrac{1}{2}\right). \quad \left(\because \Gamma(p+1) = p\,\Gamma(p)\right)$

$$\Rightarrow \Gamma\left(n+\tfrac{1}{2}\right) = \frac{\Gamma\left(n+3/2\right)}{\left(n+1/2\right)} = \frac{(2n+1)!\,\sqrt{\pi}}{2^{2n+1}\times n!} \times \left(\frac{1}{n+1/2}\right)$$

$$= \frac{\cancel{(2n+1)}(2n!)\,\sqrt{\pi} \times \cancel{2}}{(2^{2n})(\cancel{2})(n!)\,\cancel{(2n+1)}}$$

$$\Rightarrow \Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n)!\,\sqrt{\pi}}{2^{2n}\cdot n!}$$

$$= \left(n-\tfrac{1}{2}\right)!$$

☆ PROPERTIES of BESSEL's f^n :-

Results :-

$$\boxed{\begin{aligned} &1(a)\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x \\ &1(b)\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\cos x \end{aligned}}$$

$\rightarrow 1(a)$

$\rightarrow 1(b)$

Imp ☆

↳ Proof :-

(a) $$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,(p+n)!}$$

Put $p = 1/2$

$$\Rightarrow J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1/2}}{(n!)\,(n+1/2)!}$$

(M1) $$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1/2}}{(n!)\left[\frac{(2n+1)!\,\sqrt{\pi}}{n!\times 2^{2n+1}}\right]}$$

done before

(M2) Or $$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1/2}}{n!\,\Gamma\left(n+\frac{3}{2}\right)}$$

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M.2 $= \frac{(x/2)^{1/2}}{\Gamma 3/2} - \frac{(x/2)^{5/2}}{1! \times \Gamma 5/2} + \frac{(x/2)^{9/2}}{2! \, \Gamma 7/2} - \dots$

$\times$ & $\div$ $\left(\frac{x}{2}\right)^{1/2}$ by entire series

$$= \frac{(x/2)^{1/2}}{(x/2)^{1/2}} \left[\frac{(x/2)^{1/2}}{\Gamma 3/2} - \frac{(x/2)^{5/2}}{\Gamma 5/2} + \dots \right]$$

$$= \frac{1}{(x/2)^{1/2}} \left[\frac{x/2}{1/2 \, \Gamma 1/2} - \frac{(x/2)^3}{(1!)\, 3/2 \left(\frac{1}{2} \Gamma 1/2\right)} + \frac{(x/2)^5}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma 1/2} - \dots \right]$$

$$= \left(\frac{2}{x}\right)^{1/2} \left[\frac{x}{\sqrt{\pi}} - \frac{x^3}{6\sqrt{\pi}} + \frac{x^5}{60\sqrt{\pi}} - \dots \right]$$

$$= \left(\frac{2}{x}\right)^{1/2} \times \left(\frac{1}{\sqrt{\pi}}\right) \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

H.P

Il;ly,
do for $J_{-1/2}(x)$.

(2) (a) $\boxed{\dfrac{d}{dx}\left[x^{p} J_p(x)\right] = x^{p} J_{p-1}(x)} \longrightarrow$ (a)

(2) (b) $\boxed{\dfrac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)} \longrightarrow$ (b)

Proof :- (a) $J_p(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(p+n)!}$

So, $x^{p} J_p(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+2p}}{n!\,(p+n)!\cdot 2^{2n+p}}$

Now $\dfrac{d}{dx}(x^{p} J_p(x)) = \dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p}}{2^{2n+p}(n!)(p+n)!}\right]$

$= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p)\, x^{2n+2p-1}}{2^{2n+p}(n!)(p+n)!}$

$= x^{p} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p)\, x^{2n+p-1}}{2^{2n+p}(n!)(p+n)!}$

$= x^{p} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (p+n)\, x^{2n+p-1}}{2^{2n+p-1}(n!)(p+n)!}$

$= x^{p} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+(p-1)}}{2^{2n+(p-1)}(n!)(p+n-1)!}$

$= x^{p} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+(p-1)}}{2^{2n+(p-1)}(n!)(n+(p-1))!}$

$\Rightarrow \dfrac{d}{dx}\left[x^{p} J_p(x)\right] = x^{p} J_{p-1}(x)$ H.P

(b) $$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(p+n)!}$$

$$\times\, (x^{-p})$$

$$\Rightarrow x^{-p} J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(n!)\,(p+n)! \times 2^{2n+p}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{2^{2n+p} (n!)(p+n!)}$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1} (2n)}{2^{2n+p} (n!)(p+n)!}$$

To convert a summation into a regular summation $\rightarrow n \rightarrow n+1$

$$\Rightarrow \frac{d}{dx}\left[x^{-p} J_p(x)\right] = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1} (2(n+1))}{2^{2n+p+2} (n+1)!\,(p+n+1)!}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \cancel{(2)} \cancel{(n+1)}}{(2^{2n+p+1}) \cancel{(2)} \cancel{(n+1)} (n!)(p+p+1)}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+p+1} (n!)\,(n+(p+1))!}$$

$$= -x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+(p+1)}}{2^{2n+(p+1)} (n!)\,(n+p+1)!}$$

$$= -x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+(p+1)}}{n!\,(n+(p+1))!}$$

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$$\Rightarrow \frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

H.P

(2) (c) $\boxed{J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)} \rightarrow$ (c)

(2) (d) $\boxed{J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)} \rightarrow$ (d)

Proof of 2(c) using 2(a)

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \quad [\text{from (a)}]$$

$$\Rightarrow x^p J_p'(x) + J_p(x)(p x^{p-1}) = x^p J_{p-1}(x)$$

$\div$ by x^p

$$\Rightarrow J_p'(x) + J_p(x) \frac{p}{x} = J_{p-1}(x)$$

Proof of 2(d) using 2(b)

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

$$\Rightarrow x^{-p} J_p'(x) - p x^{-p-1} J_p(x) = -x^{-p} J_{p+1}(x)$$

$\div \; x^{-p}$

$$\Rightarrow J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$$

$2(c) + 2(d)$

$\Rightarrow$ 2(e) $\boxed{J_p'(x) = \dfrac{J_{p+1}(x) - J_{p-1}(x)}{2}} \rightarrow$ (e)

$2(c) - 2(d)$

2(f) $\boxed{\dfrac{2p}{x} J_p(x) = J_{p+1}(x) + J_{p-1}(x)} \rightarrow$ (f)

Q. To find :- $J_{3/2}(x)$, $J_{5/2}(x)$, - - -

Solⁿ :- Using (2)(f)

Put $p = 1/2$

$$\Rightarrow \frac{1}{x} J_{1/2}(x) = J_{3/2}(x) + J_{-1/2}(x)$$

$$\Rightarrow J_{3/2}(x) = -J_{-1/2}(x) + \frac{1}{x} J_{1/2}(x)$$

$$= -\sqrt{\frac{2}{\pi x}} \cos x + \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x$$

(from results 1(a) & 1(b))

$$\Rightarrow J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\cos x + \frac{\sin x}{x}\right]$$

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Now, put $p = 3/2$

$$\Rightarrow \frac{3}{x} J_{3/2}(x) = J_{5/2}(x) - J_{1/2}(x)$$

$$\Rightarrow J_{5/2}(x) = J_{1/2}(x) + \frac{3}{x} J_{3/2}\, x$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x \right]$$

Put $p = -1/2$ in (1)

$$\Rightarrow J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left[-\frac{\cos x}{x} - \sin x \right]$$

Put $p = -3/2$

$$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x \right]$$

Q Express $J_2(x)$, $J_3(x)$ & $J_4(x)$ in terms of $J_0(x)$ & $J_1(x)$

Ans: Using $\frac{2P}{x} J_p(x) = J_{p+1}(x) + J_{p-1}(x) \longrightarrow 2(f)$

Put $P = 1$

$$\Rightarrow \frac{2}{x} J_1(x) = J_2(x) + J_0(x)$$

$$\Rightarrow J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

Put $p = 2$

$$\Rightarrow \frac{4}{x} J_2(x) = J_1(x) + J_3(x)$$

$$\Rightarrow J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$= \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

$$\Rightarrow J_3(x) = J_1(x)\left[\frac{8}{x^2} - 1\right] - \frac{4}{x} J_0(x)$$

Put $p = 3$

$$\Rightarrow \frac{6}{x} J_3(x) = J_2(x) + J_4(x)$$

$$\Rightarrow J_4(x) = \frac{6}{x}\left[J_1(x)\left(\frac{8}{x^2} - 1\right) - \frac{4}{x} J_0(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$\Rightarrow J_4(x) = J_1(x)\left[\frac{48}{x^3} - \frac{8}{x}\right] + J_0(x)\left[1 - \frac{24}{x^2}\right]$$

Ans

Note :- For this topic, whatever type of ques. have been done, only those type of ques. will come.

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EIGEN Values of EIGEN f^{ns}

(Differential Equations)
↳ 2nd order

Q. Find the Eigen values (λ_n) & corresponding Eigen f^{ns} ($y_n(x)$) for the D.E :-

(1) ← $y'' + \lambda y = 0$; $y(0) = 0$ & $y(\frac{\pi}{2}) = 0$

→ λ : Eigen value *
↳ a real qty for which the eq^n is solvable (with given initial cond^ns)
↳ Corresponding sol^n on putting the eigen value is called Eigen function.

The auxiliary eq^n for (1) :-

$$m^2 + \lambda = 0 \qquad (y'' = m^2, y' = m, y = 1)$$

$$\Rightarrow m = \underbrace{\pm j\sqrt{\lambda}}_{\text{Imaginary pair of roots}}$$

(λ need to be +ve always ∵ when $\lambda = 0$ or $\lambda < 0$, we get only trivial sol^ns)
↳ ASSUMPTION ∀ Ques.

So, sol^n of (1) :-

$$y(x) = e^{0x}\left[c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x\right] = c_1\cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$
↳ (2)

Now, given :- $y(0) = 0$

$\Rightarrow y(0) = c_1 \cos 0 + c_2 \sin 0$

$\Rightarrow 0 = c_1 + 0$

$\Rightarrow c_1 = 0. \quad \Rightarrow \quad y(x) = c_2 \sin\sqrt{\lambda}\, x \rightarrow (3)$

& $y\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow y\left(\frac{\pi}{2}\right) = c_1 \cos\left[(\sqrt{\lambda})\frac{\pi}{2}\right] + c_2 \sin\left[(\sqrt{\lambda})\frac{\pi}{2}\right]$

$\Rightarrow 0 = 0 + c_2 \sin(\sqrt{\lambda})\frac{\pi}{2}$

↳ If $c_2 = 0$, so no sol^n will be there.

hence, $\sin(\sqrt{\lambda})\frac{\pi}{2} = 0$

$\Rightarrow \sqrt{\lambda}\frac{\pi}{2} = n\pi \; ; n \in \mathbb{Z}$

$\Rightarrow \sqrt{\lambda} = 2n \; ; n \in \mathbb{Z}$

$\Rightarrow \lambda = 4n^2 \; ; n \in \mathbb{Z}$

By assumption, λ cannot be 0 or $-$ve.

So, $n = 0, -1, -2, -3, \ldots$ are neglected

Hence,

$\lambda = 4n^2 \; ; n = 1, 2, 3, 4, \ldots$

So, λ_n: 4, 16, 36, 64, ... : Eigenvalues

The corresponding sol^n of eq^n :-

↳ for $\lambda = 4, 4, 36, 64, \ldots$

$y_n(x) = \sin 2x, \sin 4x, \sin 6x, \ldots$ (from (3))

(eigen functions)

* The linear combinⁿ of Eigen fⁿ is also an Eigen fⁿ.

Q. Find Eigenvalues & Eigenfⁿs for D.E given as:—

$$y'' + \lambda y = 0 \;\to (1), \quad \text{s.t. } y(0) = 0 \;\&\; y(L) = 0 \;\to L > 0$$

Auxiliary eqⁿ

$\Rightarrow m^2 + \lambda = 0$

$\Rightarrow m = \pm j\sqrt{\lambda}$.

Solⁿ:—

$$y(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x \longrightarrow (2)$$

Given $y(0) = 0$

$\Rightarrow 0 = c_1 \cos 0 + 0$.

$\Rightarrow c_1 = 0$.

$$\Rightarrow y(x) = c_2 \sin(\sqrt{\lambda}\,x) \longrightarrow (3)$$

Now, $y(L) = 0$

$\Rightarrow 0 = c_2 \sin(\sqrt{\lambda}\,L)$

for solⁿ to exist, $c_2 \neq 0$.

(dropping $0, Z^{-ve}$)

$\Rightarrow \sin(\sqrt{\lambda}\,L) = 0$

$= \sin n\pi \;;\; n = 1, 2, 3, 4, \ldots$

$\Rightarrow \sqrt{\lambda}\,L = n\pi$

$\Rightarrow \sqrt{\lambda} = \dfrac{n\pi}{L}$

$\Rightarrow \lambda = \left(\dfrac{n\pi}{L}\right)^2 \;;\; n = 1, 2, 3, \ldots \;\; L > 0$

$\Rightarrow \lambda_n = (\pi/L)^2, (2\pi/L)^2, (3\pi/L)^2, \ldots$: Eigenvalues

Corresponding eigenfunctions:—

$y_n(x) = \sin\dfrac{\pi x}{L}, \sin\dfrac{2\pi x}{L}, \sin\dfrac{3\pi x}{L}, \ldots$ (from (3))

$\to L > 0$

§ LINEAR SYSTEM.

↳ (only homogeneous sys)

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y + f_1(t) \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y + f_2(t) \end{aligned}\right\} \xrightarrow{\text{System}} (1)$$

here, x, y : dependent variable (depend on value of t)

t : independent variable.

In (1), if $f_1(t)$ & $f_2(t)$ are zero, its called HOMOGENEOUS SYSTEM.

So,
$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right\} \xrightarrow{\text{System}} (2)$$

Note :-

(1) If System (2) has 2 sets of sol^n :-

$$\left\{\begin{aligned} x &= x_1(t) \\ y &= y_1(t) \end{aligned}\right\} \ \& \ \left\{\begin{aligned} x &= x_2(t) \\ y &= y_2(t) \end{aligned}\right\} \rightarrow (3)$$

Then, the linear combin^n :

$$\left\{\begin{aligned} x &= c_1 x_1(t) + c_2 x_2(t) \\ y &= c_1 y_1(t) + c_2 y_2(t) \end{aligned}\right\} \rightarrow (4)$$

is also a ~~lo~~ sol^n of (2).

