MATHEMATICS II

FIRST YEAR NOTES

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Mathematics II Algebra Notes, First Edition

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Chapter - 2
Linear Algebra
System of Linear equations .
Section - 2.3

★ Elementary Transform^{ns}
The following are referred to as elementary row transform^{ns} / oper^{ns} (eros).
① Interchanging of any 2 rows.
② Multiplication of all the elements of a row by a non-zero scalar.
③ Addition or subtraction of scalar multiple of one row with any other row.

∴ Note: Replacing rows by col^{mns} in above oper^{ns}, we get elementary column oper^{ns} (ecos).

★ Equivalent matrices:
2 matrices A & B are said to be equivalent if one is obtained from the other by using elementary transform^{ns} & we write $A \sim B$.

★ Rank of a matrix
The order of the largest non vanishing minor of a matrix A is called the rank of A & is denoted by $r(A)$, $\rho(A)$ or $R(A)$

- Note:
Equivalent matrices have the same rank.

eg Find the rank of:

① $A = \begin{bmatrix} 1 & 3 & -1 \\ 4 & 5 & 1 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & 3 & -1 \\ 4 & 1 & 5 \\ 2 & 6 & -2 \end{bmatrix}$

① Consider $\begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} = 5-12 = -7 \neq 0.$

$\therefore \; r(A) = 2.$

②
$$|A| = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 1 & 5 \\ 2 & 6 & -2 \end{vmatrix}$$
$$= -32 + 54 - 22 = 0.$$

$\therefore \; r(A) \neq 3.$

Consider $\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = 1-12 = -11 \neq 0.$

$\therefore \; r(A) = 2.$

(We shall use elementary row operⁿˢ to find the rank

$$A = \begin{bmatrix} \boxed{\overset{\text{Pivot element}}{\boxed{1}} \quad 3 \quad -1} \\ 4 \quad\; 1 \quad\; 5 \\ 2 \quad\; 6 \quad\; -2 \end{bmatrix} \xrightarrow{} \text{Pivot row}$$

making upper Δlar matrix

$$\sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -11 & 9 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_2 - 4R_1 \\ R_3 - 2R_1 \end{matrix}$$

So, ☆ $\boxed{r(A) = \text{no. of non zero rows}} = 2$

** To make a matrix to	* Make it
REF	Upper Δlar matrix
RREF	Diagonal Matrix

* Reduced Row Echelon Form (RREF)
(Also called Gauss Jordon Form)

A matrix is said to be in RREF if the following condⁿˢ are satisfied:

① In any non-zero row, the all leading non zero entry is 1 (or leading 1)

② The leading non zero entry 1 should occur in the successive colᵐˢ (to the right).

③ All the elements in the column containing leading 1 are zero.

④ Zero rows, if any occur below the non zero rows.

eg: The matrix $A = \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 \end{bmatrix}$ is in RREF

The matrix $B = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \end{bmatrix}$ is not in RREF.

* Row Echelon Form (REF)
(Also called Gauss Form).

A matrix is said to be in REF if the condⁿˢ 1, 2 & 4 (of RREF) are true.

Note:

- If a matrix is in REF or RREF, then, the rank is the no. of non zero rows
- While reducing the matrix to REF or RREF, we use only elementary row operⁿˢ.

Note:

- An upper Δlar or lower Δlar matrix is also referred to be in REF (not RREF).

Section - 2.1

§* System of Linear equations.

consider sys. of eqns:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \rightarrow ①.$$

The system ① can be written as

$$AX = B \longrightarrow ①,$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

coefficient matrix — RHS constt vector — unknown vector.

The system ① is said to be consistent, if it has a soln. Otherwise, it is said to be inconsistent.

* If $B=0$, i.e, the system
  $AX=0$ is called a homogeneous sys. of eqns.
  $B \neq 0$, then, ① is called
  $(AX \neq 0)$ a non-homogeneous sys of eqns

Section - 2.2 & 2.3.

§ Sol^{ns} to Non-Homogeneous Sys. of Eqns.

Let $AX = B \longrightarrow$ ①

be a non-homogeneous sys., with m eq^{ns} in n unknowns.

S1 Form the AUGMENTED matrix $(A:B)$.

S2 Reduce the augmented matrix into
M(i) REF (Gaussian Elimin^{n} method)
M(ii) RREF (Gauss Jordon Elim method)
using elementary row oper^{ns}.

S3 (i) If $r(A:B) = r(A) = n$, the no. of unknowns, then, the sys. is CONSISTENT with a UNIQUE SOLUTION.

(ii) If $r(A:B) = r(A) < n$, the no. of unknowns, then, the sys. is CONSISTENT with an INFINITE NO. OF SOLUTIONS.

(iii) If $r(A:B) \neq r(A)$, then, sys is INCONSISTENT with NO SOLUTION.

S4 To find the sol^{n}, if it exists, we proceed as follows:
(i) In REF, we use BACK SUBSTITUTION METHOD.
(ii) In RREF, we use ACTUAL SUBSTITUTION METHOD.

eg ① The sys. $x + y = 3$, $2x + 3y = 8$ is consistent with a unique sol^{n}, $x = 1$, $y = 2$

$$\begin{aligned} x + y &= 3 \\ 2x + 3y &= 8 \end{aligned}$$

② The sys. $x + y = 3$ is consistent with many sol^{ns}.

$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$

③ The sys. $x + y = 3$ is inconsistent.

$$\begin{aligned} x + y &= 3 \\ x + y &= -1 \end{aligned}$$

Q. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 5 & 1 & 6 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

by reducing into RREF.

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 5 & 1 & 6 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

(Reducing it to diagonal matrix).

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 9 & -7 & -6 \\ 0 & 2 & -2 & -1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - 4R_1 \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 9 & 0 & 11 & 21 \\ 0 & 9 & -7 & -6 \\ 0 & 0 & -4 & 3 \end{bmatrix} \begin{matrix} 9R_1 + R_2 \\ R_2 \\ 9R_3 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 36 & 0 & 0 & 117 \\ 0 & 36 & 0 & -45 \\ 0 & 0 & -4 & 3 \end{bmatrix} \begin{matrix} 4R_1 + 11R_3 \\ 4R_2 - 7R_3 \\ \ \end{matrix}$$

Leading 1,

$$\sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 117/36 \\ 0 & \textcircled{1} & 0 & -5/4 \\ 0 & 0 & \textcircled{1} & -3/4 \end{bmatrix} \begin{matrix} R_1/36 \\ R_2/36 \\ R_3/-4 \end{matrix}$$

This matrix is in RREF. So, $r(A)$ = no. of non zero rows in the reduced matrix = 3.

Aliter. By REF

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 5 & 1 & 6 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 9 & -7 & -6 \\ 0 & 2 & -2 & -1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - 4R_1 \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 9 & -7 & -6 \\ 0 & 0 & -4 & 3 \end{bmatrix} \begin{matrix} \\ R_2 \\ 9R_3 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -7/9 & -2/3 \\ 0 & 0 & 1 & -3/4 \end{bmatrix} \begin{matrix} R_2/9 \\ \\ R_3/-4. \end{matrix}$$

The matrix is in REF &
$\rho(A) = 3$.

Q. Use Gaussian Elimin^n method to solve the following sys. of eq^{ns}. Also, if sol^{ns} ~~is~~ are infinite, specify, any 3 particular sol^{ns}.

① $-5x_1 - 2x_2 + 2x_3 = 14$
$3x_1 + x_2 - x_3 = -8$
$2x_1 + 2x_2 - x_3 = -3$.

② $3x_1 - 3x_2 - 2x_3 = 23$
$-6x_1 + 4x_2 + 3x_3 = -38$
$-2x_1 + x_2 + x_3 = -11$

① The augmented matrix (A : B) is given by

$$\left[\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 3 & 1 & -1 & -8 \\ 2 & 2 & -1 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 0 & -1 & 1 & 2 \\ 0 & 6 & -1 & 13 \end{array}\right] \begin{array}{l} 5R_2+3R_1 \\ 5R_3+2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 5 & 25 \end{array}\right] R_3+6R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2/5 & -2/5 & -14/5 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} R_1/-5 \\ R_2/-1 \\ R_3/5 \end{array}$$

∴ The reduced matrix is in REF.

So, $\rho(A) = 3$, $\rho(A:B) = 3$.

$\Rightarrow \rho(A) = \rho(A:B) = 3$, the no. of unknowns.

So, the sys. is consistent with a unique solⁿ.

Back Substitution method

From last row, $\boxed{x_3 = 5}$.

$$x_2 - x_3 = -2$$

$$\Rightarrow \boxed{x_2 = 3}$$

$$x_1 + \frac{2x_2}{5} - \frac{2}{5}x_3 = \frac{-14}{5}$$

$$\Rightarrow x_1 = \frac{-14 - 2(3) + 2(5)}{5}$$

$$= -14 - 6 + 10$$

$$\Rightarrow \boxed{x_1 = -2}$$

$\therefore$ The sol^n set is $\{x_1 = -2, x_2 = 3, x_3 = 5\}$.

* CHECK ANSWER

Substitute in ALL eq^{ns}.

* We shall solve the above problem by Gauss Jordon method.

$$A:B = \left[\begin{array}{ccc|c} \underline{-5} & -2 & 2 & 14 \\ 3 & 1 & -1 & -8 \\ 2 & 2 & -1 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 0 & \underline{-1} & 1 & 2 \\ 0 & 6 & -1 & 13 \end{array}\right] \begin{array}{l} \\ 5R_2 + 3R_1 \\ 5R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -5 & 0 & 0 & 10 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & \underline{5} & 25 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 6R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -5 & 0 & 0 & 10 \\ 0 & -5 & 0 & -15 \\ 0 & 0 & 5 & 25 \end{array}\right] \begin{array}{l} \\ 5R_2 - R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & -2 \\ 0 & \textcircled{1} & 0 & 3 \\ 0 & 0 & \textcircled{1} & 5 \end{array}\right] \begin{array}{l} R_1/-5 \\ R_2/-5 \\ R_3/5 \end{array}$$

∴ The matrix is in RREF.

So, $r(A) = 3$, $r(A:B) = 3$.

$\Rightarrow r(A) = r(A:B) = 3$, the no. of unknowns.

So, the sys. is consistent with unique sol^n.

Actual Substitution

The sol^n set is given by $\{x_1 = -2, x_2 = 3, x_3 = 5\}$.

② $$A:B = \left[\begin{array}{ccc|c} \underline{3} & -3 & -2 & 23 \\ -6 & 4 & 3 & -38 \\ -2 & 1 & 1 & -11 \end{array}\right]$$ By Gauss Jordon method.

$$\sim \left[\begin{array}{ccc|c} 3 & -3 & -2 & 23 \\ 0 & \underline{-2} & -1 & 8 \\ 0 & -3 & -1 & 13 \end{array}\right] \begin{array}{l} \\ R_2 + 2R_1 \\ 3R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 6 & 0 & -1 & 22 \\ 0 & -2 & -1 & 8 \\ 0 & 0 & \underline{1} & 2\ \cancel{-43} \end{array}\right] \begin{array}{l} 2R_1 - 3R_2 \\ \\ 2R_3 - 3R_2 \end{array}$$

$$\begin{array}{r} 69 \\ -26 \\ \hline 43 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 6 & 0 & 0 & 24 \\ 0 & -2 & 0 & 10 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 + R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & \underline{1} & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1/6 \\ R_2/-2 \\ \\ \end{array}$$

∴ The matrix is in RREF.

* Here, $r(A:B) = r(A) = \text{same} = 3 =$ no. of unknowns.

∴ The sys. is consistent with a unique solⁿ.

So, by actual substitution, the solⁿ set is
$\{ x_1 = 4, x_2 = -5, x_3 = 2 \}$.

Q. Using Gaussian eliminⁿ method, find the solⁿs, if any, of the following sys :—

① $3x_1 - 2x_2 + 4x_3 = -54$
$-x_1 + x_2 - 2x_3 = 20$
$5x_1 - 4x_2 + 8x_3 = -83$

$$\left[\begin{array}{ccc|c} 3 & -2 & 4 & -54 \\ -1 & 1 & -2 & 20 \\ 5 & -4 & 8 & -83 \end{array}\right]$$

imp ✓

② $6x_1 - 12x_2 - 5x_3 + 16x_4 - 2x_5 = -53$
$-3x_1 + 6x_2 + 3x_3 - 9x_4 + x_5 = 29$
$-4x_1 + 8x_2 + 3x_3 - 10x_4 + x_5 = 33$

① $$(A:B) = \left[\begin{array}{ccc|c} 3 & -2 & 4 & -54 \\ -1 & 1 & -2 & 20 \\ 5 & -4 & 8 & -83 \end{array}\right] \quad \boxed{\text{By RREF}}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -2 & 4 & -54 \\ 0 & \underline{1} & -2 & 6 \\ 0 & -2 & 4 & 21 \end{array}\right] \begin{array}{l} \\ 3R_2 + R_1 \\ 3R_3 - 5R_1 \end{array}$$

$\begin{array}{r} 54 \\ \times 5 \\ \hline 270 \end{array}$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 0 & -66 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 0 & 33 \end{array}\right] \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$\begin{array}{r} 83 \\ \times 3 \\ \hline 249 \end{array}$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -22 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1/3 \\ \\ R_3/33 \end{array}$$

The matrix is in RREF.

Here, $\rho(A) = 2$, $\rho(A:B) = 3$.

$\Rightarrow \rho(A) \neq \rho(A:B)$

$\therefore$ The sys. is inconsistent with no sol^n.