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Note :-

② If the Wronskian of ③ :

$$\begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}$$ is not zero, then, the 2 sol^ns given by ③ are linearly independent & ④ is called General Solⁿ of ① under this case.

Q. Solve :-

$$\left.\begin{aligned} \frac{dx}{dt} &= \underbrace{-3x+4y}_{(i)} \\ \frac{dy}{dt} &= \underbrace{-2x+3y}_{(ii)} \end{aligned}\right\} \text{Sys ①}$$

Solⁿ Now,

$$\frac{dx}{dt} = -3x+4y$$

$$\Rightarrow \frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 4\frac{dy}{dt} \quad \left(\text{differentiating w.r.t } t\right)$$

$$\Rightarrow \frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 4(-2x+3y) \quad \left[\text{from (ii)}\right]$$

$$\Rightarrow \frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 4\left[-2x+3\left[\frac{\frac{dx}{dt}+3x}{4}\right]\right]$$

$$= -3\frac{dx}{dt} - 8x + 3\left(\frac{dx}{dt}+3x\right)$$

$$\Rightarrow \frac{d^2x}{dt^2} = x \Rightarrow \frac{d^2x}{dt^2} - x = 0 \longrightarrow \text{(iii)}$$

Solⁿ done by method diff^t from back.

from (3), auxiliary eq^n: $m^2 - 1 = 0$

$\Rightarrow m = \underbrace{\pm 1}$

real & distinct roots.

$\Rightarrow x(t) = c_1 e^t + c_2 e^{-t} \longrightarrow (iv)$

Using (iv) in (i)

$\Rightarrow \frac{d}{dt}(c_1 e^t + c_2 e^{-t}) = -3(c_1 e^t + c_2 e^{-t}) + 4y$

$\Rightarrow c_1 e^t - c_2 e^{-t} + 3c_1 e^t + 3c_2 e^{-t} = 4y$

$\Rightarrow 4y = 4c_1 e^t + 2c_2 e^{-t}$

$\Rightarrow y_{(t)} = c_1 e^t + \frac{1}{2} c_2 e^{-t} \longrightarrow (v)$.

So, the req^d sol^n from (iv) & (v), we get

$$\left\{ \begin{array}{l} x(t) = c_1 e^t + c_2 e^{-t} \\ y(t) = c_1 e^t + \frac{1}{2} c_2 e^{-t} \end{array} \right\}$$

Q. $\frac{dx}{dt} = 4x - 2y \longrightarrow (i)$

$\frac{dy}{dt} = 5x + 2y \longrightarrow (ii)$

Solve the following sys. of homogeneous eq^n.

here, $\underbrace{x, y}_{\text{dependent var.}} \longrightarrow \underbrace{t}_{\text{Independent var.}}$

from (i)

$$\frac{dx}{dt} = 4x - 2y \Rightarrow y = \frac{4x - \frac{dx}{dt}}{2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 2\frac{dy}{dt}$$

$$= 4(4x-2y) - 2(5x+2y)$$

$$= 16x - 8y - 10x - 4y$$

$$\Rightarrow \frac{d^2x}{dt^2} = 6x - 12y$$

$$= 6x - \overset{6}{\cancel{12}}\left(\frac{4x - \frac{dx}{dt}}{\cancel{2}}\right)$$

$$\Rightarrow \frac{d^2x}{dt^2} = 6x - 24x + 6\frac{dx}{dt}$$

$$\Rightarrow \frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 18x = 0 \rightarrow \text{(iii)}$$

Auxiliary eqⁿ :

$$\begin{array}{r} 18 \\ \times 4 \\ \hline 72 \end{array}$$

$$m^2 - 6m + 18 = 0$$

$$\Rightarrow m = \frac{6 \pm \sqrt{36-72}}{2}$$

$$m = \underbrace{3 \pm 3i}_{\text{complex conjugate pairs}}$$

$$\Rightarrow$$

$$x(t) = e^{3t}(c_1 \cos 3t + c_2 \sin 3t) \rightarrow \text{(iv)}$$

Using (iv) in (i)

$$\Rightarrow \frac{d}{dt}\left(e^{3t}(c_1\cos 3t + c_2 \sin 3t)\right) = 4e^{3t}(c_1\cos 3t + c_2\sin 3t) - 2y$$

$$= e^{3t}(3c_1\cos 3t - c_1\sin 3t + 3c_2\sin 3t + c_2\cos 3t) = e^{3t}(4c_1\cos 3t + 4c_2\sin 3t) - 2y$$

$$\Rightarrow 3e^{3t}(c_2\cos 3t - c_1\sin 3t) = e^{3t}(c_1\cos 3t + c_2\sin 3t) - 2y$$

$$\Rightarrow y = \frac{e^{3t}[c_1\cos 3t + c_2\sin 3t - 3c_2\cos 3t + 3c_1\sin 3t]}{2}$$

$$\Rightarrow y(t) = \frac{e^{3t}}{2}\left[(c_1 - 3c_2)\cos 3t + (c_2 + 3c_1)\sin 3t\right] \to (v)$$

Hence, (iv) & (v) give req^d sol^n

Q. $\dfrac{dx}{dt} = 5x + 4y \to (i)$

$\dfrac{dy}{dt} = -x + y \to (ii)$

Using (ii)

$$\Rightarrow \frac{dy}{dt} = -x + y \Rightarrow x = y - \frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{dx}{dt} + \frac{dy}{dt}$$

$$= -(5x + 4y) + \frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\left(5\left(y - \frac{dy}{dt}\right) + 4y\right) + \frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} = -5y + 5\frac{dy}{dt} - 4y + \frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 0 \to (iii)$$

Auxiliary eq^n:-

$$m^2 - 6m + 9 = 0$$

$$\Rightarrow (m-3)^2 = 0$$

$$\Rightarrow m = \underbrace{3, 3}_{\text{real \& equal roots}}$$

$$\Rightarrow y(t) = (c_1 + c_2 t)e^{3t} \longrightarrow \text{(iv)}$$

Using (iv) in (ii)

$$\Rightarrow \frac{d}{dt}\left((c_1 + c_2 t)e^{3t}\right) = -x + (c_1 + c_2 t)e^{3t}$$

$$\Rightarrow 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t} = -x + c_1 e^{3t} + c_2 t e^{3t}$$

$$\Rightarrow x(t) = -2c_2 t e^{3t} - c_2 e^{3t} - 2c_1 e^{3t} = -e^{3t}(c_2(2t+1) + 2c_1)$$

(iv) & (v) give req^d sol^n

Laplace Transform

$$\begin{array}{ccc} (y, x) & & (y, p) \end{array}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{Differential eq}^n} \arrow[r, "\text{Laplace}", "\text{Transform}"'] & \boxed{\text{Algebraic eq}^n} \arrow[d, "\text{Solve it}"] \\
\boxed{\text{Solution of differential eq}^n} \arrow[u] & \boxed{\text{Solution of algebraic eq}^n} \arrow[l, "\text{Inverse}"', "\text{Laplace transform}"]
\end{tikzcd}$$

$$\begin{array}{ccc} (y, x) & & (y, p) \end{array}$$

Definition :

$$\boxed{L\{f(x)\} = \int_0^\infty e^{-px} f(x)\, dx}$$

↳ L : Laplace operator
↳ p : real parameter ($\equiv s$: in control sys)

→ $L\{f(x)\} = F(p)$

✩ Inverse Laplace Transform :

$$L^{-1}(F(p)) = f(x).$$

✓ Inverse LT will exist for a f^n only if LT exists.

eg: Find LT of 1

$$L(1) = \int_0^\infty e^{-px}(1)\,dx$$

$$= \int_0^\infty e^{-px}\,dx$$

$$= \left(\frac{e^{-px}}{-p}\right)_0^\infty$$

$$\Rightarrow \boxed{L(1) = \frac{1}{p}} \quad \text{or} \quad L^{-1}\left(\frac{1}{p}\right) = 1.$$

eg: $L(k) = \frac{k}{s}$; k: any constt.

* Property of Laplace Transform:

1. LT is linear:
i.e $L(a f_1(x) \pm b f_2(x)) = a L(f_1(x)) \pm b L(f_2(x))$.

2. Laplace inverse is also linear:

So, $L^{-1}(c_1 F_1(p) \pm c_2 F_2(p)) = c_1 L^{-1}[F_1(p)] + c_2 L^{-1}[F_2(p)]$

* Table

($f(x)$) f^n	Laplace Transform ($L(f(x))$)	Inverse LT $L^{-1}[F(p)]$
1. e^{ax}	$L(e^{ax}) = \frac{1}{p-a}$	$L^{-1}\left(\frac{1}{p-a}\right) = e^{ax}$
2. e^{-ax}	$L(e^{-ax}) = \frac{1}{p+a}$	$L^{-1}\left(\frac{1}{p+a}\right) = e^{-ax}$

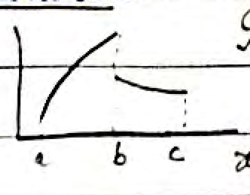
$\Gamma(n) = \int_0 e \;\; x \; dx$

$f^n(f(x))$	LT $(L[f(x)])$	Inverse LT $(L^{-1}F(p))$
3. $\sin ax$	$L(\sin ax) = \frac{a}{p^2+a^2}$	$L^{-1}\left(\frac{a}{p^2+a^2}\right) = \sin(ax)$ or $L^{-1}\left(\frac{1}{p^2+a^2}\right) = \frac{1}{a}\sin(ax)$
4. $\cos ax$	$L(\cos ax) = \frac{p}{p^2+a^2}$	$L^{-1}\left(\frac{p}{p^2+a^2}\right) = \cos(ax)$
5. $\sinh(ax)$	$L(\sinh(ax)) = \frac{a}{p^2-a^2}$	$L^{-1}\left(\frac{a}{p^2-a^2}\right) = \sinh(ax)$
6. $\cosh(ax)$	$L(\cosh(ax)) = \frac{p}{p^2-a^2}$	$L^{-1}\left(\frac{p}{p^2-a^2}\right) = \cosh(ax)$
7. x^n ~~(n : Integer)~~ $\hookrightarrow n$: any value (non integer)	$L(x^n) = \frac{\Gamma(n+1)}{p^{n+1}}$	$L^{-1}\left[\frac{\Gamma(n+1)}{p^{n+1}}\right] = x^n$.
8. x^n n : Integer	$L(x^n) = \frac{n!}{p^{n+1}}$	$L^{-1}\left(\frac{n!}{p^{n+1}}\right) = x^n$ or $L^{-1}\left(\frac{1}{p^{n+1}}\right) = \frac{x^n}{n!}$ or $L^{-1}\left(\frac{1}{p^n}\right) = \frac{x^{n-1}}{(n-1)!}$

☆ Sufficient condⁿ for a fⁿ to possess Laplace Transform.

$\boxed{1.}$ $f(x)$: should be a continuous f^n
(or) piece-wise continuous.

↳ eg:



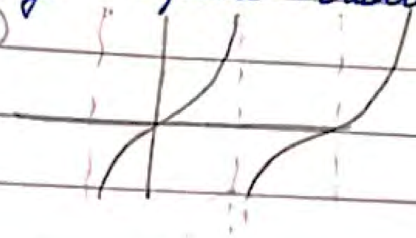
In $a \to b$: cts
$b \to c$: cts

piece wise

✓ piece wise continuity has discontinuity, but, the discontinuity should be finite.

↳ Laplace transform won't be possible for a f^n having infinite discontinuity.

eg: $\tan(x)$



: Infinite discontinuity

⇓

LT not possible

$\boxed{2.}$ $f(x)$ should be a f^n of exponential order.

i.e, $\boxed{|f(x)| \leq m e^{cx}}$

↳ for some constt, m & c.

✓ a f^n satisfying both conditions (1) & (2), only then, LT will exist.

eg find LT of following functions :-

(i) $L(\sin^2 ax)$
(ii) $L(\cos^2 ax)$
(iii) $L(4\sin x \cos x + 2e^{-x})$
(iv) $L(x^6)$

(i) $L\left(\dfrac{1-\cos 2ax}{2}\right) = \dfrac{1}{2}\left[L(1) - L(\cos 2ax)\right]$

$= \dfrac{1}{2}\left(\dfrac{1}{p} - \dfrac{p}{p^2+(2a)^2}\right)$

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(ii) $L\left(\frac{1+\cos 2ax}{2}\right) = \frac{1}{2}\left[\frac{1}{p} + \frac{p}{p^2+(2a)^2}\right]$

(iii) $L\left(2\sin 2x + 2e^{-x}\right)$

$$= 2\left[L(\sin 2x) + L(e^{-x})\right]$$

$$= 2\left[\frac{p}{p^2+2^2} + \frac{1}{p+1}\right]$$

(iv) $L(x^6) = L(x^n)$; $n=6$

$$= \frac{6!}{p^7}$$

Q. Find Inverse LT :

(a) $\frac{1}{p^2+p}$

(b) $\frac{4}{p^3} + \frac{6}{p^2+4}$

(a) $L^{-1}\left(\frac{1}{p(p+1)}\right) = L^{-1}\left(\frac{1}{p} - \frac{1}{p+1}\right)$

$$= L^{-1}\left(\frac{1}{p}\right) - L^{-1}\left(\frac{1}{p+1}\right)$$

$$= 1 - e^{-x}$$

(b) $L^{-1}\left(\frac{4}{p^3}\right) + L^{-1}\left(\frac{6}{p^2+4}\right)$

$$= 2L^{-1}\left(\frac{2!}{p^{2+1}}\right) + 3L^{-1}\left(\frac{2}{p^2+2^2}\right)$$

$= 2x^2 + 3\sin 2x$ Ans

Q. Find LT of :-

(1) $x^{-1/2}$

(2) unit step f^n $(U_a(x)$ or $U(x-a))$

(3) unit impulse f^n.