② $$(A:B) = \left[\begin{array}{ccccc|c} 6 & -12 & -5 & 16 & -2 & -53 \\ -3 & 6 & 3 & -9 & 1 & 29 \\ -4 & 8 & 3 & -10 & 1 & 33 \end{array}\right]$$

$$\sim \begin{array}{l} \\ 2R_2+R_1 \\ 6R_3+4R_1 \end{array} \left[\begin{array}{ccccc|c} 6 & -12 & -5 & 16 & -2 & -53 \\ 0 & 0 & \underline{1} & -2 & 0 & 5 \\ 0 & 0 & \frac{-2}{2} & \frac{4}{2} & \frac{-2}{2} & \frac{-14}{2} \end{array}\right]$$

153
×4
212

−198
⑭

$$\sim \left[\begin{array}{ccccc|c} 6 & -12 & -5 & 16 & -2 & -53 \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{array}\right] R_3 - R_2$$

$$\sim \left[\begin{array}{ccccc|c} \textcircled{1} & -2 & -5/6 & 8/3 & -1/3 & -53/6 \\ 0 & 0 & \textcircled{1} & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 2 \end{array}\right] \begin{array}{l} R_1/6 \\ \\ R_3/-1 \end{array}$$

This matrix is in REF

Here, $\rho(A) = 3$, $\rho(A:B) = 3$

$\therefore \rho(A) = \rho(A:B) = 3 < 5$, the no. of unknowns

$\therefore$ Sys. is consistent with infinite no. of sol^{ns}.

The diff. b/w the no. of unknowns & the rank, namely, 5-3 = 2 indicates the no. of parameters in the solⁿ set.

From the reduced sys.,

$$x_5 = 2.$$

let $x_2 = k$ & $x_4 = l$; where k & l are parameters.

From 2nd row, $x_3 - 2x_4 = 5$

$$\Rightarrow x_3 = 5 + 2x_4 = 5 + 2l.$$

$$\Rightarrow \boxed{x_3 = 5 + 2l}$$

From row 1,

$$x_1 - 2x_2 - \frac{5x_3}{6} + \frac{8x_4}{3} - \frac{x_5}{3} = -\cancel{5}\frac{53}{6}$$

$$\Rightarrow x_1 = \frac{1}{6}\left[-53 + 12x_2 + 5x_3 \div 16x_4 + 2x_5\right].$$

$$\Rightarrow x_1 = \frac{1}{6}\left[-53 + 12k + 5(5+2l) - 16l + 2(2)\right]$$

$$= \frac{1}{6}\left[-53 + 25 + 4 + 12k - 6l\right]$$

$$x_1 = -4 + \cancel{2}\,2k - l$$

$$\Rightarrow \boxed{x_1 = 2k - l - 4.}$$

∴ Solⁿ set is $\{x_1, x_2, x_3, x_4, x_5\}$

$$= \{2k - l - 4, k, 5 + 2l, l, 2\},$$

where k & l are parameters.

Particular sol^{ns}

Put $k=0$, $l=0$

$x_1 = -4, x_2 = 0, x_3 = 5, x_4 = 0, x_5 = 2$

Put $k=1$, $l=0$

$\Rightarrow x_1 = -2, x_2 = 1, x_3 = 5, x_4 = 0, x_5 = 2$

Put $k=0$, $l=1$

$\Rightarrow x_1 = -5, x_2 = 0, x_3 = 7, x_4 = 1, x_5 = 2$.

3) Q. Find the quadratic eq^n, $y = ax^2 + bx + c \longrightarrow ①$ that passes through the pts $(3,18), (2,9)$ & $(-2,13)$

As, ① passes through the given pts., we have

$$\left.\begin{aligned} 9a+3b+c &= 18 \\ 4a+2b+c &= 9 \\ 4a-2b+c &= 13 \end{aligned}\right\} \longrightarrow ②$$

The augmented matrix is given by

$$(A:B) = \left[\begin{array}{ccc|c} \underline{4} & -2 & 1 & 13 \\ 4 & 2 & 1 & 9 \\ 9 & 3 & 1 & 18 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 4 & -2 & 1 & 13 \\ 0 & \underline{4} & 0 & -4 \\ 0 & 30 & -5 & -45 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ 4R_3 - 9R_1 \end{array}$$

12
13

13
117

3 18
×4
72

$$\sim \left[\begin{array}{ccc|c} 4 & -2 & 1 & 13 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & -10 & -30 \end{array}\right] \begin{array}{l} \\ \\ \cancel{\frac{R_3}{5}}\ 2R_3 - \cancel{30}\ 15R_2 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{4} & \frac{13}{4} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & +1 & +3 \end{array}\right] \begin{array}{l} R_1/4 \\ R_2/4 \\ R_3/-10 \end{array}$$

The given matrix is in REF & $\rho(A) = \rho(A:B) = 3$, the number of unknowns. So, the given system of equations have a unique solution

By back substitution,

$$c = 3$$

$$b = -1$$

$$a \cancel{+} - \frac{b}{2} + \frac{c}{4} = \frac{13}{4}$$

$$\Rightarrow 4a + 2 + 3 = 13$$

$$\Rightarrow 4a = 8$$

$$\Rightarrow a = 2.$$

So, $\{a, b, c\} = \{2, -1, 3\}$.

So, the quadratic eqn is $y = 2x^2 - \cancel{6}\,x + 3$

Ans

Q. Use Gauss Jordan method to convert the following matrices to RREF & hence, solve the sys.

①
$$\left[\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 18 & 13 \end{array}\right]$$

②
$$\left[\begin{array}{cccc|c} -5 & -10 & -19 & -17 & 20 \\ -3 & 6 & -11 & -11 & 14 \\ -7 & 14 & -26 & -25 & 31 \\ 9 & -18 & 34 & 31 & -37 \end{array}\right]$$

⑦ The augmented matrix

$$(A:B) = \left[\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 18 & 13 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 0 & 0 & -16 & 48 \\ 0 & 0 & 18 & 21 \end{array}\right] \begin{array}{l} \\ 5R_2 - 3R_1 \\ 5R_3 + 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 18 & 21 \end{array}\right] \begin{array}{l} \\ R_2/16 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 5 & 20 & 0 & -65 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 75 \end{array}\right] \begin{array}{l} R_1 - 18R_2 \\ \\ R_3 + 18R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & -13 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1/5 \\ -R_2 \\ R_3/75 \end{array}$$

→ No need

3 16 ×5 80

18 ×5 8

65 −44 21

54 +21 75

The resolved matrix is in RREF

Here, ~~R~~ $\rho(A) \neq \rho(A:B)$. $(\rho(A)=2, \rho(A:B)=3)$

So, the ~~e~~ system is inconsistent.

It has no solution.

✱ HOMOGENEOUS SYS. OF EQNS.

Let $AX = 0 \longrightarrow ①$ be a homogeneous sys. of eq^{ns} in n unknowns. The sys. ① is always consistent with the zero sol^n, $X = 0$ (also called Trivial sol^n).

We reduce the coefficient matrix into REF or RREF & find its rank.

✓ If $r(A) < n$, the no. of unknowns, then, sys. ① will have an infinite no. of non-trivial (non-zero) sol^{ns}.

Q. Solve the following sys. completely. Also find any 3 particular sol^{ns}.

① $-2x_1 + x_2 + 8x_3 = 0$
REF $7x_1 - 2x_2 - 22x_3 = 0$
$3x_1 - x_2 - 10x_3 = 0$

② $2x_1 + 4x_2 - x_3 + 5x_4 + 2x_5 = 0$
RREF $3x_1 + 3x_2 - x_3 + 3x_4 + 0x_5 = 0$
$-5x_1 \pm 6x_2 + 2x_3 - 6x_4 - 5x_5 = 0$

① $$A = \begin{bmatrix} \underline{-2} & 1 & 8 \\ 7 & -2 & -22 \\ 3 & -1 & -10 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 8 \\ 0 & \underline{3} & 12 \\ 0 & 1 & 4 \end{bmatrix} \begin{matrix} \\ 2R_2 + 7R_1 \\ 2R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -2 & 1 & 8 \\ 0 & 3 & 12 \\ 0 & 0 & 0 \end{bmatrix} \quad 3R_3 - R_2$$

$$\sim \begin{bmatrix} (1) & -1/2 & -4 \\ 0 & (1) & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1/-2 \\ R_2/3 \\ \end{matrix}$$

The given matrix is in REF.
Here, $\varrho(A) = 2$ ($\neq 3$, the no. of unknowns).
Hence, the sys. is consistent with an infinite no. of solns.
Let $x_3 = k$ be a parameter.

$x_2 + 4x_3 = 0$

$\Rightarrow \boxed{x_2 = -4k}$

$x_1 - \frac{x_2}{2} - 4x_3 = 0$

$\Rightarrow x_1 = 4k - 2k = 2k$

$\Rightarrow \boxed{x_1 = 2k}$

So, $(x_1, x_2, x_3) = (2k, -4k, k)$, where k is a parameter.
$= k(2, -4, 1)$

② $A = \begin{bmatrix} \underline{2} & 4 & -1 & 5 & 2 \\ 3 & 3 & -1 & 3 & 0 \\ -5 & -6 & 2 & -6 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & 2 \\ 0 & \underline{-6} & 1 & -9 & -6 \\ 0 & 8\cancel{3} & -1 & 13 & 8 \end{bmatrix} \quad \begin{matrix} \\ 2R_2 - 3R_1 \\ 2R_3 + 5R_1 \end{matrix}$$

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$$\sim \begin{bmatrix} 12 & 0 & -2 & -6 & -12 \\ 0 & -6 & 1 & -9 & -6 \\ 0 & 0 & \underline{13} & -105 & -72 \end{bmatrix} \begin{matrix} 6R_1 + 4R_2 \\ \\ 3R_3 + \overset{16}{\cancel{32}}R_2 \end{matrix}$$

$$\sim \begin{bmatrix} & 0 & & & \\ 0 & & & & \\ 0 & 0 & 13 & -105 & -72 \end{bmatrix} \begin{matrix} 13R_1 + 2R_3 \\ 13R_2 - R_3 \\ \\ \end{matrix}$$

$$\begin{array}{r} 144 \\ -39 \\ \hline \end{array}$$

$$\sim \begin{bmatrix} 6 & 0 & -1 & -3 & -6 \\ 0 & 6 & 1 & -9 & -6 \\ 0 & 0 & \underline{1} & 3 & 0 \end{bmatrix} \begin{matrix} 3R_1 + 2R_2 \\ \\ 3R_3 + 4R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 6 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -12 & -6 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix} \begin{matrix} R_1 + R_3 \\ R_2 - R_3 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 & -1/2 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}. \begin{matrix} R_1/6 \\ R_2/\cancel{-12}\,-6 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix}. \quad R_2 \longleftrightarrow R_3$$

The matrix is in RREF.

$\rho(A) = 3 < 5$, the no. of unknowns.

So, $5 - 3 = 2$ will be the no. of parameters.

Let $x_4 = k$, $x_5 = l$, where k & l are parameters.

$x_3 + 3x_4 = 0$

$\Rightarrow \boxed{x_3 = -3k}$

$x_2 + 2x_4 + x_5 = 0$

$\Rightarrow x_2 + 2k + l = 0$

$\Rightarrow \boxed{x_2 = -(2k+l)}$

$x_1 = x_5$

$\Rightarrow \boxed{x_1 = l}$

$\therefore$ Solⁿ $(x_1, x_2, x_3, x_4, x_5) = (l, -(2k+l), -3k, k, l)$

where k & l are parameters

$$\left[\begin{aligned} &= k(0,-2,-3,1,0) \\ &+ l(1,-1,0,0,1) \end{aligned}\right]$$

Section - 2.4

* Finding inverse of a matrix
(If A is a sq. matrix with ~~det~~ $|A| \neq 0$, then, ~~A~~ A is said to be a non-singular matrix). Let A be a non singular matrix of order n. To find A^{-1}, we proceed as follows :-

S1. Form the augmented unit matrix
$(A : I_n)$

S2) Using elementary row operⁿˢ, reduce the matrix A into unit matrix. Then, the unit matrix gets transformed to A^{-1}.

i.e :-

$$(A : I_n) \sim (I_n : A^{-1})$$

We can find inverse only when $|A| \neq 0$.

Note:

✓ Imp. If A cannot be reduced to In (the reduced matrix will have zero rows), then A is a singular matrix & A^{-1} doesn't exist.

Q. Determine whether each of the following matrix is non singular & hence, find A^{-1}.

① $\begin{bmatrix} 3 & -1 & 4 \\ 2 & -2 & 1 \\ -1 & 3 & 2 \end{bmatrix}$ ② $\begin{bmatrix} -6 & -6 & 1 \\ 2 & 3 & -1 \\ 8 & 6 & -1 \end{bmatrix}$

③ $\begin{bmatrix} -4 & 7 & 6 \\ 3 & -5 & -4 \\ -2 & 4 & 3. \end{bmatrix}$ ④ $\begin{bmatrix} 0 & 0 & -2 & -1 \\ -2 & 0 & -1 & 0 \\ -1 & -2 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

⑤ $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$

⑤ The augmented unit matrix is

$$[A : I] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \quad \left\{ A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \right\}$$

$$\sim \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & 11 & -1 & 2 \end{array}\right] 2R_2 - R_1$$

$$\sim \left[\begin{array}{cc|cc} 22 & 0 & 8 & 6 \\ 0 & 11 & -1 & 2 \end{array}\right] 11R_1 + 3R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 4/11 & 3/11 \\ 0 & 1 & -1/11 & 2/11 \end{array}\right] \begin{array}{l} R_1/22 \\ R_2/11 \end{array}$$

$$\sim [\, I \mid A^{-1} \,].$$

∴ A^{-1} exists and is equal to $\frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$

① $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & -2 & 1 \\ -1 & 3 & 2 \end{bmatrix}$

So augmented unit matrix

$$(A : I) = \left[\begin{array}{ccc|ccc} 3 & -1 & 4 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 3 & -1 & 4 & 1 & 0 & 0 \\ 0 & -4 & -5 & -2 & 3 & 0 \\ 0 & 8 & 10 & 1 & 0 & 3 \end{array}\right] \begin{array}{l} \\ 3R_2 - 2R_1 \\ 3R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 12 & 0 & 21 & 6 & -3 & 0 \\ 0 & -4 & -5 & -2 & 3 & 0 \\ 0 & 0 & 0 & -3 & 6 & 3 \end{array}\right] \begin{array}{l} 4R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

The reduced matrix of A contains a zero row. Hence, A is a singular matrix & A^{-1} does not exist.