(4) e^{x^2}

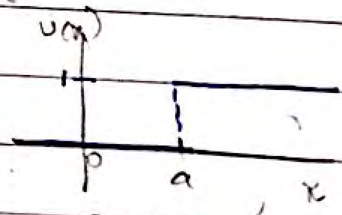
(5) e^{-1}

(1) $x^{-1/2} = x^n ; n \notin Z$

So, $L(x^{-1/2}) = \dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{p^{-1/2+1}} = \dfrac{\Gamma(1/2)}{\sqrt{p}}$

$= \dfrac{\sqrt{\pi}}{\sqrt{p}}$ Ans

(2) $L(u(x-a)) = \int_0^\infty e^{-px} u(x-a)\,dx$



$U(x-a) = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases}$

$\Rightarrow L(u(x)) = \int_0^a e^{-px}(0)\,dx + \int_a^\infty e^{-px}(1)\,dx$

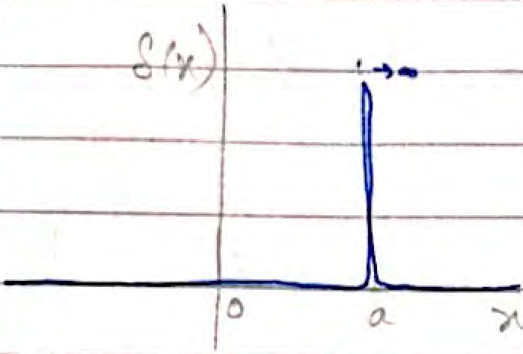
$= \left(\dfrac{e^{-px}}{-p}\right)_a^\infty$

$\Rightarrow L(u(x)) = 0 + \dfrac{e^{-pa}}{p} = \dfrac{e^{-pa}}{p}$

M1

(3) $L(\delta(x-a)) = \int_0^{\infty} e^{-px} (\delta(x))\, dx$

* Defining Impulse fn or Dirac Delta fn.



$$= \int_0^{a} e^{-p(x)} (0)\, dx + \int_a^{\infty} e^{-px} \delta(x)\, dx$$

$$\delta(x-a) = \begin{cases} 0 & ; x = a \\ \infty & ; x \neq a \end{cases}$$

$$= 0 + \int_a^{\infty} e^{-px} \delta(x)\, dx$$

Mathematically,

If

$$f_{\epsilon}(x-a) = \begin{cases} \frac{1}{\epsilon - a} & , 0 \leq x \leq \epsilon \\ 0 & , x > \epsilon \end{cases}$$

then,

$$\delta(x-a) = \lim_{\epsilon \to a} f_{\epsilon}(x-a)$$

By defin^n of Impulse fn, it exists only at $x = a$.

$$= \int_a^{\infty} e^{-p(a)} \delta(x)\, dx$$

$$= e^{-pa} \int_a^{\infty} \delta(x)\, dx.$$

Again by defin^n, area under Impulse fn $= 1$

$$\Rightarrow L(\delta(x-a)) = e^{-pa}.$$

↳ for $a = 0$

$$\Rightarrow L(\delta(x)) = 1$$

M2 $L\{f_{\epsilon}(x)\} = \int_0^{\infty} e^{-px} f_{\epsilon}(x)\, dx$

$$= \int_0^{\epsilon} e^{-px} \frac{1}{\epsilon}\, dx$$

$$= \frac{1}{\epsilon}\left[\frac{e^{-px}}{-p}\right]_0^{\epsilon} = \frac{1}{\epsilon}\left[\frac{e^{-p\epsilon} - 1}{-p}\right]$$

$$= \frac{1-e^{-p\epsilon}}{p\epsilon}$$

$$L(\delta(x)) = \int_0^\infty e^{-px}\,\delta(x)\,dx$$

$$= \int_0^\infty e^{-px}\left[\lim_{\epsilon \to 0} f_\epsilon(x)\right] dx$$

$$= \lim_{\epsilon \to 0} \int_0^\infty e^{-px} f_\epsilon(x)\,dx$$

$$= \lim_{\epsilon \to 0} L[f_\epsilon(x)]$$

$$= \lim_{\epsilon \to 0} \left(\frac{1-e^{-p\epsilon}}{p\epsilon}\right)$$

$$= \lim_{\epsilon \to 0} \left(\frac{0-(e^{-p\epsilon})(-p)}{p}\right)$$

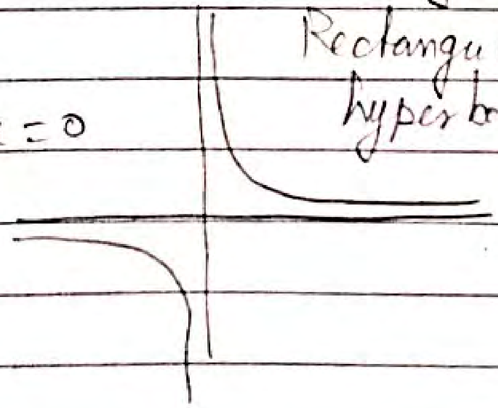
$$\Rightarrow L(\delta(x)) = \frac{p}{p} = 1$$

(5) x^{-1}

$L(x^{-1})$

Plotting $x^{-1} = y$

$\Rightarrow xy = 1$



Rectangular hyperbola

Infinite discontinuity at $x=0$

So, $L(x^{-1})$ doesn't exist.

Section - 50

APPLICATION OF LT (to solve D.E)

RESULTS

(i) $L\{y'(x)\} = pL[y(x)] - y(0)$

$\hookrightarrow y(0) = y(x)\big|_{x=0}$

(ii) $L\{y''(x)\} = p^2L[y(x)] - py(0) - y'(0)$

$\hookrightarrow y'(0) = \frac{d}{dx}y(x)\big|_{x=0}$

Shifting Rules

(iii) $L\{e^{ax}f(x)\} = F(p-a)$

$\hookrightarrow$ where $L[f(x)] = F(p)$

i.e $L\{e^{ax}f(x)\} = \Big(L[f(x)]\Big)_{\text{Replace } p \to p-a}$

(iv) $L\{e^{-ax}f(x)\} = F(p+a)$

i.e $L\{e^{-ax}f(x)\} = \Big[L(f(x))\Big]_{\text{Replace } p \to p+a}$

(a)

eg ① $L\{x^5 e^{-2x}\}$

② $L\{e^{3x}\cos 2x\}$

③ $L\{(1-x^2)e^{x}\}$

(b) ① $L^{-1}\left(\frac{6}{(p+2)^2+9}\right)$

② $L^{-1}\left(\frac{12}{(p+3)^3}\right)$

③ $L^{-1}\left(\frac{p+3}{p^2+2p+5}\right)$

(a) ① $L\{e^{-ax} f(x)\} = \{L[f(x)]\}$ Replace $p \to p+a$

here, $f(x) = x^5$

$a = 2$

So, $[L(x^5)]_{p\to p+2} = \left(\frac{5!}{p^{5+1}}\right)_{p\to p+2}$

$= \frac{5!}{(p+2)^6}$ Ans

② $L\{e^{ax} f(x)\} = \{L[f(x)]\}_{p\to p-a}$

$f(x) = \cos 2x$

$a = 3$

$\Rightarrow \left(L[\cos 2x]\right)_{p\to p-3} = \left(\frac{p}{p^2+2^2}\right)_{p\to p-3}$

$= \frac{p-3}{(p-3)^2+2^2}$ Ans

③ $L\{(1-x^2)e^{-x}\}$

$$\left(L\{(1-x^2)\}\right)_{p\to p+1} = \left[L(1) - L(x^2)\right]_{p\to p+1}$$

$$= \left(\frac{1}{p} - \frac{2!}{p^3}\right)_{p\to p+1}$$

$$= \frac{1}{p+1} - \frac{2}{(p+1)^3}.$$

Ans

* Inverse form of shifting rule:

(1) $L^{-1}[F(p-a)] = e^{ax} f(x)$

$$\Rightarrow \boxed{L^{-1}[F(p-a)] = e^{ax}\left[L^{-1}F(p)\right]}$$

(2) $\boxed{L^{-1}[F(p+a)] = e^{-ax}\left[L^{-1}F(p)\right]}$

(b) ① $L^{-1}\left(\dfrac{6}{(p+2)^2+9}\right) \equiv F(p+2)$

So, $e^{-2x} L^{-1}\left(\dfrac{6}{p^2+9}\right)$

$$= e^{-2x} \times (2)\, L^{-1}\left(\frac{3}{p^2+3^2}\right)$$

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$$\Rightarrow L^{-1}\left(\frac{6}{(p+2)^2+9}\right) = 2e^{-2x}\sin 3x$$

(2) $$L^{-1}\left(\frac{12}{(p+3)^3}\right) = L^{-1}\left(F(p+3)\right)$$

$a=3$

So, $$L^{-1}\left(\frac{12}{(p+3)^3}\right) = e^{-3x}\left(L^{-1}\left(\frac{12}{p^3}\right)\right)$$

$$= e^{-3x} \times (6)\left(L^{-1}\left(\frac{2!}{p^{2+1}}\right)\right)$$

$$= 6e^{-3x}\cdot x^2$$

(3) $$L^{-1}\left(\frac{p+3}{p^2+2p+5}\right) = L^{-1}\left(\frac{p+3}{(p+1)^2+2^2}\right)$$

$$= L^{-1}\left(\frac{(p+1)+2}{(p+1)^2+2^2}\right)$$

$$= \underbrace{L^{-1}\left(\frac{p+1}{(p+1)^2+2^2}\right) + L^{-1}\left(\frac{2}{(p+1)^2+2^2}\right)}_{L^{-1}(F(p+1))}$$

$$= e^{-x}L^{-1}\left(\frac{p}{p^2+2^2}\right) + e^{-x}L^{-1}\left(\frac{2}{p^2+2^2}\right)$$

$$= e^{-x}\cos 2x + e^{-x}\sin 2x$$

$$= e^{-x}(\cos 2x + \sin 2x)$$

Ans

Q. Solve the D.E :-

① $y' + y = 3e^{2x}$; $y(0) = 0$

let $\frac{d}{dx} = D$

$\Rightarrow (D+1)y = 3e^{2x}$.

or $\frac{d}{dx}y + y = 3e^{2x}$

LT both sides

$$\Rightarrow pY(p) - \underbrace{y(0)}_{0} + Y(p) = 3L(e^{2x})$$

$$= 3\left(\frac{1}{p-2}\right)$$

where $L(y(x)) = Y(p)$

$$\Rightarrow (p+1)Y(p) = \frac{3}{p-2}.$$

$$\Rightarrow Y(p) = L(y(x)) = \frac{3}{(p+1)(p-2)}$$

$\Rightarrow$

$$y(x) = L^{-1}\left(\frac{3}{(p+1)(p-2)}\right)$$

$$= L^{-1}\left(\frac{(p+1)-(p-2)}{(p+1)(p-2)}\right)$$

$$= L^{-1}\left(\frac{1}{p-2} - \frac{1}{p+1}\right)$$

$$= L^{-1}\left(\frac{1}{p-2}\right) - L^{-1}\left(\frac{1}{p+1}\right)$$

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$\Rightarrow y(x) = e^{2x} - e^{x}$

Ans

Aliter $y' + y = 3e^{2x}$.

$\Rightarrow (D+1)y = 3e^{2x}$

$\Rightarrow y = \frac{1}{D+1}(3e^{2x})$

$= 3\left(e^{-x}\int e^{x}\cdot e^{2x}\,dx\right)$

$= 3\left(e^{-x}\left[\frac{e^{3x}}{3} + c\right]\right)$

$= e^{2x} + 3ce^{-x}$

If $c = -\frac{1}{3}$, we get

$y(x) = e^{2x} - e^{-x}$, same as above

SOLVE THE D.E USING LT :

Q8

① → $y'' + 2y' + 5y = 3e^{-x}\sin x$; $y(0) = 0$ & $y'(0) = 3$ (Given)

Taking LT on both sides of ①

$$\Rightarrow L(y''(x)) + 2L(y'(x)) + 5L(y(x)) = 3L(e^{-x}\sin x)$$

$$= [p^2 L(y(x)) - p\,y(0) - y'(0)]_{\to 3} + 2[pL(y(x)) - \cancel{y(0)}] + 5L(y(x)) = 3L(e^{-x}\sin x)$$

$$\Rightarrow L(y(x))[p^2+2p+5] - 3 = 3L(e^{-x}\sin x) = 3L(\sin x)\Big|_{p\to p+1}$$

$p \to p+1$ shifting thm.

$$\Rightarrow L(y(x))[p^2+2p+5] - 3 = 3\left(\frac{1}{p^2+1}\right)_{p\to p+1} = 3\left(\frac{1}{(p+1)^2+1}\right)$$

$$\Rightarrow L(y(x)) = \frac{\frac{3}{(p+1)^2+1} + 3}{(p^2+2p+5)}$$

$$\Rightarrow L(y(x)) = \frac{3}{((p+1)^2+1)((p+1)^2+2^2)} + \frac{3}{[(p+1)^2+2^2]} \quad \to ②$$

Now, $L^{-1}(F(p+a)) = e^{-ax} L^{-1}(F(p))$ ↳ result

Taking $\mathcal{L}^{-1}$ on both sides of eqⁿ (2)

$$\Rightarrow y(x) = \mathcal{L}^{-1}\left(\frac{3}{((p+1)^2+1)[(p+1)^2+2^2]}\right) + \mathcal{L}^{-1}\left(\frac{3}{(p+1)^2+2^2}\right)$$

Using the result

$$= e^{-x}\mathcal{L}^{-1}\left(\frac{3}{(p^2+1)(p^2+2^2)}\right) + e^{-x}\mathcal{L}^{-1}\left(\frac{3}{p^2+2^2}\right)$$

$$= e^{-x}\mathcal{L}^{-1}\left(\frac{(p^2+2^2)-(p^2+1^2)}{(p^2+1)(p^2+2^2)}\right) + \frac{3}{2}e^{-x}\mathcal{L}^{-1}\left(\frac{2}{p^2+2^2}\right)$$

$$= e^{-x}\left[\mathcal{L}^{-1}\left(\frac{1}{p^2+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{p^2+2^2}\right)\right] + \frac{3}{2}e^{-x}\sin 2x$$

$$= e^{-x}\left(\sin x - \frac{1}{2}\sin 2x\right) + \frac{3}{2}e^{-x}\sin 2x$$

$$\Rightarrow y(x) = e^{-x}\left[\sin x + \sin 2x\right]$$

$$\left(\because \mathcal{L}^{-1}\left(\frac{a}{p^2+a^2}\right) = \sin ax\right)$$

☆ **RESULT:**

Imp. $$\boxed{\mathcal{L}\left\{\int_0^x f(x)\,dx\right\} = \frac{F(p)}{p} = \frac{\mathcal{L}(f(x))}{p}}$$

$$\hookrightarrow \mathcal{L}\left\{\int f(x)\,dx\right\} = \frac{F(p)}{p} + \frac{1}{s}f^{-1}(0)$$

$$\hookrightarrow f^{-1}(0) = \int f(x)\,dx\Big|_{x=0}$$

• Inverse form :-

Imp. $$\boxed{\mathcal{L}^{-1}\left(\frac{F(p)}{p}\right) = \int_0^x f(x)\,dx = \int_0^x \mathcal{L}^{-1}(F(p))\,dx}$$