③ Let $A = \begin{bmatrix} -4 & 7 & 6 \\ 3 & -5 & -4 \\ -2 & 4 & 3 \end{bmatrix}$

Let the augmented matrix

$$(A:I) = \left[\begin{array}{ccc|ccc} \underline{-4} & 7 & 6 & 1 & 0 & 0 \\ 3 & -5 & -4 & 0 & 1 & 0 \\ -2 & 4 & 3 & 0 & 0 & 1 \end{array}\right].$$

$$\sim \left[\begin{array}{ccc|ccc} -4 & 7 & 6 & 1 & 0 & 0 \\ 0 & 1 & +2 & 3 & 4 & 0 \\ 0 & 1 & 0 & -1 & 0 & 2 \end{array}\right] \begin{array}{l} \\ 4R_2+3R_1 \\ 2R_3-R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} -4 & 0 & -8 & -20 & -28 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & -2 & -4 & -4 & 2 \end{array}\right] \begin{array}{l} R_1-7R_2 \\ \\ R_3-R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} -4 & 0 & 0 & -4 & -12 & -8 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & -2 & -4 & -4 & 2 \end{array}\right] \begin{array}{l} R_1-4R_3 \\ R_2+R_3 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 2 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 2 & -1 \end{array}\right] \begin{array}{l} R_1/-4 \\ \\ R_3/-2 \end{array}$$

$$\sim \left[\, I \mid A^{-1} \right].$$

So, A^{-1} exists and is equal to $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 2 \\ 2 & 2 & -1 \end{bmatrix}$.

④ Let $A = \begin{bmatrix} 0 & 0 & -2 & -1 \\ -2 & 0 & -1 & 0 \\ -1 & -2 & -1 & -5 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

~~of $A = \begin{bmatrix} -1 & -2 & -1 & -5 \\ -2 & 0 & -1 & 0 \end{bmatrix}$~~

So,

$$(A : I) = \left[\begin{array}{cccc|cccc} 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -2 & -1 & -5 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} \underline{-1} & -2 & -1 & -5 & 0 & 0 & 1 & 0 \\ -2 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} -1 & -2 & -1 & -5 & 0 & 0 & 1 & 0 \\ 0 & \underline{4} & 1 & 10 & 0 & 1 & -2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] R_2 - 2R_1$$

$$\sim \left[\begin{array}{cccc|cccc} -2 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 10 & 0 & 1 & -2 & 0 \\ 0 & 0 & \underline{-2} & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & -1 & 2 & 4 \end{array}\right] \begin{array}{l} 2R_1 + R_2 \\ \\ \\ 4R_4 - R_2 \end{array}$$

$$\sim\left[\begin{array}{cccc|cccc} -4 & 0 & 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & 8 & 0 & 19 & 1 & 2 & -4 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & -2 & 4 & 8 \end{array}\right] \begin{array}{l} 2R_1 - R_3 \\ 2R_2 + R_3 \\ \\ 2R_4 + 3R_3 \end{array}$$

$$\sim\left[\begin{array}{cccc|cccc} -4 & 0 & 0 & 0 & -4 & 4 & -4 & -8 \\ 0 & 8 & -38 & 0 & -56 & 40 & -80 & -152 \\ 0 & 0 & -2 & 0 & 4 & -2 & 4 & 8 \\ 0 & 0 & 0 & 1 & 3 & -2 & 4 & 8 \end{array}\right] \begin{array}{l} R_1 - R_4 \\ R_2 - 19R_4 \\ R_3 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1/2 & 1 & 2 \\ 0 & 1 & -19/4 & 0 & 5/2 & 1/4 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & -2 & -4 \\ 0 & 0 & 0 & 1 & & & & \end{array}\right] \begin{array}{l} R_1/4 \\ R_2/8 \\ R_3/-2 \\ \\ \end{array}$$

$$\sim\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & -7 & 5 & -10 & -19 \\ 0 & 0 & 1 & 0 & -2 & 1 & -2 & -4 \\ 0 & 0 & 0 & 1 & 3 & -2 & 4 & 8 \end{array}\right] \begin{array}{l} R_1/-4 \\ R_2/8 \\ R_3/-2 \\ \\ \end{array}$$

$$\begin{array}{r} 19 \\ \times 3 \\ \hline 57 \end{array} \qquad \begin{array}{r} 19 \\ \times 4 \\ \hline \boxed{78} \end{array} \qquad \begin{array}{r} 19 \\ \times 8 \\ \hline 152 \end{array}$$

$$\sim [\, I \mid A^{-1} \,]$$

So, $A^{-1} = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -7 & 5 & -10 & -19 \\ -2 & 1 & -2 & -4 \\ 3 & -2 & 4 & 8 \end{bmatrix}$

Q Find A^{-1}, if it exists & hence, solve

$$-5x_1 + 3x_2 + 6x_3 = 4$$
$$3x_1 - x_2 - 7x_3 = 11$$
$$-2x_1 + x_2 + 2x_3 = 2$$

Let

$$A = \begin{bmatrix} -5 & 3 & 6 \\ 3 & -1 & -7 \\ -2 & 1 & 2 \end{bmatrix}$$

$$(A:I) = \left[\begin{array}{ccc|ccc} \underline{-5} & 3 & 6 & 1 & 0 & 0 \\ 3 & -1 & -7 & 0 & 1 & 0 \\ -2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} -5 & 3 & 6 & 1 & 0 & 0 \\ 0 & \underline{4} & -17 & 3 & 5 & 0 \\ 0 & -1 & -2 & -2 & 0 & 5 \end{array}\right] \begin{array}{l} \\ 5R_2+3R_1 \\ 5R_3-2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} -20 & 0 & 75 & -5 & -15 & 0 \\ 0 & 4 & -17 & 3 & 5 & 0 \\ 0 & 0 & \underline{-25} & -5 & 5 & 20 \end{array}\right] \begin{array}{l} 4R_1-3R_2 \\ \\ 4R_3+R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 0 & -15 & 1 & 3 & 0 \\ 0 & 4 & -17 & 3 & 5 & 0 \\ 0 & 0 & \underline{5} & 1 & -1 & -4 \end{array}\right] \begin{array}{l} \cancel{3R_1+R_3}\ R_1/-5 \\ \\ R_2/-5 \end{array}$$

$$\begin{array}{r} 24 \\ \cancel{51} \\ \hline 75 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & 4 & 0 & -12 \\ 0 & 20 & 0 & 32 & 8 & -68 \\ 0 & 0 & 5 & 1 & -1 & -4 \end{array}\right] \begin{array}{l} 3R_1+3R_3 \\ 5R_2+17R_3 \\ \cancel{38} \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & \overset{8/5}{\cancel{19/10}} & 2/5 & -17/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & -4/5 \end{array}\right] \begin{array}{l} R_1/4 \\ R_2/20 \\ R_3/5. \end{array}$$

$$\therefore \; A^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ \cancel{19/10}\,8/5 & 2/5 & -17/5 \\ 1/5 & -1/5 & -4/5 \end{bmatrix}$$

$$\therefore \; X = A^{-1} B$$

$$= \begin{bmatrix} 1 & 0 & -3 \\ \cancel{19/10}\,8/5 & 2/5 & -17/5 \\ 1/5 & -1/5 & -4/5 \end{bmatrix} \begin{bmatrix} 4 \\ 11 \\ 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 & 0 & -15 \\ \cancel{19/2}\,8 & 2 & -17 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ 11 \\ 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 20 - 30 \quad \cancel{38 + 22 - 34} \quad \cancel{4 - 11 + 8} \\ \overset{32}{\cancel{38}} + 22 - 34 \\ 4 - 11 \cancel{+8} - 8 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -10 \\ 20 \\ -15 \end{bmatrix}$$

So, $(x_1, x_2, x_3) = \left(-2, \overset{\textcircled{4}}{\underset{4}{\cancel{26/5}}}, \overset{-3}{\cancel{15}}\right)$

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Section - 3.4

Eigenvalues & Eigenvectors

Let A be a sq. matrix of order n. An eigenvalue of A is a scalar λ satisfying the eq^n

$$AX = \lambda X \longrightarrow ①$$

with a non zero sol^n X $(X \neq 0)$.

The non-zero vector X is called the Eigenvector of A corresponding to the Eigenvalue λ.

<u>Note 1:-</u>

An ~~eo~~ Eigenvector is a non-zero vector.

<u>Note 2:-</u>

The eq^n ① can be re-written as $AX - \lambda I X = 0$, where I is unit matrix of same order.

$$\Rightarrow \boxed{(A - \lambda I) X = 0} \longrightarrow ②$$

The sys. ② will have non-zero sol^{ns}. for X only when

$$|A - \lambda I| = 0 \longrightarrow ③$$

Eq^n ③ is called the Characteristic eq^n of A & the roots of this eq^n are the Eigenvalues of A.

We use the sys. ② to get Eigenvectors

Note 3:-

The Eigenvalues are also referred to as Characteristics roots or Latent values.

Note 4:-

If A is a matrix of order n, then, eqⁿ ③ is an nth degree polynomial eqⁿ & has only n roots, real or complex.

ex:- Find the Eigenvalues and Eigenvectors of ⑦

$$\begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix} = A, \text{ say}.$$

The characteristic eqⁿ is given by

$$|A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} -4-\lambda & 8 & -12 \\ 6 & -6-\lambda & 12 \\ 6 & -8 & 14-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (-4-\lambda)((14-\lambda)(-6-\lambda)+96)$$
$$-8(6\times(14-\lambda)+72)$$
$$-12(-48+6(6+\lambda)) = 0$$

$$\Rightarrow (4+\lambda)(-84+6\lambda-14\lambda+\lambda^2+96)$$
$$+8(84-6\lambda+72)+12(-48+36+6\lambda) = 0$$
$$\Rightarrow (4+\lambda)(12+\lambda^2-8\lambda)+8(156-6\lambda)+12(-12+6\lambda) = 0$$
$$\Rightarrow 720+4\lambda^2-32\lambda+180\lambda+\lambda^3-8\lambda^2$$
$$+96-48\lambda-144+72\lambda = 0$$
$$\Rightarrow \lambda^3-4\lambda^2\underset{-44}{\cancel{+172}}\lambda+\underset{96}{\cancel{672}} = 0.$$

$48+12\lambda+$
$4\lambda^2+\lambda^3-32\lambda$
$-8\lambda^2-48\lambda$
$+$

1720
96
196
84
0

ALITER : $A = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}$

Let the characteristic eq^n be

$\lambda^3 - C_0\lambda^2 + C_1\lambda - C_2 = 0$

C_0 = Sum of diagonal elemts.

$= -4-6+14 = 4$

C_1 = Sum of minors of diagonal elemts

$= \begin{vmatrix} -6 & 12 \\ -8 & 14 \end{vmatrix} + \begin{vmatrix} -4 & -12 \\ 6 & 14 \end{vmatrix} + \begin{vmatrix} -4 & 8 \\ 6 & -6 \end{vmatrix}$

$\Rightarrow C_1 = 4.$

$C_2 = |A|$

$= -4(-84+96) - 8(84-72) - 12(-48+36)$

$= 0.$

$\therefore$ The characteristic eq^n is

$\lambda^3 - 4\lambda^2 + 4\lambda - 0 = 0$

$\Rightarrow \lambda^3 - 4\lambda^2 + 4\lambda = 0.$

$\Rightarrow \lambda(\lambda^2 - 4\lambda + 4) = 0$

$\Rightarrow \lambda(\lambda-2)^2 = 0 \Rightarrow \lambda = 0, 2, 2$ are Eigenvalues.

* * *

The Eigenvectors are given by

$(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} -4-\lambda & 8 & -12 \\ 6 & -6-\lambda & 12 \\ 6 & -8 & 14-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \textcircled{I}$$

When $\lambda = 0$

$$\begin{pmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking any 2 independent rows, in particular, the first 2 rows, and applying cross multiplicⁿ rule,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 8 & & -12 & & -4 & & 8 \\ -6 & & 12 & & 6 & & -6 \end{array}$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} 8 & -12 \\ -6 & 12 \end{vmatrix}} = \frac{\cancel{8}x_2}{\begin{vmatrix} -12 & -4 \\ 12 & 6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & 8 \\ 6 & -6 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{96 \cancel{+} -72} = \frac{x_2}{-72+48} = \frac{x_3}{24-48}$$

$$\Rightarrow \frac{x_1}{24} = \frac{x_2}{-24} = \frac{x_3}{-24}$$

$$\Rightarrow x_1/1 = x_2/-1 = x_3/-1$$

$\therefore$ Eigenvector is $x_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

When $\lambda = 2$.

$$\begin{pmatrix} -6 & 8 & -12 \\ 6 & -8 & 12 \\ 6 & -8 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

All the rows are dependent & there is only 1 independent eqⁿ

$$-6x_1 + 8x_2 - 12x_3 = 0$$

$$\Rightarrow 3x_1 - 4x_2 + 6x_3 = 0.$$

Let $x_2 = 1, x_3 = 0$

$\Rightarrow 3x_1 = 4x_2$

$\Rightarrow x_1 = \frac{4}{3}$

$$\Rightarrow X_2 = \begin{pmatrix} 4/3 \\ 1 \\ 0 \end{pmatrix}$$

or

$$\Rightarrow X_2 = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$$

Let $x_2 = 0, x_3 = 1$

$\Rightarrow 3x_1 = -6x_3$

$\Rightarrow x_1 = -2$

$$\Rightarrow X_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

* If X is an eigenvector corresponding to an Eigenvalue λ, then, the Eigen space of λ is given by

$$E_\lambda = \{ kX \mid, k \text{ is a scalar} \}$$

In the above example, the Eigenspaces of the Eigenvalues 0 and 2 are given by

$$E_0 = \left\{ k \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \middle| , k \text{ is a scalar} \right\}$$

$$E_2 = \left\{ l \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + m \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \middle| , l \,\&\, m \text{ are scalars} \right\}$$

Q. Find the Eigenvalues & Eigenspaces of

① $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 2 \\ -3 & -3 & -2 \end{bmatrix}$ ② $\begin{bmatrix} 3 & 4 & 12 \\ 4 & -12 & 3 \\ 12 & 3 & -4 \end{bmatrix}$

③ $\begin{bmatrix} 4 & 0 & -2 \\ 6 & 2 & -6 \\ 4 & 0 & -2 \end{bmatrix}$ ④ $\begin{bmatrix} 8 & -21 \\ 3 & -8 \end{bmatrix}$

① Let $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 2 \\ -3 & -3 & -2 \end{bmatrix}$

Let $\lambda^3 - C_0\lambda^2 + C_1\lambda - C_2 = 0$ be the characteristic eq^n of A.

C_0 = Sum of diagonal elemts.

$= 1 + 3 - 2 = \textcircled{2}$.

C_1 = Sum of minors of diagonal elemts.

$= \begin{vmatrix} 3 & 2 \\ -3 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -3 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}$

$= 0 \mp 5 + 4$

$= \cancel{5} - 1$

$C_2 = |A|$

$= 0 + 4 - 6 = -2$.

∴ The ch. eq^n is $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

We find one root by inspection method,

λ	$f(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$
0	$2 \neq 0$
1	$1 - 2 - 1 + 2 = 0$

∴ $\lambda = 1$ is an Eigenvalue.

* Sum of Eigenvalues = Sum of diagonal elements.
* Product of Eigenvalues = $|A|$

By synthetic division method;

$$\begin{array}{c|cccc} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow (\lambda+1)(\lambda-2) = 0.$

$\Rightarrow \lambda = -1, 2.$

$\therefore$ Eigenvalues are $\lambda = 1, -1, 2$

The Eigenvectors are given by

$\underline{(A - \lambda I)X = 0}$

$$\Rightarrow \begin{pmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 2 \\ -3 & -3 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = 1$

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 2 \\ -3 & -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Choosing 2nd & 3rd rows,

$$\frac{x_1}{-6+6} = \frac{-x_2}{-3+6} = \frac{x_3}{-3+6}$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{-\cancel{+}3} = \frac{x_3}{3}$$

$$\Rightarrow X_1 = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}.$$

~~When~~ ∴ The Eigenspace is

$E_1 = \{ k X_1 \mid, k$ is a scalar $\}$

$= \{ k \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} \mid, k$ is a scalar $\}$.

When $\lambda = -1$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 4 & 2 \\ -3 & -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{x_1}{-2+4} = \frac{-x_2}{4+1} = \frac{x_3}{8+1}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-5} = \frac{x_3}{9}$$

$$\therefore X_2 = \begin{pmatrix} 2 \\ -5 \\ 9 \end{pmatrix}$$

∴ The Eigenspace is

$E_2 = \{ l\cancel{k} X_2 \mid, l$ is a scalar $\}$

$= \{ l \begin{pmatrix} 2 \\ -5 \\ 9 \end{pmatrix} \mid, l$ is a scalar $\}$

When $\lambda = 2$

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 2 \\ -3 & -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{x_1}{-2+1} = \frac{-x_2}{-2+1} = \frac{x_3}{-1+1}$$

$$\Rightarrow X_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

∴ The Eigenspace is

$$E_3 = \{ m X_3 \mid, m \text{ is a scalar} \}$$

$$= \left\{ m\begin{pmatrix}-1\\+1\\0\end{pmatrix} \middle|, m \text{ is a scalar} \right\}$$

② Let $A = \begin{bmatrix} 3 & 4 & 12 \\ 4 & -12 & 3 \\ 12 & 3 & -4 \end{bmatrix}$

Let $\lambda^3 - C_0\lambda^2 + C_1\lambda - C_2 = 0$ be the characteristic eqⁿ of A.

C_0 = Sum of diagonal elemts.
$= -13$

C_1 = Sum of the minors of diagonal elements

$= (48-9) + (-156) + (-52)$

$= 39 - 208$

$C_1 = -169$

$C_2 = 3(39) - 4(-16-36) + 12(12+144)$.

$\Rightarrow C_2 = 2197$

∴ Characteristic eqⁿ is

$$\lambda^3 + 13\lambda^2 - 169\lambda - 2197 = 0.$$

λ	$f(\lambda) = \lambda^3 + 13\lambda - 169\lambda - 2197$
13	0

∴ $\lambda = 13$ is an eigenvalue.

By synthetic division method,

$$\begin{array}{r|rrrr} 13 & 1 & 13 & -169 & -2197 \\ & 0 & 13 & 338 & 2197 \\ \hline & 1 & 26 & 169 & \boxed{0.} \end{array}$$

$\Rightarrow \lambda^2 + 26\lambda + 169 = 0$

$\Rightarrow \lambda = \dfrac{-26 \pm \sqrt{169 \times 4 - 169 \times 4}}{2}$ 13×13×4

$\Rightarrow \lambda = -13, -13$

So, Eigenvalues are $\lambda = 13, -13, -13$.

To find Eigenvectors, they are given by

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 4 & 12 \\ 4 & -12-\lambda & 3 \\ 12 & 3 & -4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow (I)$$

When $\lambda = 13$

$$\Rightarrow \begin{bmatrix} -10 & 4 & 12 \\ 4. & -25 & 3 \\ 12 & 3 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking 1st 2 rows

$$\frac{x_1}{12+300} = \frac{x_2}{78} = \frac{x_3}{234}$$

$$\Rightarrow \frac{x_1}{\underset{\underset{4}{\cancel{106}}}{\cancel{312}}} = \frac{x_2}{\underset{\underset{1}{\cancel{39}}}{\cancel{78}}} = \frac{x_3}{\underset{\underset{3}{\cancel{117}}}{\cancel{234}}} \Rightarrow X = \begin{pmatrix} 312 \\ 78 \\ 234 \end{pmatrix} \equiv \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

The Eigenspace of an Eigenvalue λ having an Eigenvector X is denoted & defined by

$$E_\lambda = \{ kX \mid , k \text{ is a scalar} \}$$

$\therefore$ Eigenspace of the Eigenvalue 13 is given by

$$E_{13} = \left\{ k \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \middle| \, k \text{ is a scalar} \right\}$$

For $\lambda = -13$

$$\begin{pmatrix} 16 & 4 & 12 \\ 4 & 1 & 3 \\ 12 & 3 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

All the rows are dependent, & we have

$4x_1 + x_2 + 3x_3 = 0$,

Let $x_2 = 4$, $x_3 = 0$

$\Rightarrow x_1 = -1$

$\Rightarrow X_2 = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$

$x_2 = 0$, $x_3 = 4$

$\Rightarrow x_1 = -3$.

$X_3 = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$

$\therefore$ Eigenspace is given by

$$E_{-13} = \{ l X_2 + m X_3 \mid , l, m \text{ are scalars} \}$$

$$= \left\{ l \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} + m \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \middle| , l, m \text{ are scalars} \right\}$$

④ $A = \begin{bmatrix} 8 & -21 \\ 3 & -8 \end{bmatrix}$

Let the characteristic eqn be

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -21 \\ 3 & -8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)(-8-\lambda) + 63 = 0$$

$$\Rightarrow \cancel{64 + \lambda^2} \; (64-\lambda^2)(-1) + 63 = 0$$

$$\Rightarrow 63 = 64 - \lambda^2$$

$$\Rightarrow \lambda = \underline{\pm 1}.$$

$\therefore$ Eigenvalues are ± 1.

The Eigenvectors are given by

$$[A - \lambda I](x) = 0$$

$$\Rightarrow \begin{pmatrix} 8-\lambda & -21 \\ 3 & -8-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow (I)$$

When $\lambda = 1$

$$\Rightarrow \begin{pmatrix} 7 & -21 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow x_1 - 3x_2 = 0$$

Let $x_2 = 1 \Rightarrow x_1 = 3$.

$\therefore X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\therefore$ Eigenspace

$$E_1 = \{ kX_1 \mid k \text{ is scalar} \}$$

$$= \left\{ k\begin{pmatrix} 3 \\ 1 \end{pmatrix} \middle|, k \text{ is scalar} \right\}$$

For $\lambda = -1$

$\Rightarrow \begin{pmatrix}9 & -21\\ 3 & -7\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

$\Rightarrow 3x_1 - 7x_2 = 0$

Let $x_1 = 7 \Rightarrow x_2 = 3$

So, $X_2 = \begin{pmatrix}7\\ 3\end{pmatrix}$

So, Eigenspace is given by

$E_{-1} = \{ l X_2 \mid l \text{ is scalar} \}$

$\Rightarrow E_{-1} = \{ l \begin{pmatrix}7\\ 3\end{pmatrix} \mid l \text{ is scalar} \}$

③ $A = \begin{bmatrix}4 & 0 & -2\\ 6 & 2 & -6\\ 4 & 0 & -2\end{bmatrix}$

Let $\lambda^3 - C_0\lambda^2 + C_1\lambda - C_2 = 0$ be the characteristic eq^n of A.

C_0 = Sum of diagonal elemts

$\Rightarrow C_0 = 4.$

C_1 = Sum of minors of diagonal elemts

$= -4 + (0) + 8$

$C_1 = 4$

$C_2 = |A|$

$= 4(-4) \cancel{+} -0 \cancel{+} -2(-8)$

$= -16 + 16 = 0$

So, characteristic eq^n of A is

$$\lambda^3 - 4\lambda^2 + 4\lambda = 0.$$

$$\Rightarrow \lambda(\lambda^2 - 4\lambda + 4) = 0$$

$$\Rightarrow \lambda(\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 0, 2, 2$$

So, Eigenvalues are 0, 2 & 2.

Eigenvectors are given by

$$(A - \lambda I)(X) = 0$$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 0 & -2 \\ 6 & 2-\lambda & -6 \\ 4 & 0 & -2-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 0$

$$\Rightarrow \begin{pmatrix} 4 & 0 & -2 \\ 6 & 2 & -6 \\ 4 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

⇒ Choosing 1st & 2nd row

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{12} = \frac{x_3}{8}$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

So, Eigenspace is given by

$$E_0 = \{kX_1 \mid, k \text{ is scalar}\}$$
$$= \left\{k \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \middle| k \text{ is scalar}\right\}$$

When $x = 2$.

$$\Rightarrow \begin{pmatrix} 2 & 0 & -2 \\ 6 & 0 & -6 \\ 4 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x_1 - x_3 = 0$

$\Rightarrow \quad \underline{\underline{x_1 = x_3}}.$, x_2 can take any real value

Put $x_1 = x_3 = 2$

$x_2 = 0$

$\therefore \quad X_2 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$x_1 = x_3 = 0$

$x_2 = 5$

$\Rightarrow X_3 = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

So, Eigenspace is given by

$$E_2 = \{ l X_2 + m X_3 \mid l, m \text{ are scalars} \}$$

$$\Rightarrow E_2 = \left\{ l \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + m \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} \middle| \; l, m \text{ are scalars} \right\}$$

★ Properties of Eigenvalues & Eigenvectors

P1) The sum of the Eigenvalues = Sum of diagonal elemts (called as trace of matrix)

$\Rightarrow$ Sum of Eigenvalues = $tr(A)$

P2) Product of Eigenvalues = $|A|$.

P3) If A is a singular matrix, then, atleast one of the Eigenvalues = 0 & vice versa.

P4) If A is a real, symmetric matrix, then, all the Eigenvalues of A are real.

P5) If λ is an Eigenvalue of A, then,

(i) λ^2 is an Eigenvalue of A^2.

(ii) λ^n is an Eigenvalue of A^n for any integral power of n.

(iii) $1/\lambda$ is an Eigenvalue of A^{-1}, if it exists.

(iv) $k\lambda$ is an Eigenvalue of kA, for any scalar.

* The Eigenvectors remain the same.

If $\lambda = 0$ ⇒ A^{-1} doesn't exis ∵ its eigenvalue is ∞. that means it is not singular ⇓ If A is not singular, then one of the eigenvalues (atleast 1) is zero

eg:- Find the Eigenvalues & Eigenvectors of A^4, if $A = \begin{bmatrix} 8 & -21 \\ 3 & -8 \end{bmatrix}$.

Eigenvalues of A	-1	1
Eigenvalues of A^4	$(-1)^4 = 1$	$(1)^4 = 1$
Eigenvectors	$\begin{pmatrix} 7 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

P6) The Eigenvalues of a Δlar or a diagonal matrix are the diagonal elemts. of the matrix

ex:- The Eigenvalues of

$$A = \begin{bmatrix} 4 & -6 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix} \text{ are } \lambda = 4, 1, -5.$$

P3) If A is a singular matrix, then, atleast one of the Eigenvalues = 0 & vice versa.

P4) If A is a real, symmetric matrix, then, all the Eigenvalues of A are real.

Imp P5) If λ is an Eigenvalue of A, then,

(i) λ^2 is an Eigenvalue of A^2

(ii) λ^n is an Eigenvalue of A^n for any integral power of n.

(iii) $1/\lambda$ is an Eigenvalue of A^{-1}, if it exists.

(iv) $k\lambda$ is an Eigenvalue of kA, for any scalar k

* The Eigenvectors remain the same.

If $\lambda = 0 \Leftarrow$ $\Rightarrow A^{-1}$ doesn't exis $\therefore$ its eigenvalue is ∞. that means It is not singular $\Downarrow$ If A is not singular, then one of the eigenvalues (atleast 1) is zero

eg:- Find the Eigenvalues & Eigenvectors of A^4, if $A = \begin{bmatrix} 8 & -21 \\ 3 & -8 \end{bmatrix}$

Eigenvalues of A	-1	1
∴ Eigenvalues of A^4	$(-1)^4 = 1$	$(1)^4 = 1$
Eigenvectors	$\begin{pmatrix} 7 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

P6) The Eigenvalues of a Δlar or a diagonal matrix are the diagonal elemts. of the matrix.

ex:- The Eigenvalues of

$$A = \begin{bmatrix} 4 & -6 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix} \text{ are } \lambda = 4, 1, -5.$$

Chapter - 4.

Vector Spaces

Section - 4.1

* A non empty set V together with the oper^{ns} $\oplus$, called the vector addition & $\odot$, called the scalar multiplic^{n}. satisfying the following properties, is called a vector space.

Ⓐ ▨ $\forall\ u, v \in V$, $u \oplus v \in V$ (Closure property)

▨ $\forall\ u, v, w \in V$,

(1) $u \oplus v = v \oplus u$

(2) $(u \oplus v) \oplus w = u \oplus (v \oplus w)$

$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (3) $\bar{0} \oplus u = u \oplus \bar{0} = u$; $\bar{0}$ is called additive identity.

(4) $-u \oplus u = u \oplus -u = \bar{0}$, $-u$ is additive inverse of u.