↳ $f(x) = \mathcal{L}^{-1}[F(p)]$

Q. Solve the following D.E using LT :

$$y'' + y' = x^3 \quad , \quad y(0) = 0 \ \& \ y'(0) = 0$$

Taking LT both sides

$$\Rightarrow \mathcal{L}(y''(x)) + \mathcal{L}(y'(x)) = \mathcal{L}(x^3)$$

$$\Rightarrow \left(p^2\mathcal{L}(y(x)) + p\,\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right) + p\mathcal{L}(y(x)) - \underbrace{y(0)}_{0} = \mathcal{L}(x^3)$$

$$\Rightarrow p^2\mathcal{L}(y(x)) + p\mathcal{L}(y(x)) = \frac{3!}{p^{3+1}}$$

$$\Rightarrow \mathcal{L}(y(x))\left[p^2 + p\right] = \frac{6}{p^4}$$

$$\Rightarrow \mathcal{L}(y(x)) = \frac{6}{p^5(p+1)}$$

$$\Rightarrow y(x) = \mathcal{L}^{-1}\left(\frac{6}{p^5(p+1)}\right)$$

M1

$$= \mathcal{L}^{-1}\left[\frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{p^4} + \frac{E}{p^5} + \frac{F}{p+1}\right]$$

$* \quad 3! \times \frac{4}{4} = \frac{1}{4} \times 4!$

Idea $* \quad n! \times \frac{n+1}{n+1} = \left(\frac{1}{n+1}\right) \times (n+1)!$

(M2) Extending the formula :-

$$\mathcal{L}^{-1}\left(\frac{F(p)}{p}\right) = \int_0^x \mathcal{L}^{-1}(F_{(p)})\,dx$$

as

$$\mathcal{L}^{-1}\left(\frac{F(p)}{p^2}\right) = \int_0^x \int_0^x \mathcal{L}^{-1}(F_{(p)})\,dx\,dx$$

$$\boxed{L^{-1}\left[\frac{F(p)}{p^n}\right] = \int_0^x \cdots \int_0^x L^{-1}(F_{(p)})\,dx\,dx \ldots dx}$$

(M3) Now,

$$y(x) = \mathcal{L}^{-1}\left(\frac{6}{(\overline{p+1}-1)^5(\overline{p+1})}\right)$$

$$= e^{-x}\mathcal{L}^{-1}\left[\frac{6}{(p-1)^5\,p}\right]$$

$$= e^{-x}\mathcal{L}^{-1}\left[\frac{\frac{6}{(p-1)^5}}{p}\right] \longrightarrow (1)$$

$$\Rightarrow y(x) = e^{-x}\left(\frac{1}{4}\right)\mathcal{L}^{-1}\left[\frac{\frac{4!}{(p-1)^{4+1}}}{p}\right]$$

Now $\mathcal{L}^{-1}\left[\frac{6}{(p-1)^5}\right] = \frac{1}{4}\left(L^{-1}\left(\frac{4!}{(p-1)^{4+1}}\right)\right)$

$$= \frac{1}{4}(e^{x})\left[L^{-1}\left(\frac{4!}{p^{4+1}}\right)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{6}{(p-1)^5}\right) = \left(\frac{e^x}{4}\right)\cdot x^4 \rightarrow (A)$$

So, from (1)

$$y(x) = e^{-x} L^{-1}\left[\frac{\frac{6}{(p-1)^5}}{p}\right] \quad \frac{6}{(p-1)^5} \equiv F(p)$$

$$= e^{-x} L^{-1}\left(\frac{F(p)}{p}\right)$$

$$= e^{-x}\int_0^x \left(\frac{e^x \cdot x^4}{4}\right) dx \quad \text{(from (A))}$$

$$= \frac{e^{-x}}{4}\int_0^x (e^x \cdot x^4)\, dx$$

$$= \frac{e^{-x}}{4}\Big[e^x \cdot x^4 - e^x \cdot (4x^3) + e^x(12x^2) - e^x(24x) + e^x(24)\Big]_0^x$$

$$= \frac{e^{-x}}{4}\Big[e^x\left[x^4 - 4x^3 + 12x^2 - 24x + 24\right] - 24\Big]$$

$$\Rightarrow y(x) = \frac{x^4 - 4x^3 + 12x^2 - 24x + 24}{4} - 6e^{-x}$$

Using (M2)

$$y(x) = 6\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x L^{-1}\left(\frac{1}{p+1}\right)(dx)^5$$

$$= 6\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x (e^{-x})(dx)^5$$

$$= 6\int_0^x\int_0^x\int_0^x\int_0^x \left[\frac{e^{-x}}{(-1)^1}\right]_0^x (dx)^4$$

$$= 6\int_0^x\int_0^x\int_0^x\int_0^x \left(\frac{e^{-x} - 1}{(-1)^1}\right)(dx)^4$$

$$= 6\int_0^x\int_0^x\int_0^x \left[\frac{\frac{e^{-x}}{-1} - x}{-1}\right]_0^x (dx)^3$$

$$= 6\int_0^x\int_0^x\int_0^x \left[\frac{\left(\frac{e^{-x}}{-1} - x\right) - \left(-1\right)}{-1}\right] (dx)^3$$

$$= 6\int_0^x\int_0^x\int_0^x \left(e^{-x} + x - 1\right)(dx)^3.$$

$$= 6\int_0^x\int_0^x \left[\frac{e^{-x}}{-1} + \frac{x^2}{2} - x\right]_0^x (dx)^2$$

$$= 6\int_0^x\int_0^x \left[\left(\frac{e^{-x}}{-1} + \frac{x^2}{2} - x\right) - (-1)\right](dx)^2$$

$$= 6\int_0^x \left[\frac{e^{-x}}{1} + \frac{x^3}{6} - \frac{x^2}{2} + x\right]_0^x dx$$

$$= 6\int_0^x \left[\left(e^{-x} + \frac{x^3}{6} - \frac{x^2}{2} + x\right) - 1\right] dx$$

$$= 6\left[\frac{e^{-x}}{-1} + \frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} - x\right]_0^x$$

$$= 6\left[\frac{e^{-x}}{-1} + \frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} - x + 1\right]$$

$$\Rightarrow y(x) = -6e^{-x} + \frac{x^4 - 4x^3 + 12x^2 - 24x + 24}{4}$$

(Same as in M3)

Ans

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§ DERIVATIVES & INTEGRALS OF LT

RESULTS :-

① If $\mathcal{L}\{f(x)\} = F(p)$,

then, $\mathcal{L}\{x f(x)\} = -F'(p) \quad \left(= -\frac{d}{dp}(F(p))\right)$

② $\mathcal{L}(x^2 f(x)) = (-1)^2 F''(p)$

③ $\mathcal{L}(x^n f(x)) = (-1)^n F^{(n)}(p)$

$\hookrightarrow F^{(n)}p = \frac{d^n}{dp^n} F(p)$

④ If $\mathcal{L}\{f(x)\} = F(p)$,

then, $\mathcal{L}\left\{\frac{f(x)}{x}\right\} = \int_p^\infty F(p)\,dp$.

Q. Find :- ① $\mathcal{L}\{x \sin(ax)\}$ & $\mathcal{L}(x\cos(ax))$

② $\mathcal{L}\{x^{3/2}\}$, using $\mathcal{L}\{x^{-1/2}\}$

① . $\mathcal{L}\{x \sin ax\}$

we know $\mathcal{L}(x f(x)) = -F'(p)$

$= -\frac{d}{dp} F(p)$

$= -\frac{d}{dp}\mathcal{L}(f(x))$

$\Rightarrow \mathcal{L}(x \sin ax) = -\frac{d}{dp}\mathcal{L}(\sin ax)$

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$$= -\frac{d}{dp}\left(\frac{a}{a^2+p^2}\right)$$

$$= (-a)\left[\frac{-2p}{(a^2+p^2)^2}\right]$$

$$\Rightarrow \mathcal{L}(x\sin ax) = \frac{2ap}{(a^2+p^2)^2}.$$

$$\Rightarrow x\sin(ax) = \mathcal{L}^{-1}\left(\frac{2ap}{(a^2+p^2)^2}\right)$$

$$\Rightarrow x\sin ax = 2a\,\mathcal{L}^{-1}\left(\frac{p}{(a^2+p^2)^2}\right)$$

$$\Rightarrow * \boxed{\mathcal{L}^{-1}\left(\frac{p}{(p^2+a^2)^2}\right) = \frac{1}{2a}(x\sin ax)}$$ useful result

Now, $\mathcal{L}(x\cos ax) = -\frac{d}{dp}\mathcal{L}(f(x))$

So,

$$= -\frac{d}{dp}\mathcal{L}(\cos ax)$$

$$= -\frac{d}{dp}\left(\frac{p}{p^2+a^2}\right)$$

$$= -\left[\frac{(p^2+a^2)(1) - p(2p)}{(p^2+a^2)^2}\right]$$

$$\Rightarrow \mathcal{L}(x\cos ax) = \frac{p^2-a^2}{(p^2+a^2)^2}$$

$$\Rightarrow * \boxed{\mathcal{L}^{-1}\left(\frac{p^2-a^2}{(p^2+a^2)^2}\right) = x\cos ax}$$

useful result

(2) $\mathcal{L}\{x^{-1/2}\} = \mathcal{L}\{x^n\};\ n \notin \mathbb{Z}$

$$= \frac{\Gamma(n+1)}{p^{n+1}} \quad \text{(result written before)}$$

$\hookrightarrow$ valid $\forall\ n \in \mathbb{R}$

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i) $\mathcal{L}\{x^{-1/2}\} = \dfrac{\boxed{(-1/2+1)}}{p^{-1/2+1}} = \dfrac{\boxed{(1/2)}}{p^{1/2}} = \dfrac{\sqrt{\pi}}{\sqrt{p}} \rightarrow (1)$,

Now, finding

M1 direct
$$\mathcal{L}\{x^{3/2}\} = \frac{\boxed{3/2+1}}{p^{3/2+1}} = \frac{3/2\,\boxed{3/2}}{p^{5/2}} = \frac{3/2 \cdot 1/2 \cdot \sqrt{\pi}}{p^{5/2}}$$

M2 as asked
$$\mathcal{L}\{x^{1/2}\} = \mathcal{L}\{x \cdot x^{-1/2}\}$$
$$= -\frac{d}{dp}\mathcal{L}\{x^{-1/2}\}$$
$$= -\frac{d}{dp}\left\{\frac{\sqrt{\pi}}{\sqrt{p}}\right\}$$
Result from (1). So, as asked in ques. we have used it.
$$= -\sqrt{\pi}\left(\frac{-1}{2}p^{-3/2}\right)$$
$$\Rightarrow \mathcal{L}\{x^{1/2}\} = \frac{\sqrt{\pi}}{2p^{3/2}} \rightarrow (2)$$

Now, $\mathcal{L}\{x^{3/2}\} = \mathcal{L}\{x \cdot x^{1/2}\}$
$$= -\frac{d}{dp}\mathcal{L}\{x^{1/2}\}$$
$$= -\frac{d}{dp}\left\{\frac{\sqrt{\pi}}{2p^{3/2}}\right\} \quad \text{(from (2))}$$
$$\Rightarrow \mathcal{L}\{x^{3/2}\} = \frac{3}{4p^2}\sqrt{\frac{\pi}{p}}$$
Ans

Q Find :- ③ $\mathcal{L}^{-1}\left\{\dfrac{1}{(p^2+a^2)^2}\right\}$

④ $\mathcal{L}\{x^2 \sin ax\}$

Sol^n:- ③ $\mathcal{L}^{-1}\left\{\dfrac{1}{(p^2+a^2)^2}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{2a^2}\dfrac{\{(p^2+a^2)-(p^2-a^2)\}}{(p^2+a^2)^2}\right\}$

$$= \frac{1}{2a^2}\left\{\mathcal{L}^{-1}\left(\frac{1}{p^2+a^2}\right) - \mathcal{L}^{-1}\left(\frac{p^2-a^2}{(p^2+a^2)^2}\right)\right\}$$

$$= \frac{1}{2a^2}\left(\frac{1}{a}\sin(ax) - \underbrace{x\cos ax}_{\text{derived before.}}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(p^2+a^2)^2}\right) = \frac{1}{2a^3}\left(\sin ax - ax\cos ax\right)$$

④ $\mathcal{L}\{x^2 \sin ax\} = \mathcal{L}\{x f(x)\}$; $f(x) = x\sin ax$

$$= (-1)^1 \frac{d}{dp} F(p)$$

$$= (-1) \frac{d}{dp} \mathcal{L}\{x \sin ax\}$$

$$= (-1)\left(\frac{d}{dp}\left(\frac{2ap}{(p^2+a^2)^2}\right)\right)$$

$$= -\left[\frac{(p^2+a^2)^2(2a) - 2ap(2)(p^2+a^2)(2p)}{(p^2+a^2)^4}\right]$$

$$\Rightarrow \mathcal{L}\{x^2 \sin ax\} = \frac{6ap^2 - 2a^3}{(p^2+a^2)^3}$$

§ CONVOLUTION & CONVOLUTION Theorem

Consider 2 fns

$f(t) \circledast g(t)$ → symbol for convolution

result of Integral

$h(t)$

Defin^n ✫ $f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$ (or u)

$= \int_0^t f(\tau) g(t-\tau)\, d\tau$.

$= g(t) * f(t)$; u, τ : dummi variable

Theorem ✫ $\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

$\Rightarrow \mathcal{L}\{f(t) * g(t)\} = F(p) \cdot G(p)$

✫

↳ $F(p) = \mathcal{L}\{f(t)\}$
$G(p) = \mathcal{L}\{G(p)\}$

Result ✫ $\mathcal{L}^{-1}[F(p) \cdot G(p)] = f(t) * g(t)$

✫

↳ $f(t) = \mathcal{L}^{-1}(F(p))$
$g(t) = \mathcal{L}^{-1}(G(p))$

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Q. Find the convolution for the following pairs of f^n:-
(a) $1, \sin(at)$
(b) t, e^{at}
(c) $\sin(at), \sin(bt)$

(a) Let $f(t) = 1 = t^0$, $g(t) = \sin(at)$

Now, $$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau$$
$$= \int_0^t (t-\tau)^0 \sin(a\tau)\, d\tau$$
$$= -\left[\frac{\cos a\tau}{a}\right]_0^t$$
$$\Rightarrow f(t) * g(t) = -\left[\frac{\cos at}{a} - 1\right] = \frac{1-\cos at}{a}.$$

(b) Let $f(t) = t$, $g(t) = e^{at}$

So, $$f(t) * g(t) = \int_0^t (t-\tau) e^{a\tau}\, d\tau$$

M1 Use Bernoulli's formula.

M2 $$= \int_0^t t.e^{a\tau}\, d\tau - \int_0^t \tau e^{a\tau}\, d\tau$$
$$= t\left[\frac{e^{a\tau}}{a}\right]_0^t - \tau\left[\frac{e^{a\tau}}{a}\right]_0^t - \int_0^t \frac{e^{a\tau}}{a}\, d\tau$$
$$= t\left(\frac{e^{at}-1}{a}\right) - \tau\left(\frac{e^{at}-1}{a}\right) - \left[\frac{e^{a\tau}}{a^2}\right]_0^t$$
$$= t\left(\frac{e^{at}-1}{a}\right) - \tau\left(\frac{e^{at}-1}{a}\right) - \left(\frac{e^{at}-1}{a^2}\right)$$
$$= \frac{1}{a^2}\left(e^{at} - at - 1\right)$$

(c) $f(t) = \sin(at)$, $g(t) = \sin(bt)$.

$$f(t) * g(t) = \int_0^t \sin(at - a\tau)\sin(b\tau)\, d\tau$$

$$= \int_0^t \frac{1}{2}\left[\cos((at - a\tau) - b\tau) - \cos((at - a\tau) + b\tau)\right] d\tau$$

$$= \frac{1}{2}\int_0^t \left[\cos[at - (a+b)\tau] - \cos[at - (a-b)\tau]\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(at - (a+b)\tau)}{-(a+b)} - \frac{\sin(at - (a-b)\tau)}{-(a-b)}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin(-bt)}{-(a+b)} - \frac{\sin(+bt)}{-(a-b)}\right) - \left(\frac{\sin at}{-(a+b)} - \frac{\sin at}{-(a-b)}\right)\right]$$

$$= \frac{1}{2}\left[\sin bt\left(\frac{1}{a+b} + \frac{1}{a-b}\right) + \sin at\left(\frac{1}{a+b} - \frac{1}{a-b}\right)\right]$$

$$= \frac{1}{2}\left[\sin bt\left(\frac{2a}{a^2 - b^2}\right) + \sin at\left(\frac{-2b}{a^2 - b^2}\right)\right]$$

$$= \frac{1}{(a^2 - b^2)}\left[a \sin bt - b \sin at\right]$$

Ans

Q. Verify convolution theorem for the above pairs of f^n.

i.e, $\mathcal{L}\{f(t) * g(t)\} = F(p) \cdot G(p)$

(a) $f(t) = 1$, $g(t) = \sin at$

$$LHS = \mathcal{L}\{1 * \sin at\}$$

$$= \mathcal{L}\left[\frac{1 - \cos at}{a}\right] = \frac{1}{a}\mathcal{L}\{1\} - \frac{1}{a}\mathcal{L}(\cos at)$$

$$= \frac{1}{a}\left[\frac{1}{p} - \frac{p}{p^2+a^2}\right]$$

$$= \frac{1}{a}\left[\frac{p^2 - p^2 + a^2}{p(p^2+a^2)}\right]$$

$$= \frac{a}{p(p^2+a^2)}$$

RHS :- $F(p) \cdot G(p)$

$$= \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$$

$$= \mathcal{L}(1) \cdot \mathcal{L}\{\sin at\}$$

$$= \left(\frac{1}{p}\right)\left(\frac{a}{p^2+a^2}\right)$$

$$= \frac{a}{p(p^2+a^2)}$$

$= LHS.$

Hence, verified.

(b) $f(t) = t,\ g(t) = e^{at}$

$$LHS = \mathcal{L}\{t * e^{at}\} = \mathcal{L}\left\{\frac{1}{a^2}(e^{at} - at - 1)\right\}$$

$$= \frac{1}{a^2}\left\{\mathcal{L}(e^{at}) - a\mathcal{L}(t) - \mathcal{L}(1)\right\}$$

$$= \frac{1}{a^2}\left[\frac{1}{p-a} - \frac{a}{p^2} - \frac{1}{p}\right]$$

$$\Rightarrow LHS = \frac{1}{p^2(p-a)}$$

RHS :- $\mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

$= \mathcal{L}\{t\} \cdot \mathcal{L}\{e^{at}\}$

$= \left(\frac{1}{p^2}\right)\left(\frac{1}{p-a}\right)$

$=$ LHS.

So, verified

(c) $f(t) = \sin(at)$, $g(t) = \sin(bt)$

LHS $= \mathcal{L}\{\sin(at) * \sin(bt)\}$

$= \mathcal{L}\left\{\frac{1}{a^2-b^2}\left(a\sin bt - b\sin at\right)\right\}$

$= \frac{a}{a^2-b^2}\mathcal{L}(\sin bt) - \frac{b}{a^2-b^2}\mathcal{L}(\sin at)$

$= \left(\frac{a}{a^2-b^2}\right)\left(\frac{b}{p^2+b^2}\right) - \left(\frac{b}{a^2-b^2}\right)\left(\frac{a}{p^2+a^2}\right)$

$= \frac{ab}{a^2-b^2}\left[\frac{a^2-b^2}{(p^2+a^2)(p^2+b^2)}\right]$

$= \frac{ab}{(p^2+a^2)(p^2+b^2)}$

RHS $= F(p) \cdot G(p)$

$= \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t))$

$= \mathcal{L}(\sin at) \cdot \mathcal{L}(\sin bt)$

$= \frac{a}{p^2+a^2} \cdot \frac{b}{p^2+b^2}$

$= \frac{ab}{(p^2+a^2)(p^2+b^2)} =$ LHS.

Verified ✓

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Q. Use convolution thm, solve following DEs :-

1) $y'' + 5y' + 6y = 5e^{3t}$; $y(0) = y'(0) = 0$

2) $y'' - y' = t^2$, $y(0) = y'(0) = 0$

1) Applying LT on both sides:-

$$\mathcal{L}\left(y''(t) + 5\,y'(t) + 6y(t)\right) = \mathcal{L}(5e^{3t})$$

$$\Rightarrow \left[p^2 \mathcal{L}\{y(t)\} - p\,\cancelto{0}{(y(0))} - \cancelto{0}{y'(0)}\right]$$

$$+ 5\left[p\mathcal{L}(y(t)) - \cancelto{0}{y(0)}\right]$$

$$+ 6\,\mathcal{L}\{y(t)\} = 5\underbrace{\left(\frac{1}{p-3}\right)}_{\mathcal{L}(e^{3t})}$$

$$\Rightarrow \mathcal{L}(y(t))\left[p^2 + 5p + 6\right] = \frac{5}{p-3}$$

$$\Rightarrow \mathcal{L}\{y(t)\} = \frac{5}{(p+3)(p+2)(p-3)}$$

M1 Apply partial fraction.

M2 Use Convolution theorem

(M2)

$$y(t) = \mathcal{L}^{-1}\left[\frac{5}{(p^2-3^2)(p+2)}\right]$$

$$= 5\,\mathcal{L}^{-1}\left[\left(\frac{1}{(p^2-3^2)} \cdot \frac{1}{(p+2)}\right)\right]$$

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$$= 5\left[\mathcal{L}^{-1}\left(\frac{1}{p^2-3^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{p+2}\right)\right]$$

$$= 5\left[\frac{1}{3}\sinh(3t) * e^{-2t}\right]$$

$$= \frac{5}{3}\left[\int_0^t \sinh(3t')\, e^{-2(t-\tau)}\, d\tau\right]$$

$$= \frac{5}{3}\left[\int_0^t e^{(2\tau-2t)}\left(\frac{e^{3\tau}-e^{-3\tau}}{2}\right) d\tau\right]$$

$$= \frac{5}{6}\left[\int_0^t e^{-2t}\, e^{2\tau}\left(e^{3\tau}-e^{-3\tau}\right) d\tau\right]$$

$$= \frac{5}{6}\, e^{-2t}\left[\int_0^t e^{2\tau}\left(e^{3\tau}-e^{-3\tau}\right) d\tau\right]$$

$$- \frac{5}{6}\left[\int_0^t \left(e^{5\tau}-e^{-\tau}\right) d\tau\right]$$

$$= \frac{5}{6}\, e^{-2t}\left[\left(\frac{e^{5t}}{5}-\frac{e^{-t}}{-1}\right)-\left(\frac{1}{5}+1\right)\right]$$

$$= \frac{5}{6}\, e^{-2t}\left[\frac{e^{5t}}{5}+e^{-t}-\frac{6}{5}\right]$$

$$= \frac{e^{3t}}{6} + \frac{5e^{-3t}}{6} - e^{-2t}$$

2) $\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(t^2)$

$$\left[p^2\mathcal{L}(y(t)) - p\,\underbrace{y(0)}_{\to 0} - \underbrace{y'(0)}_{\to 0}\right] - \left[p\mathcal{L}(y(t)) - \underbrace{y(0)}_{\to 0}\right] = \frac{2}{p^3}$$

$$\Rightarrow \mathcal{L}(y(t))\left[p^2 - p\right] = \frac{2}{p^3}$$

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$$\Rightarrow \mathcal{L}(y(t)) = \frac{2}{p^4(p-1)}$$

M1 Convolution

M2 Any other direct method :-
- shifting rule
- Partial fraction...

(M2)

$$\mathcal{L}(y(t)) = \mathcal{L}^{-1}\left(\frac{2}{[(p-1)+1]^4 (p-1)}\right)$$

$$= 2e^t \mathcal{L}^{-1}\left(\frac{1}{(p+1)^4 p}\right)$$

$$= 2e^t \mathcal{L}^{-1}\left(\frac{1/(p+1)^4}{p}\right)$$

(M1)

$$y(t) = 2\mathcal{L}^{-1}\left[\left(\frac{1}{p^4}\right)\left(\frac{1}{p-1}\right)\right]$$

$$= 2\left[\mathcal{L}^{-1}\left(\frac{1}{p^4}\right) * \mathcal{L}^{-1}\left(\frac{1}{p-1}\right)\right]$$

$$= 2\left[\frac{t^3}{3!} * e^t\right]$$

$$= \frac{2}{6}\left[t^3 * e^t\right]$$

$$= \frac{1}{3}\int_0^t \tau^3 e^{t-\tau}\, d\tau$$

$$\Rightarrow y(t) = 2e^t - \frac{t^3}{3} - \frac{t^2}{2} - 2t - 2.$$

end of chapter

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Ch-

FOURIER SERIES

→ only possible for periodic f^n

§ Defin^n: Periodic f^{ns}:

If $f(x+T) = f(x)$, then, $f(x)$ is periodic with period T.

examples:-

$f(x)$	$\sin x$	$\cos x$	$\sin 2x$	$\cos 2x$
Period	2π	2π	π	π

*NOTE: Fourier series are expressed only for Periodic f^{ns}.

eg: non periodic f^{ns} can be made periodic by giving a cond^n:-

$$f(x) = x^2 \; ; \; f(x+2\pi) = f(x)$$

§ Fourier Series defin^n:-

The Fourier series of $f(x)$ defined in $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

→ $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

→ $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$

→ $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$

$\Big\} n \in \mathbb{Z}$

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* **Note**:- If ~~fo~~ Fourier series is defined in $(0, 2\pi)$, its given as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

↳ same as before

Fourier coefficients or Euler's consts.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx \end{aligned}\right\} n \in \mathbb{Z}$$

* Note:- If $f(x)$ is EVEN f^n, the Fourier series from $(-\pi, \pi)$ is defed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx)$$

$$\to a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$\to a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \;;\; n \in \mathbb{Z}$$

$$\to b_n = 0 \quad \forall\, n.$$

* Note:- If $f(x)$ is ODD f^n, the Fourier series ~~fo~~ defined in $(-\pi, \pi)$ is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\to a_0 = 0 = a_n \quad \forall\, n$$

$$\to b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\,dx \;;\; n \in \mathbb{Z}$$

Also called:
Cosine Half Range series

✯ FOURIER COSINE SERIES for $f(x)$
↳ defined in HALF INTERVAL: $(0, \pi)$

just like even f^n (in $-\pi, \pi$)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

↳ $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

↳ $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \; ; n \in Z$

✯ FOURIER SINE SERIES for $f(x)$
↳ defined in HALF INTERVAL: $(0, \pi)$ or $(-\pi, 0)$ ✯

just like what we get for ODD f^n (in $(-\pi, \pi)$)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

↳ $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

✯ NOTE: If the interval of Fourier Series expansion is $(-l, l)$, then, the full range Fourier series of $f(x)$ is given as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

↳ $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

↳ $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

↳ $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$\Big\}\, n \in Z$

* e^x : Neither even nor odd f^n.

Fourier series

* NOTE: Even & ODD expansion for $f(x)$ is possible in $(-l, l)$ also,
  if $f(x)$ is even in $(-l, l)$ or
  $f(x)$ is odd in $(-l, l)$

* NOTE: Fourier Cosine series & Sine series can be defined in a similar way for half interval $(0, l)$.

Q. Find Fourier series of:

$$f(x) = x^2 + x \quad \text{in } (-\pi, \pi)$$

Sol^n:- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\,dx = \frac{2\pi^2}{3}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\cos nx\,dx$

By property of Definite integrals

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; f(x) \text{ is even} \\ 0 & , f(x) \text{ is odd} \end{cases}$$

↳ where ODD f^n :

$f(-x) = -f(x) \quad \forall x$

EVEN f^n :

$f(-x) = f(x)$

eg: $x, x^3, \tan x, \sin x$

eg: $x^2, |x|, \cos x$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \underbrace{x^2 \cos(nx)}_{\text{even} \times \text{even} = \text{even}}\,dx + \int_{-\pi}^{\pi} \underbrace{x\cos(nx)}_{\text{odd} \times \text{even} = \text{odd}}\,dx\right]$$

So, $a_n = \frac{1}{\pi}\left[2\int_0^{\pi} x^2 \cos(nx)\,dx + 0\right]$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

From Bernoulli's expansion.

$$= \frac{2}{\pi}\left[\left[0 + \frac{2\pi \cos n\pi}{n^2} - 0\right] - 0\right]$$

$$\Rightarrow a_n = \frac{4}{n^2}\left[(-1)^n\right] \qquad \left[\because \cos n\pi = (-1)^n,\ \sin n\pi = 0\right]$$

So, now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \underbrace{x^2 \sin nx}_{\substack{(\text{even})\times(\text{odd}) \\ = (\text{odd})}}\,dx + \int_{-\pi}^{\pi} \underbrace{x \sin nx}_{\substack{(\text{odd})\times(\text{odd}) \\ = (\text{even})}}\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + 2\int_0^{\pi} x \sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi \cos n\pi}{n} + 0\right) - 0\right]$$

$$\Rightarrow b_n = \frac{-2\cos n\pi}{n} = -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}.$$

So, using values of a_0, a_n & b_n in Fourier series :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[(a_n \cos nx) + (b_n \sin nx)\right]$$

* Problems can come as:
Find Fourier series & hence, deduce value of this series . . . .