Ⓑ $\forall$ scalars α, $u \in V$

$\alpha \odot u \in V$ (Closure property)

(1) $\alpha \odot (u \oplus v) = \alpha \odot u \oplus \alpha \odot v$

(2) $(\alpha + \beta) \odot u = (\alpha \odot u) \oplus (\beta \odot u)$

(3) $(\alpha\beta) \odot u = \alpha \odot (\beta \odot u) = \beta \odot (\alpha \odot u)$

(4) $1 \odot u = u$

R^n

eg: Show that $\mathbb{R}^3$ is a vector space ~~wr~~ under usual add^n & scalar multiplic^n.

By defin^n, $\mathbb{R}^3 = \{ x, y, z \mid x, y, z \in R \}$

Let $(u, v) \in \mathbb{R}^3$ & α be a scalar.

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$.

(i) $u \oplus v = (x_1, y_1, z_1) \oplus (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

(ii) $\alpha \odot u = \alpha \odot (x_1, y_1, z_1)$

$= (\alpha x_1, \alpha y_1, \alpha z_1)$.

With these oper^{ns} of vector add^n & scalar multiplic^n, we shall show that $\mathbb{R}^3$ is a vector space.

$\forall (u, v) \in \mathbb{R}^3 \quad u \oplus v = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in R^3$.

$\therefore$ R^3 is closed under $\oplus$.

Consider

$u \oplus v = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in \mathbb{R}^3$

$= (x_2 + x_1, y_2 + y_1, z_2 + z_1)$

$= (x_2, y_2, z_2) \oplus (x_1, y_1, z_1) = v \oplus u$.

$\Rightarrow u \oplus v = v \oplus u \quad \forall (u, v) \in V$.

$\therefore$ Property A (1) is true.

Consider ~~w~~ $w = (x_3, y_3, z_3)$ & $u, v, w \in R^3$.

$u \oplus v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$(u \oplus v) \oplus w = ((x_1 + x_2), y_1 + y_2, z_1 + z_2) \oplus (x_3, y_3, z_3)$

$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3, z_1 + z_2 + z_3)$.

Now

$$(U \oplus V) \oplus W = (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$$
$$= (x_1, y_1, z_1) \oplus (x_2+x_3, y_2+y_3, z_2+z_3)$$
$$= U \oplus ((x_2, y_2, z_2) \oplus (x_3, y_3, z_3))$$
$$= U \oplus (V \oplus W)$$

So, A(2) is proved.

Here,

$$\bar{O} = (0, 0, 0)$$
$$\bar{O} + \bar{U} = (0,0,0) \oplus (x_1, y_1, z_1)$$
$$= (0 + x_1, 0 + y_1, 0 + z_1)$$
$$= (x_1, y_1, z_1)$$
$$= \bar{U}.$$

So, A(3) is proved.

For $U = (x_1, y_1, z_1)$

$-U = (-x_1, -y_1, -z_1)$ is additive inverse.

Consider

$$U \oplus -U = (x_1, y_1, z_1) \oplus (-x_1, -y_1, -z_1)$$
$$= (x_1 - x_1, y_1 - y_1, z_1 - z_1)$$
$$= \bar{O}.$$

$\therefore$ $-U$ is additive inverse.

So, A(4) is proved.

Let α, β be scalars & $U = (x_1, y_1, z_1) \in \mathbb{R}^3$.

LHS

$$(\alpha+\beta) \odot U = (\alpha+\beta) \odot (x_1, y_1, z_1)$$
$$= [(\alpha+\beta)x_1, (\alpha+\beta)y_1, (\alpha+\beta)z_1]$$
$$= [\alpha x_1 + \beta x_1, \alpha y_1 + \beta y_1, \alpha z_1 + \beta z_1]$$
$$= [(\alpha x_1, \alpha y_1, \alpha z_1) \oplus (\cancel{\alpha}\beta x_1, \beta y_1, \beta z_1)]$$

$= \alpha \odot (x_1, y_1, z_1) \oplus \beta \odot (x_1, y_1, z_1)$

$= \alpha \odot (U) \oplus \beta \odot V$.

$= RHS$.

Hence, $B(2)$ is proved.

Consider a scalar α & $U = (x_1, y_1, z_1) \in \mathbb{R}^3$.

So, $\alpha U = \{\alpha x_1, \alpha y_1, \alpha z_1) \in \mathbb{R}^3$.

So, $\mathbb{R}^3$ is closed under $\odot$.

Consider ~~a~~ ~~&~~ $U = (x_1, y_1, z_1)$ & $V = (x_2, y_2, z_2)$

So, $\alpha \odot (U \oplus V) = \alpha \odot [(x_1, y_1, z_1) \oplus (x_2, y_2, z_2)]$

$= \alpha \odot (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= [\alpha(x_1 + x_2) \cancel{+}, \alpha(y_1 + y_2), \alpha(z_1 + z_2)]$

$= (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2, \alpha z_1 + \alpha z_2)$

$= (\alpha x_1, \alpha y_1, \alpha z_1) \oplus (\alpha x_2, \alpha y_2, \alpha z_2)$

$= [\alpha \odot (x_1, y_1, z_1)] \oplus [\alpha \odot (x_2, y_2, z_2)]$

$= (\alpha \odot U) \oplus (\alpha \odot V)$.

$= RHS$.

So, $B.①$ is proved.

Consider Scalars α & β.

So, $(\alpha\beta) \odot U = \alpha\beta \odot (x_1, y_1, z_1) = \cancel{\alpha\beta}$

$= (\alpha\beta x_1, \alpha\beta y_1, \alpha\beta z_1)$

$= \alpha \odot (\beta x_1, \beta y_1, \beta z_1)$

$= \alpha \odot (\beta \odot (x_1, y_1, z_1))$

$= \alpha \odot (\beta \odot U) = RHS$.

So, $B③$ is proved.

Consider $U = (x_1, y_1, z_1)$

$1 \cancel{\$} \odot U = 1 \odot (x_1, y_1, z_1)$

$= (1x_1, 1y_1, 1z_1)$

$= (x_1, y_1, z_1)$.

$\Rightarrow 1 \odot (u) = u = \text{RHS}$.

So, B(4) is proved.

$\therefore$ $\mathbb{R}^3$ is the vector space under usual addⁿ & scalar multiplicⁿ.

eg ②: Let $V = \mathbb{R}^3$.

Define the vector addⁿ $\oplus$ & scalar multiplicⁿ $\odot$ as follows:-

$(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1, y_1 + y_2, z_2)$.

$\alpha \odot (x_1, y_1, z_1) = (\alpha x_1, 1, z_1)$.

Show that $\mathbb{R}^3$ is not a vector space under these operⁿˢ.

M1) Let $u, v \in \mathbb{R}^3$ &

$u = (1, 5, 10)$ & $v = (1, 1, -1)$

$u \oplus v = (1, 5, 10) \oplus (1, 1, -1)$

$= (1, 6, -1)$

$v \oplus u = (1, 1, -1) \oplus (1, 5, 10)$

$= (1, 6, 10)$

Clearly, $u \oplus v \neq v \oplus u$.

So, A(1) is not true.

$\Rightarrow$ $\mathbb{R}^3$ is not a vector space.

M2) Let $u, v \in \mathbb{R}^3$ s.t

$u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

So, $u \oplus v = (x_1, y_1 + y_2, z_2)$

$v \oplus u = (x_2, y_1 + y_2, z_1)$

Clearly, $u \oplus v \neq v \oplus u$.

So, A(1) is not true

Hence, $\mathbb{R}^3$ is not a vector space.

M3) $\underline{1 \odot u = u}$

Let $u = (1, -1, 5) \in \mathbb{R}^3$.

$1 \odot u = (1, 1, 5)$.

$\neq u$.

∴ B④ is not true.

⇒ ~~B~~ $\mathbb{R}^3$ is not a vector space.

eg③ $\mathbb{R}^n$ is a vector space with usual add^n & scalar multiplic^n.

eg④ Check whether, $\mathbb{R}^+$, the set of +ve real nos. is a VS. with usual add^n & scalar multiplic^n.

→ Additive inverses do not exist in $\mathbb{R}^+$ ($-u$ is additive inverse of u in $\mathbb{R}^+$ & $-u \notin \mathbb{R}^+$)

→ $\mathbb{R}^+$ is not closed under scalar multiplic^n. So, its not a vector space (∵ take α as −ve)

eg.* How to make $\mathbb{R}^+$ as vector space?

Let $V = \mathbb{R}^+$, the set of +ve real nos.

Define vector add^n $\oplus$ & scalar multiplic^n $\odot$ as follows:

$u \oplus v = uv$

$\alpha \odot u = u^{\alpha}$

Show $\mathbb{R}^+$ is a vector space.

Clearly, ~~∀~~ the oper^{ns} belong to $\mathbb{R}^+$.

∴ $\mathbb{R}^+$ is closed under vector add^n & scalar multiplic^n.

So, PⒶ & PⒷ are true.

A① :- By $u \oplus v = uv$

$= vu$

$= v \oplus u , \forall (u, v) \in \mathbb{R}^+$

$\therefore$ A① is true.

A② :- let $u, v, w \in \mathbb{R}^+$.

So, consider $u \oplus (v \oplus w) = u \oplus (vw)$

$= uvw$

$= (uv) w$

$= (u \oplus v) w$

$= (u \oplus v) \oplus w$.

So, A② is true.

$\forall u \in \mathbb{R}^+$, $1 \in \mathbb{R}^+$, which is the ~~addi~~ additive identity s.t

$u \oplus 1 = u \cdot 1 = u$. $\left\{ \text{Here } \bar{0} \text{ is } 1 \right\}$.

$\therefore$ A③ is true.

$\forall$ u in $\mathbb{R}^+$, $\exists$ $1/u \in \mathbb{R}^+$ s.t

$u \oplus \frac{1}{u} = u \cdot \frac{1}{u} = 1$.

$\therefore$ A④ is true.

Prove: B② $(\alpha+\beta) \odot u = (\alpha \odot u) \oplus (\beta \odot u)$

Let α, β be scalars & $u \in \mathbb{R}^+$.

Consider LHS.

$(\alpha+\beta) \odot u = u^{\alpha+\beta}$.

$= u^{\alpha} \cdot u^{\beta}$

$= (u^{\alpha}) \oplus (u^{\beta})$

$= (\alpha \odot u) \oplus (\beta \odot u)$

$=$ RHS.

$\therefore$ ~~B2~~ B2 is true.

IIly, we can show B①, B③ & B④.
∴ $\mathbb{R}^+$ is a vector space under the vector addⁿ ⊕ & scalar multiplicⁿ ⊙.

eg. Consider the set (M) of all matrices of order 2 over the set of Real nos. Then, M is a vector space w.r.t the usual addⁿ & scalar multiplicⁿ.

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \text{ are real} \right\}.$$

Let A, B ∈ M, s.t, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$

$$\therefore A \oplus B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

$$\Rightarrow A \oplus B = \begin{pmatrix} a+a_1 & b+b_1 \\ c+c_1 & d+d_1 \end{pmatrix}$$

$$\& \ \alpha \odot A = \alpha \odot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}.$$

With above operⁿˢ, M is a VS.

eg Let $N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ \begin{array}{l} d \text{ is an integer,} \\ \text{in particular, } d = 10 \end{array} \right\}$.

Check whether N is a VS, with usual addⁿ & scalar multiplicⁿ.

P Ⓑ :- let $\alpha = 1/100$.

$$\text{So, } \alpha \odot N = \alpha \odot \begin{pmatrix} a & b \\ c & \cancel{d}10 \end{pmatrix}$$

$$= \cancel{\alpha}\begin{pmatrix} a/100 & b/100 \\ c/100 & 1/10 \end{pmatrix}$$ Now, d ≠ 10.
So, Ⓑ is false.

ex:- Check whether the set of all singular matrices of order 2 is a vector space with the usual add^n & scalar multiplic^n.

Here, $S = \{A \in M_{22} \mid |A| = 0\}$.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$.

So, $A+B = \begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix}$

$|(A+B)| = -3 \neq 0$.

So, $(A+B) \notin S$.

So, S is not closed under vector add^n & it is not a vector space.

ex. In $\mathbb{R}^2$, define the vector add^n $\oplus$ & scalar multiplic^n $\odot$ as follows:-

$(x, y) \oplus (w, z) = (x+w-2, y+z+3)$

$\alpha \odot (x, y) = (\alpha x - 2\alpha + 2, \alpha y + 3\alpha - 3)$

Check whether $\mathbb{R}^2$ is a vector space under these oper^{ns}.

Let $u, v \in \mathbb{R}^2$ with $u = (x_1, y_1)$, $v = (x_2, y_2)$

By defin^n,

$u \oplus v = (x_1, y_1) \oplus (x_2, y_2)$

$= (x_1 + x_2 - 2, y_1 + y_2 + 3) \in \mathbb{R}^2$.

$\therefore$ $u \oplus v \in \mathbb{R}^2$.

So, $\mathbb{R}^2$ is closed under $\oplus$.

~~$\alpha \oplus v =$~~

Consider $v \oplus u = (x_2, y_2) \oplus (x_1, y_1)$

$= (x_2 + x_1 - 2, y_2 + y_1 + 3)$

$= u \oplus v$

$\therefore$ A1 is true.

Also, we can prove that $\oplus$ is associative.

i.e $(U_1 \oplus U_2) \oplus U_3 = U_1 \oplus (U_2 \oplus U_3)$

LHS $((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$

$\Rightarrow (x_1 + x_2 - 2, y_1 + y_2 + 3) \oplus (x_3, y_3)$

$= (x_1 + x_2 - 2 + x_3 - 2, y_1 + y_2 + y_3 + 3 + 3)$

$= (x_1 + x_2 + x_3 - 4, y_1 + y_2 + y_3 + 6)$

RHS $(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3))$

$= (x_1, y_1) \oplus (x_2 + x_3 - 2, y_2 + y_3 + 3)$

$= (x_1 + x_2 + x_3 - 4, y_1 + y_2 + y_3 + 6)$

= LHS.

So, the associative property is also true.

Illy, we can prove all ~~other~~ other properties

Hence, $\mathbb{R}^3$ is a vector space under given oper^{ns} of vector add^n & scalar multiplic^n.

* To find the additive identity ($\bar{O}$), we proceed as follows :-

Let $U = (x, y) \in \mathbb{R}^2$. Consider $0 \odot U$.