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$$\Rightarrow x^2+x = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\cos nx\right) + \sum_{n=1}^{\infty}\left(\frac{-2(-1)^n}{n}\sin nx\right)$$

Extra; If its asked → deduce that :

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

should be a cts pt. always.

Put $x=0$ → Idea:- $\cos nx$ has $1/n^2$ term. So, make others zero.

$$\Rightarrow 0^2+0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}(1) + 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

$$\Rightarrow \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots = -\frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$$

H.P

—o—

✫ NOTE : DIRICHLET'S CONDⁿ

The Fourier series of $f(x)$ converges to the f^n value $f(x)$ at all pts. of continuity & converges to $\frac{f(x_-)+f(x_+)}{2}$ at all pts of discontinuity.

↳ $f(x_-)$ = LHS limit of $f(x)$

↳ $f(x_+)$ = RHS limit of $f(x)$.

Date: 19/5/13

Q. Find the Fourier series of

(a) $$f(x) = \begin{cases} 0 & \text{in } (-\pi \leq x < 0) \\ x^2 & \text{in } 0 \leq x \leq \pi \end{cases}$$
where $f(x+2\pi) = f(x)$

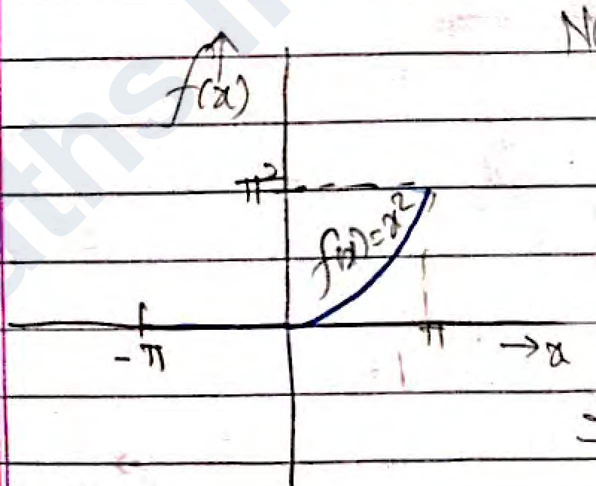
(b) deduce $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$,

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$.

(c) Sketch the graph of $f(x)$ in $-5\pi \leq x \leq 5\pi$

Sol^n (a) The Fourier series of any f^n in the interval $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$



Now,
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x^2\,dx\right]$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}. \quad \longrightarrow (1)$$

& $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\cos nx\,dx + \int_0^{\pi} \underset{u}{x^2} \underset{v}{\cos nx}\,dx\right]$$

Applying Bernoulli's formula,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$v_1 = \int v\,dx,\quad v_2 = \int v_1\,dx, \ldots$$
$$u' = \frac{d}{dx}u,\quad u'' = \frac{d}{dx}u', \ldots$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\cancel{\pi^2\frac{\sin n\pi}{n}} + 2\pi\frac{\cos n\pi}{n^2} - \cancel{2\frac{\sin n\pi}{n^3}}\right) - \left(0 + 0 - \cancel{2\frac{\sin(0)}{n^3}}\right)\right]$$

We know, $\sin n\pi = 0$ & $\cos n\pi = (-1)^n$

$$\Rightarrow a_n = \frac{1}{\cancel{\pi}}\left[2\cancel{\pi}\frac{(-1)^n}{n^2}\right] = \frac{2(-1)^n}{n^2};\quad n = 1, 2, \ldots$$

$$\Rightarrow a_n = \frac{2(-1)^n}{n^2};\quad n = 1, 2, 3, \ldots \longrightarrow (2)$$

New,

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\overbrace{\int_{-\pi}^{0}(0)\sin nx\,dx}^{\to 0} + \int_0^{\pi}\overset{u}{x^2}\,\underset{v}{\sin nx}\,dx\right]$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

cts : continuous

$\sin n\pi = 0$

$$= \frac{1}{\pi}\left[\left[\frac{-\pi^2(-1)^n}{n} + (0) + \frac{2(-1)^n}{n^3}\right] - \left(0 + (0)\frac{2}{n^3}\right)\right]$$

$\sin 0 = 0$

$$\Rightarrow b_n = \frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^3}\left[(-1)^n - 1\right] \quad ; n = 1, 2, 3, \ldots$$

$\longrightarrow$ (3)

So, the req^d F.S from (1), (2) & (3) is given as :-

$$f(x) = \underbrace{\frac{1}{2}\left(\frac{\pi^2}{3}\right)}_{a_0} + \sum_{n=1}^{\infty}\left[\overset{a_n}{\frac{2(-1)^n}{n^2}}\cos nx\right]$$

$$+ \sum_{n=1}^{\infty}\left[\underbrace{(-1)^{n+1}\frac{\pi}{n} + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]}_{b_n}\right]\sin nx$$

$\longrightarrow$ Valid when $x \in (-\pi, \pi)$

(b) Idea :- We need a series of the form

$$\left.\begin{array}{l} 1 - \frac{1}{2^2} + \frac{1}{3^2} \ldots \\ 1 + \frac{1}{2^2} + \frac{1}{3^2} \ldots \end{array}\right\} \text{i.e., } \frac{1}{n^2}$$

So, n^2 in denominator is req^d
Other terms in $f(x)$ should be got as zero or 1.

Now,
We can see from graph that $f(x)$ is cts at $x = 0$.

So, Put $x = 0$ in F.S

$$\Rightarrow f(0) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}(\overset{1}{\cos n \cdot 0}) + \sum_{n=1}^{\infty}\left[\quad\right]\overset{0}{\sin(n \cdot 0)}$$

$$\Rightarrow 0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}$$

$$\Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

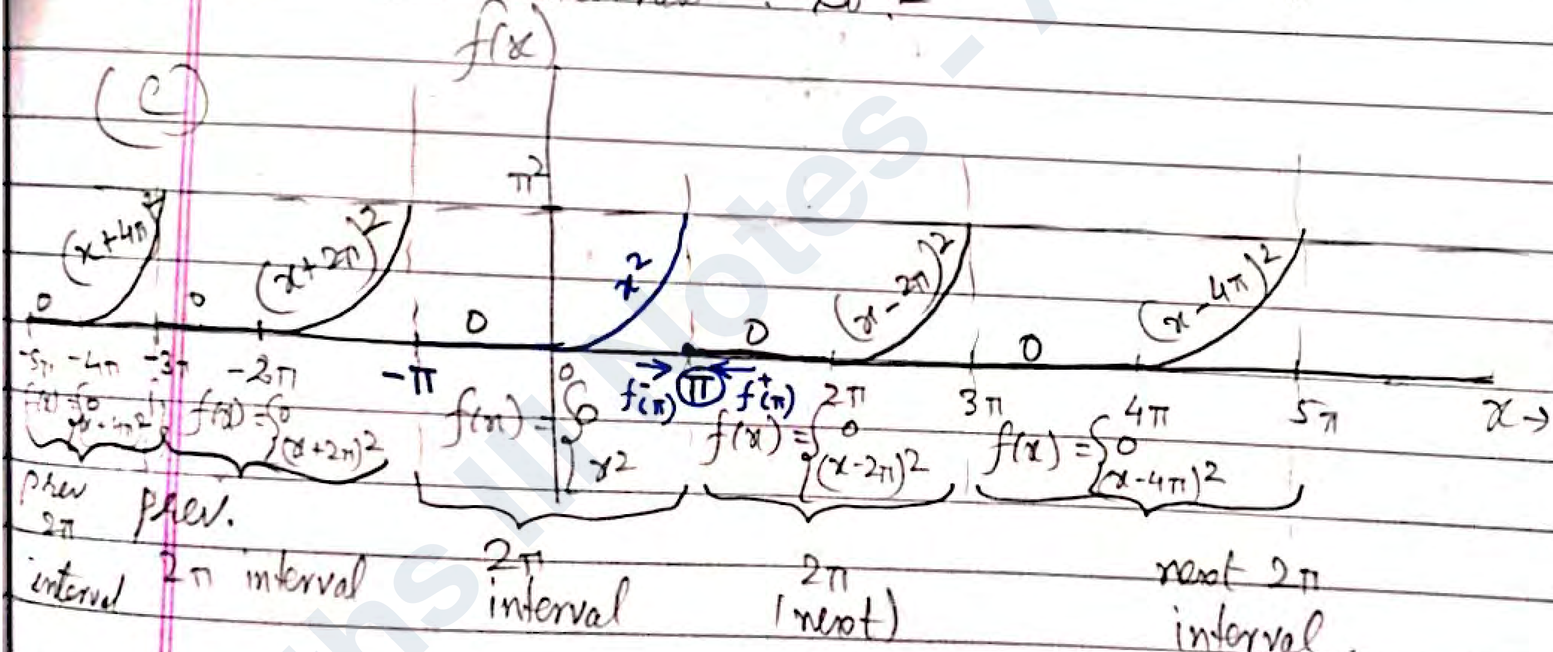
$$\Rightarrow \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{---}$$

Now, seeing the graph of $f(x)$ from $(-5\pi, 5\pi)$,

We know $f(x+2\pi) = f(x)$ (Given)

So, $f(x)$ is repeated after every 2π intervals. So :-

(C)



Considering $f(x)\big|_{x=\pi}$, $f^-(\pi) = \pi^2$ & $f^+(\pi) = 0$

So, $x = \pi$ is pt. of discontinuity.

So, By Dirichlet's condⁿ, the fⁿ converges to

$$\frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos n\pi + \underbrace{\sum_{n=1}[\;]\sin n\pi}_{\to 0}$$

$$\Rightarrow \frac{\pi^2 + 0}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

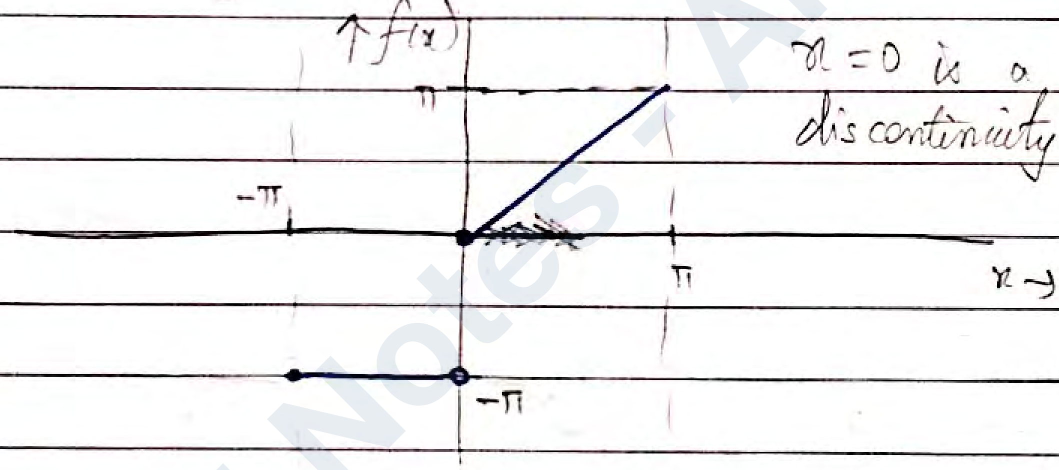
(b) $\Rightarrow \dfrac{\pi^2}{2} - \dfrac{\pi^2}{6} = 2\sum_{n=1}^{\infty} \dfrac{(-1)^{2n}}{n^2}$.

$$\Rightarrow \frac{1}{2}\left(\frac{\pi^2}{3}\right) = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Q Find the F.S of the f^n.

$$f(x) = \begin{cases} -\pi & ; -\pi \leqslant x < 0 \\ x & ; 0 \leqslant x \leqslant \pi \end{cases}$$



$x = 0$ is a pt. of discontinuity

Now, $a_0 = -\dfrac{\pi}{2}$.

$$a_n = \frac{1}{\pi}\left\{\frac{(-1)^n - 1}{n^2}\right\}$$

$$b_n = \frac{1 - 2(-1)^n}{n}$$

by solving using formula

$n = 1, 2, 3, \dots$

Simplifying

$$a_n = \begin{cases} \dfrac{-2}{\pi n^2} & n : \text{odd} \\ 0 & n : \text{even} \end{cases}$$

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$$b_n = \begin{cases} \frac{3}{n} ; & n = odd \\ -\frac{1}{n} ; & n = even \end{cases}$$

So, the reqᵈ F.S of $f(x)$ in $(-\pi, \pi)$ is :-

$$f(x) = \underbrace{\frac{1}{2}\left(\frac{-\pi}{2}\right)}_{a_0} + \sum_{n=1}^{\infty} \underbrace{\left[\frac{(-1)^n - 1}{n^2}\right]}_{a^n} \cos nx + \sum_{n=1}^{\infty} \underbrace{\left[\frac{1 - 2(-1)^n}{n}\right]}_{b^n} \sin nx$$

Valid for $x \in (-\pi, \pi)$
$n = 1, 2, 3, - - - -$

Put $x = 0$ in above series
Pt of discontinuity (from graph)

$\therefore$ $f(0^-) = -\pi$
$f(0^+) = 0$,

So, by Dirichlet's condⁿ, $f(x)$ reduces to

$$\frac{f(0^-) + f(0^+)}{2} = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)\cos 0 + \sum_{n=1}^{\infty}\left(\quad\right)\sin 0$$

$$\Rightarrow \frac{-\pi + 0}{2} = -\frac{\pi}{4} + \sum_{n=odd}\left(\frac{-2}{\pi n^2}\right) + \sum_{n=even}(0)$$

$$\Rightarrow \left(-\frac{\pi}{4}\right)\left(\frac{-\pi}{2}\right) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + - - - -$$

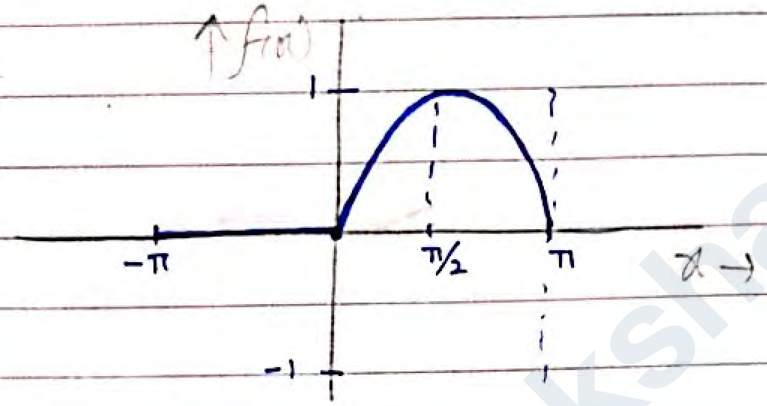
$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + - - - - -$$

Aliter :- $x = \pi$ is also a pt. of discontinuity (seen by extension of graph). So we get same result in it.

Q) F.S for

$$f(x) = \begin{cases} 0 & ; -\pi \leq x < 0 \\ \sin x & ; 0 \leq x \leq \pi \end{cases}$$

Graph :-



Using formula for finding Fourier coeff :-

$$a_0 = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(x)\cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\cos nx \, dx + \int_{0}^{\pi} \sin x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} \sin x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} \frac{1}{2}\left[\sin(n+1)x - \sin(n-1)x\right] dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{-(-1)^{n+1}}{n+1} + \frac{(1)^{n-1}}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$\left(\because \cos n\pi = (-1)^n \Rightarrow \cos(n+1)\pi = (-1)^{n+1}\right)$$

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$$\Rightarrow a_n = \frac{1}{2\pi}\left[\left[\frac{-(-(-1)^n)}{n+1} + \frac{(-(-1)^n))}{n-1}\right] + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right].$$

$$\left(\because (-1)^{n+1} = -(-1)^n. \quad (-1)^{n-1} = -(-1)^n\right)$$

$$\Rightarrow a_n = \frac{1}{2\pi}\left[(-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right] + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[\left((-1)^n + 1\right)\left[\frac{1}{n+1} - \frac{1}{n-1}\right]\right]$$

$$= \frac{1}{\cancel{2}\pi}\left[\frac{-\cancel{2}}{(n+1)(n-1)} \times \left((-1)^n + 1\right)\right]$$

$$\Rightarrow a_n = -\frac{\boxed{(-1)^n + 1}}{\pi(n^2-1)} = \begin{cases} 0, & n \text{ is odd} \\ \dfrac{-2}{\pi(n^2-1)}, & n \text{ is even} \end{cases}$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin(nx)\,dx + \int_0^{\pi} \underbrace{\sin(x)}_{\Downarrow \sin B}\,\underbrace{\sin(nx)}_{\Downarrow \sin A}\,dx\right]$$

$$\left(\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]\right)$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \frac{1}{2}\left[\cos(n-1)x - \cos(n+1)x\right]dx\right]$$

$$\Rightarrow b_n = \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$\hookrightarrow$ not valid when $n = 1$

$$\left(\because \frac{1}{n-1} \longrightarrow \infty\right)$$

So, $b_n = \dfrac{1}{2\pi}\left[\dfrac{\overset{\to 0}{\sin(n-1)\pi}}{n-1} - \dfrac{\overset{\to 0}{\sin(n+1)\pi}}{n+1}\right.$

$\left. - \left(\dfrac{\overset{\to 0}{\sin 0}}{n-1} - \dfrac{\overset{\to 0}{\sin 0}}{n+1}\right)\right]$; not valid for $n=1$

$\Rightarrow b_n = 0 \;\; \forall$ values of $n - \{1\}$

Finding b_n for $n=1$ directly
i.e b_1, by Euler's formula

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(1\cdot x)\,dx$$

[Put $n=1$ in b_n formula]

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin x\,dx + \int_{0}^{\pi}\sin x\cdot\sin x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx\right]$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[(\pi - 0) - (0-0)\right]$$

$$\Rightarrow b_1 = \frac{1}{2} = b_n\Big|_{n=1}$$

So, the req^d Fourier series for $f(x)$:-

$$f(x) = \frac{(2/\pi)}{2} + \sum_{n=2,4,6\ldots}^{\infty}\frac{-2}{\pi(n^2-1)}\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Idea: Now, we have $\sum_{n=2,4,6\ldots}^{\infty}$. To convert it to $\sum_{n=1,2,3,\ldots}$

do this :- $n \longrightarrow 2n$ in eqn.

IIly, for $\sum_{n=1,3,5,\ldots}^{\infty} \longrightarrow \sum_{n=1,2,3,\ldots}^{\infty}$

do :- $n \longrightarrow (2n+1)$

So, $$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{\boxed{(2n)^2-1}}\cos(2nx) + b_1\sin(1x) + \sum_{n=2} b_n \sin nx$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}\cos 2nx + \frac{1}{2}\sin x + \sum_{n=2}^{\infty} 0.\sin nx$$

$$\Rightarrow f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}\cos 2nx + \frac{1}{2}\sin x$$

↳ valid for $x \in (-\pi, \pi)$

Q. Find FS for the fn,

$f(x) = \cos\left(\frac{x}{2}\right)$ in $-\pi \leq x \leq \pi$

↳ even fn

↳ symmetric interval

check these 2.

So, FS of even fn in $[-\pi, \pi]$ is :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$ ($\exists$ no sine terms & $b_n = 0$)

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So, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$ (for even f^n)

$= \frac{2}{\pi}\int_{0}^{\pi} \cos\left(\frac{x}{2}\right)dx = \frac{2}{\pi}\left[\frac{\sin(x/2)}{1/2}\right]_0^{\pi}$

$= \frac{4}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin 0\right]$

$\Rightarrow a_0 = \frac{4}{\pi}$

& $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi} \cos\left(\frac{x}{2}\right)\cos(nx)\,dx$

$= \frac{2}{\pi}\left[\int_0^{\pi}\frac{1}{2}\left[\cos\left(n+\frac{1}{2}\right)x + \cos\left(n-\frac{1}{2}\right)x\right]dx\right]$

$= \frac{1}{\pi}\left[\frac{\sin\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)} + \frac{\sin\left(n-\frac{1}{2}\right)x}{\left(n-\frac{1}{2}\right)}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\left[\frac{\sin\left(n+\frac{1}{2}\right)\pi}{\left(n+\frac{1}{2}\right)} + \frac{\sin\left(n-\frac{1}{2}\right)\pi}{n-\frac{1}{2}}\right] - (0)\right]$

$\rightarrow \sin\left(n\pi + \frac{\pi}{2}\right) = \sin(n\pi)\cos\frac{\pi}{2} + \cos(n\pi)\sin\frac{\pi}{2}$

$= \cos(n\pi) = (-1)^n$

$\rightarrow \sin(n\pi - \pi/2) = \sin n\pi \cos\frac{\pi}{2} - \cos(n\pi)\sin\pi/2$

$= -\cos n\pi = -(-1)^n$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{(-1)^n}{(n+1/2)} + \frac{-(-1)^n}{(n-1/2)}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n-1}\right]$$

$$= \frac{2(-1)^n}{\pi}\left[\frac{2n-1-2n-1}{4n^2-1}\right]$$

$$\Rightarrow a_n = -\frac{4}{\pi}(-1)^n\left[\frac{1}{4n^2-1}\right]$$

So, FS becomes,

$$f(x) = \cos\frac{x}{2} = \frac{(4/\pi)}{2} + \sum_{n=1}^{\infty}\left[\frac{-4(-1)^n}{\pi(4n^2-1)}\right]\cos nx$$

$$\Rightarrow \cos\frac{x}{2} = \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{(-4)(-1)^n}{\pi(4n^2-1)}\cos nx$$

Ans

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* Note:-

If $f(x) = \begin{cases} g_1(x) & \text{in } -\pi \leq x < 0 \\ g_2(x) & \text{in } 0 \leq x \leq \pi \end{cases}$

&

If $g_1(-x) = g_2(x)$ or $g_2(-x) = g_1(x)$

then, $f(x)$ is even in $(-\pi, \pi)$

&

If $g_1(-x) = -g_2(x)$ or $g_2(-x) = -g_1(x)$

then, $f(x)$ is odd in $(-\pi, \pi)$.

Q. Find FS for given f^n :-

$$f(x) = \begin{cases} x + \pi/2 & ; -\pi \leq x < 0 \\ -x + \pi/2 & ; 0 \leq x \leq \pi \end{cases}$$

let $g_1(x) = x + \frac{\pi}{2}$ & $g_2(x) = -x + \frac{\pi}{2}$

& $g(-x) = -x + \frac{\pi}{2} = g_2(x)$

So, $g(-x) = g_2(x)$

So, $f(x)$ is even f^n

$\Rightarrow b_n = 0$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(-x+\frac{\pi}{2}\right)dx$

$$= \frac{2}{\pi}\left[-\frac{x^2}{2} + \frac{\pi x}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\frac{\pi^2}{2} + \frac{\pi^2}{2}\right) - (0-0)\right]$$

$\Rightarrow a_0 = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(-x+\frac{\pi}{2}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[-\int_0^{\pi} x\cos nx\,dx + \left(\frac{\pi \sin nx}{2n}\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{x\sin nx}{n} - \frac{\cos nx}{n} + \frac{\pi}{2n^2}\sin nx\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\pi\frac{\sin n\pi}{n} - \frac{\cos n\pi}{n^2} + \frac{\pi}{2n}\sin n\pi - \left[0 - \frac{\cos n(0)}{n^2} + 0\right]\right]$$

$$\Rightarrow a_n = \left[\left[0 - \frac{(-1)^n}{n^2} + 0\right] + \frac{1}{n^2}\right] \times \frac{2}{\pi}$$

$$= \frac{2}{\pi n^2}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0 & n : \text{even} \\ \frac{4}{\pi n^2} & n : \text{odd.} \end{cases}$$

$$\text{So } f(x) = \frac{0}{2} + \sum_{n=1,3,5} \frac{4}{\pi n^2} \cos nx$$

Put $n \to 2n+1$

$$\text{So, } f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \left[\cos(2n+1)x\right]$$

Q. Given $f(x) = \pi - x$

(a) Find cosine series in $0 \leq x \leq \pi$ — i.e. for even fn

(b) Sine series in $0 \leq x \leq \pi$ — i.e., for odd fn

(a) cosine series of $f(x)$ in $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$\text{here, } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\frac{(\pi - x)^2}{2(-1)}\right]_0^{\pi} = \frac{1}{\pi}\left[0 - \left(\frac{\pi^2}{-1}\right)\right] = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx \, dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\left(0 - \frac{(-1)^n}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right]$$

$\therefore$ Reqd cosine series for $(\pi - x)$ is

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty}\left(\frac{4}{\pi n^2}\right)\cos nx$$

$$n \to 2n+1$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi(2n+1)^2}\cos(2n+1)x.$$

Sine series of $f(x)$ in $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

So, $b_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin(nx)\, dx$

$$= \frac{2}{\pi}\left[(\pi - x)\left[-\frac{\cos nx}{n}\right] - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

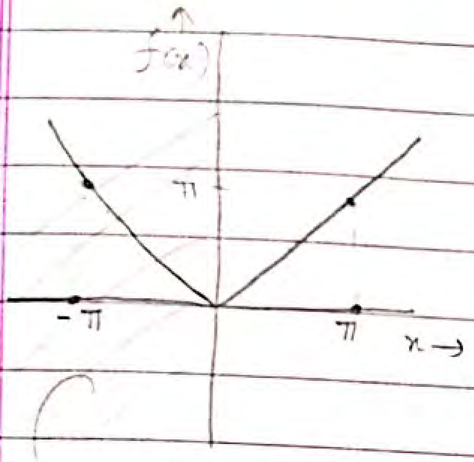
$$= \frac{2}{\pi}\left[\left(0 + \frac{\pi}{n}\right)\right] = \frac{2}{n}.$$

$\therefore$ Sine series :-

$$\pi - x = \sum_{n=1}^{\infty}\left(\frac{2}{n}\right)\sin(nx)$$

Q) Find the Fourier cosine, sine series for $f(x) = |x|$ in $0 \leq x \leq \pi$

$$f(x) = |x| = \begin{cases} x & ; x > 0 \supset x \in (0, \pi) \\ -x & ; x < 0 \end{cases}$$



Fourier cosine series:-

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \pi$$

The f^n is even. So, consider only one part for sine & cosine series

$$a_n = \frac{2}{\pi}\int_0^{\pi} |x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$\Rightarrow a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow a_n = \begin{cases} 0 & , n = \text{even} \\ \frac{-4}{\pi n^2} & , n = \text{odd} \end{cases}$$

Req^d Fourier cosine series:

$$|x| = \frac{\pi}{2} + \sum_{n=\text{odd}} \left(\frac{-4}{\pi n^2}\right) \cos nx$$

Valid $\forall x \in [0, \pi]$

$$= \frac{\pi}{2} \sum_{n=1}^{\infty} \left[\frac{-4}{\pi (2n+1)^2} \cos(2n+1)x\right]$$

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★ Note: The regular Fourier Series for $f(x) = |x|$ in $-\pi \leq x \leq \pi$ is exactly same as that of above.

i.e Cosine series of $|x|$ in $[0, \pi]$ is same as Regular FS of $|x|$ in $[-\pi, \pi]$.

Now,

Fourier sine series of $|x|$ in $0 \leq x \leq \pi$

Sine series of any f^n, $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

$\hookrightarrow a_0 = 0 = a_n$ for sine series

Now, $b_n = \frac{2}{\pi}\int_0^{\pi} |x| \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx$$

$(\because |x| = x, x > 0)$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left[\frac{\sin nx}{n^2}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n} - 0\right]$$

$$= \frac{-2(-1)^n}{n}$$

$\therefore$ reqd Fourier sine series

$$|x| = \sum_{n=1}^{\infty}\left(\frac{-2(-1)^n}{n}\right)\sin nx$$

So, $|x| = +2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right]$

Ans

* $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
* $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Q Find FS of.