$0 \odot U = 0 \odot (x, y)$

$= (0x - 2(0) + 2, 0.y + 3(0) - 3)$

$= (2, -3)$

$\therefore \bar{O} = (2, -3)$ is the additive identity.

Clearly, $\bar{O} \oplus U = (2, -3) \oplus (x, y)$

$= (2 + x - 2, -3 + y + 3)$

$= (x + y) = U$

So, A③ is true.

* To find the additive inverse for a vector in $\mathbb{R}^2$, we proceed as follows :-

Let $v = (x, y) \in \mathbb{R}^2$.

Consider $(-1) \odot v$.

$= (-1) \odot (x, y)$

$= (-x - 2(-1) + 2, -1.y + 3(-1) - 3)$

$\Rightarrow -v = (-x + 4, -y - 6)$

Here, $-v$ is the additive inverse of v in $\mathbb{R}^2$.

(Since, $v \oplus -v = \bar{0}$).

So, A4 is true.

$\to (x, y) \oplus (-x + 4, -y - 6)$

$= (x - x + 4 - 2, y - y - 6 + 3)$

$= (2, -3) = \bar{0}$.

Clearly, $\alpha \odot v \in \mathbb{R}^2$ for any scalar α & $v \in \mathbb{R}^2$. $\therefore \mathbb{R}^2$ is closed under $\odot$. So, P $\textcircled{B}$ is true.

Let α, β be scalars & $v = (x, y) \in \mathbb{R}^2$.

We shall show that

$(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v)$

LHS $(\alpha + \beta) \odot (x, y)$

$= [(\alpha + \beta) x - 2(\alpha + \beta) + 2, (\alpha + \beta) y + 3(\alpha + \beta) - 3)$

$= (\alpha x + \beta x - 2\alpha - 2\beta + 2, \alpha y + \beta y + 3\alpha + \cancel{\beta}\, 3\beta - 3)$

RHS

$(\alpha \odot v) \oplus (\beta \odot v)$

$= (\alpha \odot (x, y)) \oplus (\beta \odot (x, y))$

$= (\alpha x - 2\alpha + 2, \alpha y + 3\alpha - 3) \oplus (\beta x - 2\beta + 2, \beta y + 3\beta - 3)$

$= (\alpha x + \beta x - 2\alpha - 2\beta + 2 + \cancel{2 - 2},$
$\alpha y + \beta y + 3\alpha + 3\beta - 3 - \cancel{3 + 3}) = $ LHS.

So, B(2) is true.

lly, we can show that B1, B3 & B4 are also true & hence, $\mathbb{R}^2$ is a vector space with the above oper^n.

ex :- In $\mathbb{R}$, define add^n & scalar multiplic^n as follows :-

$$x \oplus y = 2(x+y)$$

$$\alpha \odot x = \alpha x, \ \forall x, y \in \mathbb{R} \ \& \text{ scalar } \alpha.$$

Check for vector space.

Let $(x, y, z) \in \mathbb{R}$. Consider,

$$\cancel{x \oplus} \ (x \oplus y) \oplus z = (2(x+y)) \oplus z$$

$$= 2(2x + 2y + z)$$

$$= 4x + 4y + 2z \longrightarrow ①$$

$$x \oplus (y \oplus z) = x \oplus (2(y+z))$$

$$= 2(x + 2y + 2z)$$

$$= 2x + \cancel{2} \cdot 4y + 4z \longrightarrow ②$$

From ① & ②

$$x \oplus (y \oplus z) \neq (x \oplus y) \oplus z.$$

So, A② is false.

Hence, S is not a vector space for given oper^{ns}.

Results: In a vector space 'B',

(i) $\alpha \bar{0} = (\alpha \odot \bar{0}) = \bar{0}$, for any scalar α

(ii) $0 \odot u = \bar{0}$; $\forall u \in V$.

(iii) If $\alpha \odot u = \bar{0}$, then, either $\alpha = 0$ or $u = \bar{0}$.

(iv) $(-1) \odot u = -u$.

Proof :- $\alpha \odot \bar{0} = \bar{0}$.

Let $\alpha \odot \bar{0} = \alpha \odot (\bar{0} \oplus \bar{0})$

$= \alpha \odot \bar{0} \oplus \alpha \odot \bar{0}$

$\Rightarrow \alpha \odot \bar{0} - \alpha \odot \bar{0} = \alpha \odot \bar{0} \oplus (\alpha \odot \bar{0} - \alpha \odot \bar{0})$

$\Rightarrow \bar{0} = \alpha \odot \bar{0} \oplus \bar{0}$

$\Rightarrow \boxed{\alpha \odot \bar{0} = \bar{0}}$, H.P

Section - 4.2

★ SUBSPACES

Let V be a vector space. A non empty subset 'S' is said to be a subspace of 'V' if 'S' is a vector space w.r.t the same add^n & scalar multiplic^n

Imp. ★★ A non empty subset 'S' of a vectorspace 'V' is a subspace of 'V' iff the following properties are true:

(i) $U \oplus v \in S$ whenever $U, v \in S$.

(ii) $\alpha \odot U \in S$ " $U \in S$ & α : scalar.

Note ★ If S is a subspace of B, then,

$\bar{0} \in S$.

If $\bar{0} \notin S$, then, S is NOT a subspace.

Note that it is important to find $\bar{0}$. It varies.

eg :- Let $V = \mathbb{R}^3$.

Check whether the following are subspaces of V with usual add^n & scalar multiplic^n.

① $S = \{ (x_1, x_2, x_3) \mid x_3 = 0 \}$

② $S = \{ (x_1, x_2, x_3) \mid x_1 = 2 \}$.

③ $S = \{ (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1 \}$.

④ $S = \{ (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0 \}$.

⑤ $S = \{ (x_1, x_2, x_3) \mid x_1 / x_2 = \sqrt{2} \}$

⑥ $S = \{ (x_1, x_2, x_3) \mid x_1 = \sqrt{2}\, x_2 \}$.

① Clearly, $\bar{0} = (0, 0, 0) \in S$. as $x_3 = 0$.

Let $u, v \in S$ with $u = (x_1, x_2, x_3)$ & $v = (y_1, y_2, y_3)$ & $x_3 = 0 = y_3$

$$u \oplus v = (x_1, x_2, x_3) \oplus (y_1, y_2, y_3)$$
$$= (x_1 + y_1, x_2 + y_2, 0) \quad \because x_3 = 0 = y_3$$
$$\in S.$$

So, $u \oplus v \in S$.

Let $u = (x_1, x_2, x_3)$ & α be a scalar

$$\alpha \odot u = \alpha \odot (x_1, x_2, x_3), \; x_3 = 0$$
$$\Rightarrow \alpha \odot u = (\alpha x_1, \alpha x_2, \alpha x_3)$$
$$= (\alpha x_1, \alpha x_2, 0)$$
$$\in S.$$

So, $\alpha \odot u \in S$.

So, ① S is a subspace of V.

② Clearly, $\bar{0} = (0,0,0) \notin S$ as $x_1 = 0 \, (\neq 2)$

So, S, is not a subspace.

③ Clearly, $\bar{0} = (0,0,0) \notin S$.

$\therefore x_1 + x_2 + x_3 = 1 \, (\neq 0)$.

So, S is not a subspace.

④ ~~Here, $\bar{0} = (0,0,0) \in S$ as $x_1 + x_2 + x_3 = 0$~~

M2 Let $\alpha = 10$.

Consider $\alpha \odot u = 10 \odot u = (10x_1, 10x_2, 10x_3)$

Here, $10x_1 + 10x_2 + 10x_3 = 10(x_1 + x_2 + x_3)$
$= 10 \, (\neq 1)$.

So, $10u \notin S$.

$\therefore$ S is not a subspace.

④ Here, $\bar{0} = (0,0,0) \in S$ as $x_1 + x_2 + x_3 = 0$.

Consider $\alpha \odot u = \alpha \odot (x_1, x_2, x_3)$
$= (\alpha x_1, \alpha x_2, \alpha x_3)$.

Now, $\alpha x_1 + \alpha x_2 + \alpha x_3 = \alpha(x_1 + x_2 + x_3)$
$= 0$.

So, Property 'B' is true $(\alpha \odot u \in S)$

Now, Consider $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$

$\Rightarrow x_1 + x_2 + x_3 = 0$ & $y_1 + y_2 + y_3 = 0$

$u \oplus v = ($ ~~$x_1 + y_1$~~ $(x_1, x_2, x_3) \oplus (y_1, y_2, y_3)$

~~$\Rightarrow u \oplus v = x_1 + x_2, y_1 + y_2, z_1 +$~~

$u \oplus v = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

Now

$(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3)$

$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3)$

$= 0 + 0$

$= 0$. So, Property A is true $(u \oplus v \in S)$

Hence, 'S' is a subspace.

⑤ ~~S~~ By defin^{n} of S,

$x_2 = 0$. So, $\bar{0} = (0,0,0) \notin S$.

& hence, S is not a subspace.

⑥ S is a subspace.

imp Q. $S = \{ (x_1, x_2, x_3) \mid x_1 = 2x_2$ (Or) $x_1 = x_3 \}$.

Let $u = (2,1,0)$

$v = (5,3,5)$ (~~10, 5,~~ 10)

Then, $u, v \in S$

Consider $u \oplus v = (7,4,5)$

$\Rightarrow u \oplus v \notin S$.

So, it is not a subspace.

imp Q. $S = \{ (x_1, x_2, x_3) \mid x_1 = 2x_2$ (and) $x_1 = x_3 \}$.

Let $u, v \in S$, $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$

$$\left.\begin{array}{l} \Rightarrow x_1 = 2x_2 \ \& \ x_1 = x_3 \\ \& \ y_1 = 2y_2 \ \& \ y_1 = y_3 \end{array}\right\} \longrightarrow ①.$$

Consider $u \oplus v = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

Now,

$$2(x_2 + y_2) = 2x_2 + 2y_2$$
$$= x_1 + y_1$$

$$\& \quad x_3 + y_3 = x_1 + y_1.$$

$\therefore u \oplus v \in S$.

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Q. Check whether the following are subspaces of $\mathbb{R}^2$ under the usual add^n & scalar multiplic^n.
The set of all vectors of the form

① $(1, a)$ ④ $(a, b) \mid a+b=1$

② $(a, 2a)$ ⑤ $(a, b) \mid a+b=0$

③ ~~E~~ The set of all pts. lying above the line $y = 2x-5$.

① Here, $S = \{(1, a)\}$

Clearly, $\bar{0} \notin S$, as the x-coordinate is $0 \, (\neq 1)$.

$\therefore$ S is not a subspace.

② Let $S = \{(a, 2a)\}$

$0 = 2 \cdot 0 \Rightarrow (0,0) = \bar{0} \in S$

Let $u, v \in S$, $u = (~~x,~~ ~~x~~ a, 2a)$, $v = (b, 2b)$

$u \oplus v = (a, 2a) \oplus (b, 2b)$

$= (a+b, 2(a+b))$

~~=~~ Here, clearly, $y = 2x$.

So, $u \oplus v \in S$.

Let α be a scalar.

So, $\alpha \odot u = \alpha \odot (a, 2a)$

$= (\alpha a, \alpha(2a))$

$= (\alpha a, 2(\alpha a))$

Clearly, $\alpha \odot u \in S$.

So, S is a ~~vector spa~~ subspace of $\mathbb{R}^2$.

③ Here, $S = \{(x, y) \mid y > 2x-5\}$.

Clearly, $\bar{0} \in S$, $0 > -5$.

Let $v = (-1, 0)$. Then, $0 > -2(-1)-5$

$\Rightarrow v \in S$.

Let $\alpha = -10$.

Consider $\alpha \odot v = (-10) \odot (-1, 0)$

$= (10, 0)$

Here, $x = 10$, $y = 0$

$0 \not> (20 - 5)$

$\Rightarrow \alpha \odot v \notin S$.

So, S is not a subspace of $\mathbb{R}^2$.

Q. ⑥ Set of all pts. lying on the line $y = 2x-5$.

$\rightarrow$ ✗ $\bar{0} \notin S$

⑦ Set of all pts. lying on the line $y = 2x$

$\rightarrow$ ✓

⑧ In M_{22}, let S be the set of all matrices of the form

(i) $\begin{pmatrix} a & -a \\ b & 0 \end{pmatrix}$

(ii) matrices having $tr(A) = 0$.

Check whether S is a subspace of M_{22} with usual addⁿ & scalar multiplicⁿ.

(i) Here $\bar{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Here, $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} b = -a \\ d = 0 \end{matrix} \right\}$.

Clearly, $\bar{0} \in S$.

Let $U, V \in S$.
with $U = \begin{pmatrix} a_1 & -a_1 \\ b_1 & 0 \end{pmatrix}$, $V = \begin{pmatrix} a_2 & -a_2 \\ b_2 & 0 \end{pmatrix}$

So, $U \oplus V = \begin{pmatrix} a_1+a_2 & -(a_1+a_2) \\ b_1+b_2 & 0 \end{pmatrix}$

Clearly, $U \oplus V \in S$.

Let α be any scalar.

So, $\alpha \odot U = \alpha \odot \begin{pmatrix} a_1 & -a_1 \\ b_1 & 0 \end{pmatrix}$

$= \begin{pmatrix} \alpha a_1 & -\alpha a_1 \\ \alpha b_1 & 0 \end{pmatrix}$

Clearly, $\alpha \odot U \in S$.

So, S is a subspace.

(ii) Let $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a+d=0 \right\}$

Here, $\bar{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ Now, $tr(\bar{0}) = 0+0 = 0$.

Clearly, $\bar{0} \in S$.

Now, let $U, V \in S$ s.t

$U = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}$, $V = \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix}$

$U \oplus V = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & -(a_1+a_2) \end{pmatrix} \in S$. $(\because (a_1+a_2)+(-1)(a_1+a_2) = 0$.

So, $U \oplus V \in S$.

Consider some scalar α.

So, $\alpha \odot U = \alpha \odot \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & -\alpha a_1 \end{pmatrix} \in S$.

Here, $tr(\alpha \odot U) = 0$.

So, $\alpha \odot U \in S$. So, it is a subspace.