Sol
a) $\sin^2 x$
b) $\cos^2 x$
c) $\sin^3 x$
d) $\cos^3 x$ in $(-\pi, \pi)$

Simplificn (a) $\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$

(b) $\cos^2 x = \frac{1+\cos 2x}{2}$

(c) $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

(d) $\cos^3 x = \frac{\cos 3x}{4} + \frac{3}{4}\cos x$

Q. Find sine series of the fn,
$f(x) = \cos x$ in $0 \leq x \leq \pi$
$\cos x \Rightarrow$ even. So,

$$b_n = \frac{2}{\pi}\int_0^\pi \cos x \sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^\pi \left[\frac{\sin(n+1)x + \sin(n-1)x}{2}\right] dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\left(\frac{-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right) - \left(\frac{-1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1}\right) + \left(\frac{1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2n((-1)^n + 1)}{n^2 - 1}\right]$$

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$$\Rightarrow b_n = \begin{cases} 0 & ; n = \text{odd} \\ \dfrac{4n}{\pi(n^2-1)} & ; n = \text{even} \end{cases}$$

$\hookrightarrow$ not valid when $n=1$

for $n=1$, finding b_1 directly

Put $n=1$ in b_n formula,

$$\Rightarrow b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(1x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos x \sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2\pi}{2} - \left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2} - \frac{1}{2}\right] = 0$$

So, the Fourier sine series of $\cos x$ in $[0,\pi]$ is

$$\cos(x) = \sum_{n=2,4,6,\ldots} \frac{4n}{\pi(n^2-1)} \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{4(2n)}{\pi[(2n)^2-1]} \sin(2n)x$$

$$\Rightarrow \cos x = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n}{(4n^2-1)} \sin(2nx)$$

Ans

☆ $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

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Problems :- Interval changes from $(-\pi, \pi) \to (-l, l)$

Q. Find FS for $\cos(\pi x)$ in $-1 \leq x \leq 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

↳ FS of $f(x)$ in $(-l, l)$

here, $l = 1$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

Finding coeffs :-

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{1}\int_{-1}^{1} \cos \pi x\,dx$$

$$= \left[\frac{\sin \pi x}{\pi}\right]_{-1}^{1} = 0.$$

Now, $\because$ $\cos \pi x$ is even in $(-1, 1)$

↳ symmetric interval

So, $b_n = 0$

$$\text{Now, } a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= 2\int_0^1 (\cos \pi x)\cos(n\pi x)\,dx$$

$$= \frac{2}{2}\int_0^1 \left[\cos(\pi x + n\pi x) + \cos(\pi x - n\pi x)\right] dx$$

$$= \int_0^1 \left(\left[\cos(n+1)\pi x\right] + \left[\cos(n-1)\pi x\right]\right) dx$$

$$= \left[\frac{\sin(n+1)\pi x}{(n+1)\pi} + \frac{\sin(n-1)\pi x}{n-1}\right]_0^1$$

↳ not valid for $n = 1$; valid $\forall n - \{1\}$

* eg. $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $-\pi \leq x \leq \pi$

↳ neither even nor odd.

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$\Rightarrow a_n = 0$

~~Solving separately for~~ a_1 ($n = 1$)

$$\Rightarrow a_1 = 2\int_0^1 (\cos \pi x)\cos(1.\pi x)\, dx = 1$$

So, F.S becomes

$$\cos(\pi x) = 0 + a_1 \cos(\pi x) + 0$$

$$\Rightarrow \cos(\pi x) = 1 . \cos \pi x .$$

↳ Basically, cosine series for cosine of x is same f^n.

Q(a) $f(x) = \begin{cases} 1 + x & ; -1 \leq x < 0 \\ 1 - x & ; 0 \leq x < 1 \end{cases}$

(b) $f(x) = |x|$; $-2 \leq x \leq 2$.

(a) $g_1(-x) = 1 - x = g_2(x)$

So, $f(x)$ is even.

$\Rightarrow b_n = 0$

$$a_0 = 2\int_0^1 (1 - x)\, dx = 1$$

$$a_n = \frac{2}{1}\int_0^1 f(x) \cos\left(\frac{n\pi x}{1}\right) dx = -4$$

Final F.S =

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{1}\right)$$

$\rightarrow -4$

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(b) here, $l = 2$
& its even fn.
So, $b_n = 0$
& $a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\int_0^2 |x|\, dx = 2$

$$a_n = \frac{2}{2}\int_0^2 x \cdot \cos\left(\frac{n\pi x}{2}\right)dx = \begin{cases} 0 & ; n = \text{even} \\ \frac{-8}{\pi^2 n^2} & ; n = \text{odd} \end{cases}$$

So, FS becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1,3,5\ldots}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow f(x) = 1 + \sum_{n=1,2,3\ldots}^{\infty} \frac{-8}{\pi^2(2n)^2}\cos(\pi x)$$

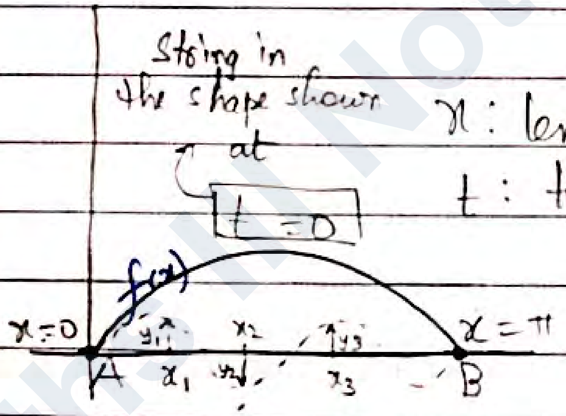
Application of Fourier Series

I. ONE DIMENSIONAL WAVE EQN.

$$\boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}} \longrightarrow (1)$$

$\to$ y $\to$ depends on x & t.

y: dependent variable; x & t: ~~Ide~~ Independent Variables



x: length, t: time } y: displacement of string at x & at time t..

Displacement y_1, y_2, y_3 at lengths x_1, x_2 & x_3.

Assumptions:
- Tension is only force
- No gravit^n $\exists$

So, by Newton's 2nd Law
$$F = ma = m\frac{d^2y}{dt^2}$$

Assumptions:
Ends A & B are tight & won't move $\Rightarrow$

Boundary Cond^n
(i) $y(0, t) = 0$
(ii) $y(\pi, t) = 0$
} $\exists$ no displacement (value of y) $\forall$ time t at $x=0$ & $x=\pi$.

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Initial Cond^ns (t=0)

(iii) $\frac{dy}{dt}(x,0) = 0$; $0 \leq x \leq \pi$ (i.e., at time t = 0, the vel. of string = 0 & it has the shape shown)

(iv) $y(x, 0) = f(x)$; $0 \leq x \leq \pi$ (the value of fn at t = 0 is f(x))

* **Method for solving questions:**
Solve eq^n ① satisfying the cond^ns (i), (ii), (iii) & (iv).

S1) Solve eq^n ① by method of separ^n of variables & applying cond^ns (i), (ii), (iii), ~~(iv)~~ gives general sol^n of ①.

So,
$$y_{(x,t)} = \sum_{n=1}^{\infty} B_n \sin(nx)\cos(nat) \longrightarrow ②$$

S2) Apply cond^n (iv) in ②

$$\Rightarrow y(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx)\cos(na(0))$$

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) = f(x) \; ; \; 0 \leq x \leq \pi \quad ③$$

Now, express f(x) as the Fourier sine series in the interval $0 \leq x \leq \pi$

So,
$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \; ④ ; \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \; ⑤$$
$n \in \mathbb{Z}^+$

Comparing (3) & (4), we get

$$B_n = b_n$$
$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \quad (\text{from } (5))$$

Substituting this B_n in (2), we get reqd ans.

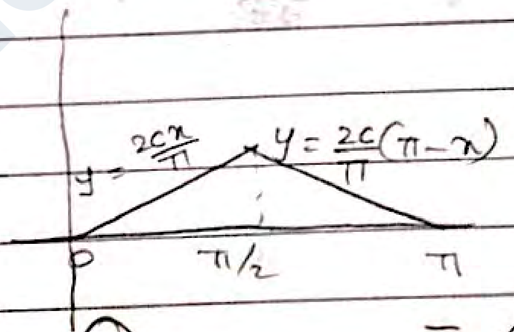
Q. Solve the vibrating string problem when initial shape is given by

Case (a) $f(x) = \begin{cases} \dfrac{2cx}{\pi} & 0 \leq x \leq \dfrac{\pi}{2} \\ \dfrac{2c(\pi - x)}{\pi} & \dfrac{\pi}{2} \leq x \leq \pi \end{cases}$

self Case (b) $f(x) = \dfrac{1}{\pi}x(\pi - x)$; $0 \leq x \leq \pi$

Sol^n

Case (a) :-



≡ a string of length π & its held from centre

Wave eq^n, from (1)

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \rightarrow (1)$$

Satisfying the cond^ns —

(i) $y(0,t) = 0$

(ii) $y(\pi, t) = 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0$

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(iv) $y(x,0) = \begin{cases} \dfrac{2cx}{\pi} & 0 \le x \le \pi/2 \\ \dfrac{2c}{\pi}(\pi - x) & \dfrac{\pi}{2} \le x \le \pi \end{cases}$

The general solⁿ of DE (1) under condⁿˢ (i), (ii) & (iii) we got

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)\cos(nat) \longrightarrow (2)$$

Applying (iv) in (2)

$$\Rightarrow y(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx)\cos(0) = \begin{cases} \dfrac{2cx}{\pi}, & (0, \frac{\pi}{2}) \\ \dfrac{2c}{\pi}(\pi - x), & (\frac{\pi}{2}, \pi) \end{cases} \longrightarrow (3)$$

Express the fⁿ $y(x,0)$ into sine series in $(0,\pi)$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) \longrightarrow (4)$$

$$\hookrightarrow b_n = \frac{2}{\pi}\int_0^{\pi} y(x,0)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{2cx}{\pi}\sin(nx)\,dx + \int_{\pi/2}^{\pi} \frac{2c}{\pi}(\pi - x)\sin(nx)\right]$$

$$\Rightarrow b_n = \frac{2}{\pi}\left(\frac{2c}{\pi}\right)\left[x\left(\frac{-\cos nx}{n}\right) + (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left((\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right)_{\pi/2}^{\pi}$$

$$\Rightarrow b_n = \frac{4c}{\pi^2}\left[\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2}\right) + \frac{1}{n^2}\left(\sin\frac{n\pi}{2}\right) - (0+0)\right] + \left[(0-0) - \left(-\frac{\pi}{2n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2}\right)\right]\right]$$

$$\Rightarrow b_n = \frac{8c}{\pi^2 n^2} \sin\left(\frac{n\pi}{2}\right)$$

Comparing (3) & (4),

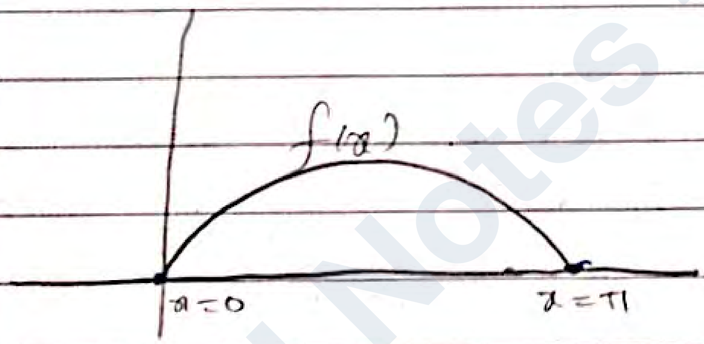
$$B_n = b_n = \frac{8c}{\pi^2 n^2} \sin\left(\frac{n\pi}{2}\right)$$

Substitute the value of B_n in general sol (2)

$$\Rightarrow y(x,t) = \sum_{n=1}^{\infty} \underbrace{\left[\frac{8c}{\pi^2 n^2} \sin\left(\frac{n\pi}{2}\right)\right]}_{B_n} (\sin nx)(\cos nat)$$

Self

Case (b) $f(x) = \frac{1}{\pi} x(\pi - x)$



II ONE DIMENSIONAL HEAT EQUATION:-

$$\boxed{\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}} \longrightarrow (1)$$

$$\alpha^2 = \frac{K}{\rho C}$$

K → conductivity

ρ → Resistivity

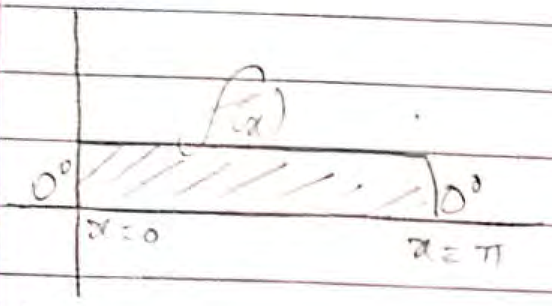
C → specific heat capacity

u : temperature

x : distance

t : time

- This is derived from Fourier law of heat conduction.



Consider the extreme ends of the wire.
If the entire body is heated up & only the ends are then cooled to 0°C, then, ∃ heat distribⁿ from center to sides. Hence, a fⁿ of initial heat distribⁿ will exist, called as $f(x)$.

Boundary condⁿˢ:-
(i) $u(0,t) = 0$
(ii) $u(\pi,t) = 0$
(iii) $u(x,0) = f(x)$
$\quad \hookrightarrow 0 < x < \pi$

The general solⁿ of ① under condⁿ (i) & (ii)

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(nx) \longrightarrow ②$$

Now,
Apply (iii) in ② & use Fourier sine series to get the value of C_n.

Q. Solve 1D heat eqⁿ under the boundary condⁿˢ:-
(i) $u(0,t) = 0$
(ii) $u(\pi,t) = 0$
(iii) $u(x,0) = kx(\pi - x)$, $0 < x < \pi$

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Sol^n :- The 1D heat eq^n is given as

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial t^2} \longrightarrow (1)$$

The general sol^n of (1) under cond^{ns} (i) & (ii) is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(nx) \longrightarrow (2)$$

Applying cond^n (iii) in (2)

$$u(x,0) = \sum_{n=1}^{\infty} C_n e^{0} \sin(nx) = kx(\pi - x)$$
$\hookrightarrow 0 < x < \pi$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin(nx) = kx(\pi - x) \rightarrow (3)$$
$\hookrightarrow x \in (0,\pi)$

Expanding $u(x,0)$ into half range sine series in $(0,\pi)$

$$\Rightarrow u(x,0) = k\,x(\pi - x) = \sum_{n=0}^{\infty} b_n \sin(nx) \rightarrow (4)$$

From (3) & (4)

$$C_n = b_n$$

But $b_n = \frac{2}{\pi}\int_0^{\pi} kx(\pi - x)\sin(nx)\,dx$

$$\Rightarrow C_n = b_n = \begin{cases} 0 & ; n : \text{even} \\ \frac{8u}{\pi n^3} & ; n : \text{odd} \end{cases}$$

Substituting C_n in ②

$\Rightarrow$

$$U(x,t) = \sum_{n=1}^{\infty} \left(\frac{8u}{\pi n^3}\right) e^{-n^2\pi^2 t} \sin(n\pi x)$$

(final sol^n)

End of Course