Q. Check whether $S = \begin{pmatrix} a & -a+1 \\ b & 0 \end{pmatrix}$ is a subspace or not.

Clearly, $\bar{0} \notin S$. So, it is not a subspace.

Q. Check $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b \in \mathbb{Z} \right\}$.

Clearly, it is not closed under scalar multiplicⁿ ~~& α = 1/2~~

Let $\alpha = \frac{1}{4}$ for matrix $U = \begin{pmatrix} 1 & 2 \\ 5 & -6 \end{pmatrix}$.

So, $\alpha \odot U = \alpha \odot \begin{pmatrix} 1 & 2 \\ 5 & -6 \end{pmatrix}$

$\Rightarrow \alpha \odot U = \begin{pmatrix} 1/4 & 1/2 \\ 5/4 & -3/2 \end{pmatrix}$.

Clearly, $1/4$ & $1/2 \notin \mathbb{Z}$.

So, $\alpha \odot U \notin S$.

So, S is not a subspace.

Q. In $P_5 = \{ p \in P \mid \text{degree } p \leqslant 5 \}$.

Let S = S.

① $S = \{ p \in P_5 \mid P(3) = 0 \}$.

② $S = \{ p \in P_5 \mid P(3) = 10 \}$

③ $S = \{ p \in P_5 \mid p(3) = p(1) \}$.

④ $S = \{ p \in P_5 \mid p(3) = 1 + p(1) \}$.

⇒ ⑤ $S = \{ p \in P_5 \mid \deg . p = 5 \}$.

Note.

The set P denotes the set of all polynomials over the field of real nos.

With usual add^n & scalar multiplic^n of polynomials, P is a vector space.

$P_5 = \{ p \in P \mid \deg p \leq 5 \}$.

Then, P_5 is a vector space with usual add^n & scalar multiplic^n.

The zero polynomial, the additive identity $(\bar{0})$ is defined as

$$\bar{0}(x) = 0$$
$$= 0 + 0x + 0x^2 + \dots + 0x^5 \text{ in } P_5.$$

① By defin^n :-

$\bar{0}(x) = 0$ for any x.

$\Rightarrow \bar{0}(3) = 0$.

So, $\bar{0} \in S$.

Let $p, q \in S$.

$\Rightarrow p(3) = 0$ & $q(3) = 0$.

Consider

$$(p \oplus q)(3) = p(3) + q(3)$$
$$= 0 + 0$$
$$= 0.$$

$\therefore p \oplus q \in S$

Consider a scalar α & $p \in S$. $\Rightarrow p(3) = 0$.

So, $(\alpha \odot p)(3) = \alpha \odot [p(3)] = \alpha \odot 0$
$= 0$,

So, $\alpha \odot p \in S$. So, it is a subspace.

★ Note:- $(p \oplus q)(x) = p(x) + q(x)$

$(\alpha \odot p)(x) = \alpha\, p(x)$

② Let $p, q \in S$ then, $p(3)=10$ & $q(3)=10$.
Consider

$$(p \oplus q)(3) = p(3) + q(3)$$
$$= 10 + 10$$
$$= 20 (\neq 10).$$

So, $(p \oplus q) \notin S$.
So, S is not a ~~subsla~~ subspace.

③ By defin^n, $\bar{0}(x) = 0 \;\forall\; x$.
$\Rightarrow \bar{0}(3) = 0 = \bar{0}(1)$
$\therefore \bar{0} \in S$.

Let $p, q \in S \Rightarrow \left.\begin{array}{l} p(3) = p(1) \\ \& \; q(3) = q(1) \end{array}\right\}$ ①.

$$\text{So, } (p \oplus q)(3) = p(3) + q(3)$$
$$= p(1) + q(1) \text{ (from ①)}$$
$$= (p \oplus q)(1)$$

So, $(p \oplus q) \in S$.

Let α be a scalar & $p \in S$.
So, $p(3) = p(1)$

$$(\alpha \odot p)(3) = \alpha (p)(3)$$
$$= \alpha (p)(1)$$
$$= (\alpha \odot p)(1)$$

So, $(\alpha \odot p) \in S$.
So, S is a subspace.

④ $S = \{p \in P_5 \mid p(3) = 1 + p(1)\}$.

~~H~~ Here, $\bar{0}(x)$

So, $\bar{0}(3) \overset{\text{should be}}{=}$ ~~0~~. equal to

& $\bar{0}(1) = 0$.

But $\bar{0}(3) = 1 + (\bar{0})(1)$

So, $\bar{0} \notin S$.

Hence, S is not a subspace.

⑤ $S = \{p \in P_5 \mid \deg p = 5\}$

By definⁿ, $\bar{0}(x) = 0 + 0x + 0x^2 + \text{---} + 0x^5$

$\therefore \bar{0} \in S$.

Now, let $p, q \in S$ s.t

$p(x) = -2x^5 + 5x^4 + 3$.

$q(x) = 2x^5 + 3x^4 + 9$

$\Rightarrow (p \oplus q)(x) = 8x^4 + 12$.

$\deg (p \oplus q)(x) = 4 \ (\neq 5)$.

So, $(p \oplus q) \notin S$.

So, it is not a subspace.

Note:- ✱ Let F denote the set of all real valued fⁿˢ; defined the addⁿ & scalar multiplicⁿ on F as follows :-

(i) $(f \oplus g)(x) = f(x) + g(x)$

(ii) $(\alpha \odot f)(x) = \alpha f(x)$; $\forall x \in \mathbb{R}$.

Then, F is a vector space w.r.t $\oplus$ & $\odot$.

Q. Check whether the set of all real valued f^{ns} with

(i) $f(1)=0$

(ii) $f(1/2)=0$

(iii) $f(1/2)=2$.

is a subspace of F with usual add^n & scalar multiplic^n.

(i) $S=\{f\in F \mid f(1)=0\}$

By defin^n, $\bar{0}(x)=0 \;\forall x$

$\Rightarrow \bar{0}(1)=0$

$\Rightarrow \bar{0}\in S$

Let $f,g\in S.$ $\Rightarrow f(1)=0=g(1) \longrightarrow ①$

Consider $(f\oplus g)(1)=f(1)+g(1)$

$=0+0$

$=0$

$\therefore (f\oplus g) \in S.$

Let α be a scalar, $f\in S.$

$(\alpha\odot f)(1)=\alpha f(1)$

$=\alpha(0)=0$

$\therefore (\alpha\odot f)\in S.$

So, S is a subspace.

(ii) S is also a subspace (Same as (i))

(iii) ~~$f(1/2)=2$~~.

$S=\{f\in F \mid f(1/2)=2\}$.

Clearly, $\bar{0}(x)=0\;(\neq 2)$.

So, $\bar{0}\notin S$.

Hence, S is not a subspace.

Problem (XV)

Q. Show that the set of all vectors of the form $[2a-3b, a-5c, a, 4c-b, c]$ form a subspace of $\mathbb{R}^5$ with usual add^n & scalar multiplic^n.

Clearly,

$$\bar{0} = (2.0-3.0, 0-5.0, 0, 4.0-0, 0) \in S.$$

$$\Rightarrow \bar{0} \in S.$$

Let u, v ~~be~~ $\in S$ s.t.

$$u = [2a_1-3b_1, a_1-5c_1, a_1, 4c_1-b_1, c_1]$$

$$v = [2a_2-3b_2, a_2-5c_2, a_2, 4c_2-b_2, c_2].$$

$$u \oplus v = [2(a_1+a_2)-3(b_1+b_2),$$
$$(a_1+a_2)-5(c_1+c_2),$$
$$a_1+a_2,$$
$$4(c_1+c_2)-(b_1+b_2),$$
$$c_1+c_2] \in S.$$

So $u \oplus v \in S$.

~~So its a subs~~

Let α be any scalar & $u \in S$.

$$\text{So, } \alpha \odot u = (\alpha \odot (2a_1-3b_1, a_1-5c_1, a_1, 4c_1-b_1, c_1)$$
$$= \begin{cases} \alpha(2a_1-3b_1), \alpha(a_1-5c_1), \alpha a_1, \alpha(4c_1-b_1), \\ \alpha c_1 \end{cases}$$

Clearly $\alpha \odot u \in S$.

So, S is a subspace.

Section - 4.3

✱ SPAN OF A SUBSET

→ Linear Combinⁿ (l.c.):

Let S be a non empty subset of a vector space V. Let $v \in V$.

v is said to be a l.c. of vectors in S if v can be expressed as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n V_n,$$

where, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars.

& $v_1, V_2, \ldots, v_n \in S$.

eg ① Let $V = \mathbb{R}^3$

& $S = \{(1,1,-1), (2,3,4), (0,0,1)\}$

Then, the vector $v = (3,4,3)$ is a ~~ln~~ l.c. of vectors in S.

$$v = (3,4,3)$$
$$= 1.(1,1,-1) + 1.(2,3,4) + 0.(0,0,1)$$

✱ Span:

Let S be a non empty subset of a vector space V. The set of all possible l.c. of vectors in S is called Span of S.

Notation:

span (S) or span S or [S].

For the set S given in eg ①, we can write

$\text{span } S = \{\alpha_1(1,1,-1) + \alpha_2(2,3,4) + \alpha_3(0,0,1) \mid \alpha_1, \alpha_2 \,\&\, \alpha_3 \in \mathbb{R}\}$

$= \{(\alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, -\alpha_1 + 4\alpha_2 + \alpha_3) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$

Note:- If $S = \{v_1, v_2, --- v_n\} \subset V$

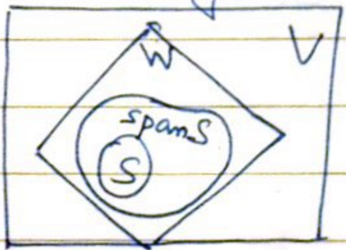
Then, $\text{span } S = [S] = \alpha_1 v_1 + \alpha_2 \overset{v_2}{+} \alpha_3 v_3 + \cdots \alpha_n v_n \mid \alpha_1, \alpha_2 \,.\, — \alpha_n \in \mathbb{R}\}$.

* PROPERTIES OF span S

P1) If S is a subset of a vector space V, then, $S \subseteq \text{span } S$.

P2) ~~&~~ span S is a subspace of V.

P3) If W is a subspace of V, containing S, then, $\text{span } S \subseteq W$.



P4) span S is the smallest subspace of V containing S.

From the Problem ⊛,

$S = \{(2a-3b, a-5c, a, 4c-b, c) \mid a, b, c \in \mathbb{R}\}$

is a subspace of $\mathbb{R}^5$ with usual add^n & scalar multiplic^n.

We can write S as

$S = \{ a(2, 1, 1, 0, 0) + b(-3, 0, 0, -1, 0) + c(0, -5, 0, 4, 1) \mid a, b, c \in \mathbb{R} \}$.

Let $U = \{ (2, 1, 1, 0, 0), (-3, 0, 0, -1, 0), (0, -5, 0, 4; 1) \}$

Then $\boxed{S = \text{span } U}$.

Q. Check whether the vector $(-4, 5, -13) \in \text{span} S$ if $S = \{ (1, -2, -2), (3, -5, 1), (-1, 1, 5) \}$

Let $v = (-4, 5, -13)$

$u_1 = (1, -2, -2)$

$u_2 = (3, -5, 1)$

$u_3 = (-1, 1, 5)$

Let $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \longrightarrow (A)$

$\Rightarrow (-4, 5, -13) = \alpha_1 (1, -2, -2) + \alpha_2 (3, -5, 1) + \alpha_3 (-1, 1, 5)$

$\Rightarrow (-4, 5, -13) = (\alpha_1 + 3\alpha_2 - \alpha_3, -2\alpha_1 - 5\alpha_2 + \alpha_3, -2\alpha_1 + \alpha_2 + 5\alpha_3)$

$$\Rightarrow \left.\begin{aligned} \alpha_1 + 3\alpha_2 - \alpha_3 &= -4 \\ -2\alpha_1 - 5\alpha_2 + \alpha_3 &= 5 \\ -2\alpha_1 + \alpha_2 + 5\alpha_3 &= -13 \end{aligned}\right\} \longrightarrow ①$$

The sys ① ~~A~~, if it has real solⁿ for α_1, α_2 & α_3, then, $v \in \text{span} S$.

$$(A : B) = \left[\begin{array}{ccc|c} 1 & 3 & -1 & -4 \\ -2 & -5 & 1 & 5 \\ -2 & 1 & 5 & -13 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & 7 & 3 & -21 \end{array}\right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 10 & 0 \end{array}\right] R_3 - 7R_2$$

Here,

$\rho(A) = 3 = \rho(A:B) =$ no of scalars

$\therefore$ The sys. is consistent with unique solⁿ.

$\therefore$ $v \in$ span S.

To get Ⓐ, solve for α_1, α_2 & α_3.

From the reduced matrix,

$\alpha 10_3 = 0 \Rightarrow \boxed{\alpha_3 = 0}$

$\alpha_2 - \alpha_3 = -3$

$\Rightarrow \boxed{\alpha_2 = -3}$

$\alpha_1 + 3\alpha_2 - \alpha_3 = -4$

$\Rightarrow \alpha_1 - 9 = -4$

$\Rightarrow \boxed{\alpha_1 = 5}$

So, $v = 5u_1 - 3u_2 + 0u_3$.

$\Rightarrow v = 5(1, -2, -2) - 3(3, -5, 1) + 0(-1, 1, 5)$

Q. Let $S = \{x^3 - 2x^2 + x - 3,\ 2x^3 - 3x^2 + 2x + 5,\ 4x^2 + x - 3,\ 4x^3 - 7x^2 + 4x - 1\}$,

be a subset of P,

Check whether $3x^3 - 8x^2 + 2x + 16 \in$ span S.

& if so, express it in terms of vectors in S.

Let $p(x) = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 p_4 \hookrightarrow ①$

$\to$ coeff. of x^3 in row 1

$$(A:B) = \begin{array}{c} x^3 \\ x^2 \\ x \\ c \end{array}\left[\begin{array}{cccc|c} \underline{1} & 2 & 0 & 4 & 3 \\ -2 & -3 & 4 & -7 & -8 \\ 1 & 2 & 1 & 4 & 2 \\ -3 & 5 & -3 & -1 & 16 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 3 \\ 0 & \underline{1} & 4 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 11 & -3 & 11 & 25 \end{array}\right] \begin{array}{l} \\ R_2+2R_1 \\ R_3-R_1 \\ R_4+3R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 3 \\ 0 & 1 & 4 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -47 & 0 & 47 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4-11R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 3 \\ 0 & 1 & 4 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4/47 \end{array}$$

$$\sim \begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \\ \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 3 \\ 0 & 1 & 4 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \\ R_4+R_3 \end{array}$$

$\rho(A) = 3 = \rho(A:B) < 4$, the no. of scalars

So, sys. is consistent with many sol^{ns}.

$\therefore\ p(x) = 3x^3 - 8x^2 + 2x + 16 \in \text{span } S$.

To find l.c ①,
we solve reduced sys.

$\Rightarrow \boxed{\alpha_3 = -1}$

& $\alpha_2 + 4\alpha_3 + \alpha_4 = -2$

$\alpha_1 + 2\alpha_2 + 4\alpha_4 = 3$

let $\boxed{\alpha_4 = 1}$ (choosing any parameter)

then, $\alpha_2 = -2 - 4\alpha_3 - 4\alpha_4$

$= -2 + 4 - 1$

$\Rightarrow \boxed{\alpha_2 = 1}$

$\alpha_1 = 3 - 2\alpha_2 - 4\alpha_4$

$\Rightarrow \boxed{\alpha_1 = -3}.$

$\therefore \; p(x) = -3\,p_1(x) + 1(p_2(x)) - 1\,p_3(x) + 1.p_4(x)$

↳ ∃ many other sol^{ns} based on choice of variable values taken.

Q. Show that the vectors in
$S = \{ \underset{u_1}{(1, 3, -1)}, \underset{u_2}{(2, 7, -3)}, \underset{u_3}{(4, 8, -7)} \}$ span $\mathbb{R}^3$.

↳ means that all the elements of S can be used to make all elemts of $\mathbb{R}^3$.

Let $v = (a, b, c)$ be any vector in $\mathbb{R}^3$.

Let $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \longrightarrow ①$

If sys ① has a solⁿ v for α_1, α_2 & α_3 then, $v \in$ span S. & hence, the set S spans the vector space $\mathbb{R}^3$.

$$(A:B) = \left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 3 & 7 & 8 & b \\ -1 & -3 & -7 & c \end{array}\right]$$

* Spans means generates

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & 1 & -4 & b-3a \\ 0 & -1 & -3 & c+a \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3+R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & 1 & -4 & b-3a \\ 0 & 0 & -7 & b+c-2a \end{array}\right] \begin{array}{l} \\ \cancel{R_2+R} \\ R_3+R_2 \end{array}$$

or vectors in S.

$r(A) = r(A:B) = 3$. = the no. of scalars.

So, sys. is consistent with unique solⁿ & hence, $v \in \text{span}(S)$.

∴ The set S spans the vector space $\mathbb{R}^3$.

(Express the vector $v = (1, 3, 4)$ in terms of v_1, v_2 & v_3.

To find relⁿ ①, we proceed as follows:

$$-7\alpha_3 = b+c-2a$$

$$\Rightarrow \boxed{\alpha_3 = \frac{b+c-2a}{-7} = \frac{2a-b-c}{7}}$$

$$\alpha_2 - 4\alpha_3 = b-3a$$

$$\Rightarrow \alpha_2 = b-3a+\frac{4}{7}(2a-b-c)$$

$$= \frac{7b-4b-21a+8a-4c}{7}$$

$$\Rightarrow \boxed{\alpha_2 = \frac{3b-13a-4c}{7}}$$

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = a$$

$$\Rightarrow \alpha_1 = a - \frac{2}{7}(3b-13a-4c) - \frac{4}{7}(2a-b-c)$$

$$\Rightarrow \alpha_1 = \frac{7a-6b+26a+8c-8a+4b+4c}{7}$$

$$\Rightarrow \boxed{\alpha_1 = \frac{25a - 2b + 12c}{7}}$$

$\therefore \quad v = \alpha_1 U_1 + \alpha_2 U_2 + \alpha_3 U_3.$

$$\therefore \boxed{(\underset{a}{1}, \underset{b}{3}, \underset{c}{4}) = \frac{67}{7} U_1 - \frac{20}{7} U_2 - \frac{5}{7} U_3}.$$

Q. Show that the vectors $\{ x^2 + x + 1, x + 1, 1 \}$ span the vector space P_2

$P_2 = \{ p \in P \mid \deg P \leq 2 \}.$

Let $p \in P_2$ with

$p(x) = ax^2 + bx + c.$

Let $p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x)$,

where $p_1 x = x^2 + x + 1$

$p_2(x) = x + 1$

$p_3(x) = 1.$

Now, $(A \vdots B) = \begin{array}{c} x^2 \\ x \\ c \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 1 & 1 & 0 & b \\ 1 & 1 & 1 & c \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b-a \\ 0 & 1 & 1 & c-a \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b-a \\ 0 & 0 & 1 & c-b \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

Here, $r(A) = r(A:B) = 3 =$ no. of vectors in S.
So, sys. is consistent with unique sol^n
$\therefore \forall\ p(x) \in \text{span } S$.
$\therefore$ The vectors span P_2.

Section 4.4

LINEAR INDEPENDENCE

Let V be a vector space. Then, the vectors $v_1, v_2, ---, v_n$ in V are said to be linearly dependent (LD) if $\exists$ ~~scars~~ scalars $\alpha_1, \alpha_2, \text{———}, \alpha_n$; $\boxed{\text{not}}$ ALL of them zero, s.t. $\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_n v_n = \bar{0}$.

If the vectors are not LD, then, they are said to be linearly independent (LI), i.e.,

if $\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_n v_n = \bar{0}$

then, $\alpha_1 = \alpha_2 = \alpha_3 = ---.. = \alpha_n = \textcircled{0}$.

For ~~LI~~ linear Independence

eg:- Let $u = (1, 0, -1)$, $v = (2, 5, 3)$, $w = (2, 0, -2)$

Then, $w = 2u$, or

$$1 w + (-2) u = \bar{0}.$$

$\Rightarrow$ u & w are LD.

The vectors (v & w) & (u & v) are LI..

Q. Check whether the following set is linearly independent

$$S = \{ \underset{u_1}{(1,2,1)}, \underset{v_2}{(-1,1,0)}, \underset{v_3}{(5,-1,2)} \} ; V = \mathbb{R}^3$$

Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \bar{0} \longrightarrow ①$

$\alpha_1 \odot (1,2,1) + \alpha_2 \odot (-1,1,0) + \alpha_3 \odot (5,-1,2) = (0,0,0)$

$\Rightarrow (\alpha_1 - \cancel{2}\alpha_2 + 5\alpha_3 , 2\alpha_1 + \alpha_2 - \alpha_3 , \alpha_1 + 2\alpha_3) = (0,0,0)$.

$$\Rightarrow \left.\begin{aligned} \alpha_1 - \alpha_2 + 5\alpha_3 &= 0 \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= 0 \\ \alpha_1 \quad + 2\alpha_3 &= 0 \end{aligned}\right\} \longrightarrow ①$$

Now,

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 5 \\ 0 & 3 & -11 \\ 0 & 1 & -3 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 5 \\ 0 & 3 & -11 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ 3R_3 \cancel{+R_1} - \cancel{3}R_2 \end{matrix}$$

Here, $r(A)$ = no. of vectors (or scalars).

So sys① has only zero solⁿ.

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$.

So, S is LI

Note :-

The above method is referred to as <u>independent test method</u>.

S1) Write vectors given, as column vectors, to form A.

S2) Reduce A to an upper Δlar matrix.

S3) If $r(A) = n$, the no. of vectors in S (or scalars) then, all the scalars are zero & S is LI.

~~S4~~ If $r(A) < n$, the sys. has non zero solns for scalars & S is LD.

Q. Which of the following sets are LI ?
$V \in \mathbb{R}^3$

① $S = \{(4,2,1);\ (-1,3,7);\ (0,0,0)\}$

② $S = \{(2,-5,1)(1,1,-1)(0,2,-3)(2,2,6)\}$

③ $S = \{(2,-1,3)(4,-1,6)(-2,0,2)\}$

④ $S = \{(5,-2,3)(-4,1,-7)(7,-4,-5)\}$

①. $$A = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 7 & 0 \end{bmatrix}.$$

∵ S contains a $\bar{0} = (0,0,0)$, it is LD.

$$\left(\because S = \cancel{B}\, 0.(4,2,1) + 0.(-1,3,7) + \underset{\neq 0}{\underset{\downarrow}{\textcircled{5}}}(0,0,0) = \bar{0} \right)$$

Notes :-

① The sets $S = \{\bar{0}\}$ or any set consisting $\bar{0}$ like $S = \{v_1, v_2, \dots, \bar{0}\}$ are always $\boxed{LD}$

② If $S = \{v_1\}$; $v_1 \neq \bar{0}$, then S is always $\boxed{LI}$

② Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = \bar{0} \longrightarrow ①$

$$A = \begin{bmatrix} 2 & 1 & 0 & 2 \\ -5 & 1 & 2 & 2 \\ 1 & -1 & -3 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 7 & 4 & 14 \\ 0 & -3 & -6 & 10 \end{bmatrix} \begin{matrix} \\ 2R_2 + 5R_1 \\ 2R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 7 & 4 & 14 \\ 0 & 0 & -30 & 112 \end{bmatrix} \begin{matrix} \\ \\ 7R_3 + 3R_2 \end{matrix}$$

Here, $r(A) = 3$ ($\neq 4$, the no. of scalars)

So, the sys ~~is~~ has non zero solns for $\alpha_1, \alpha_2, \alpha_3$ & α_4.

$\therefore$ S is LD.

③ Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \bar{0} \longrightarrow ①$.

$$A = \begin{bmatrix} 2 & 4 & -2 \\ -1 & -1 & 0 \\ 3 & 6 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 10 \end{bmatrix} \begin{matrix} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{matrix}$$

Here, $\varepsilon(A) = 3 = n$, the no. of vectors in S.
So, sys ① has only zero sol^n.
$\alpha_1 = \alpha_2 = \alpha_3 = \bar{0}$.
So, S is LI.

④ Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \bar{0} \longrightarrow ①$

$$A = \begin{bmatrix} \underline{5} & -4 & \cancel{6}\ 7 \\ -2 & 1 & -4 \\ 3 & -7 & -5 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 5 & -4 & 7 \\ 0 & \underline{-3} & -6 \\ 0 & -23 & -46 \end{bmatrix} \begin{matrix} \\ 5R_2 + 2R_1 \\ 5R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 5 & -4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ 3R_3 - 23R_2 \end{matrix}$$

Here, $\varepsilon(A) = 2$ ($\neq 3$, the no. of scalars)
So, sys. has non zero sol^{ns} for α_1, α_2 & α_3.
So, sys is LD.

Q. Check which of the following sets are LI in $\mathbb{R}^4$ using the independent test method.

① $S = \{ \overset{v_1}{(1, 3, -2, 4)} \overset{v_2}{(3, 11, -2, -2)} \overset{v_3}{(2, 8, 3, -9)}$
$(3, 11, -8, 5) \}$

② $S = \{ (1, 0, 0, 0) (1, 1, 0, 0) (1, 1, 1, 1) (0, 0, 1, 1) \}$

① Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = \bar{0}$

$$\text{So, } \bar{A} = \begin{bmatrix} 1 & 3 & 2 & 3 \\ 3 & 11 & 8 & 11 \\ -2 & -2 & 3 & -8 \\ 4 & -2 & -9 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 4 & 7 & -2 \\ 0 & -14 & -17 & -7 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \\ R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & -3 & 7 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \\ R_4 + 7R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 + R_3 \end{matrix}$$

So, $\rho(A) = 4 =$ no. of scalars.
So, $\exists$ only one value 0 for scalars $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.
So, S is LI.

② Let $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = \bar{0}$.

$$So, A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 - R_3$$

Here, $\rho(A) = 3$ ($\neq 4$, the no. of scalars)
So, $\alpha_1, \alpha_2, \alpha_3$ & α_4 will have non zero values (sol^ns).
So, S is LD.

Q. Find, which of the following ~~sts~~ sets are LI?

① $S = \{x^2-1, x+1, x-1\}$, $V = P_2$.
② $S = \{1, x+x^2, x-x^2, 3x\}$, $V = P_2$.
③ $S = \{x^2+x+1, x^2-1, x^2+1\}$, $V = P_2$.
④ $S = \{1+x^2-x^3, 2x-1, x+x^3\}$, $V = P_3$.
⑤ $S = \{3x^3+2x+1, x^3+x, x-5, x^3+x-10\}$, $V = P_3$

① Let $P_1 = x^2-1$

$P_2 = x+1$

$P_3 = x-1$

So, let $\alpha_1 P_{1(x)} + \alpha_2 P_{2(x)} + \alpha_3 P_{3(x)} = \bar{0}$ $\longrightarrow$ ①

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} R_3 - R_2$$

Here, $r(A) = 3 = n$, the no. of scalars.

So, $\exists$ only zero value for the scalars

$\Rightarrow$ $\alpha_1 = \alpha_2$ & $\alpha_3 = 0$

So, S is LI in P_2.

② Let $P_1 = 1$

$P_2 = x + x^2$

$P_3 = x - x^2$

$P_4 = 3x$.

Let $\alpha_1 P_1(x) + \alpha_2 P_2(x) + \alpha_3 P_3(x) + \alpha_4 P_4(x) = \bar{0}$ $\longrightarrow$ ①

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -2 & -3 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$

Here, $r(A) = 3$ ($\neq 4$, the no. of scalars).
So, $\exists$ many non-zero solns for scalars.
& S is LD

let $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 p_4 = \bar{0} \longrightarrow ①$

5) $$A = \begin{array}{c|cccc|} x^3 & 3 & 3 & 0 & 1 \\ x^2 & 0 & 0 & 0 & 0 \\ x & 2 & 1 & 1 & 1 \\ c & 1 & 0 & -5 & -10 \end{array}$$

$$\sim \begin{bmatrix} 3 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 3 & 1 \\ 0 & -3 & -15 & -31 \end{bmatrix} \begin{matrix} \\ \\ 3R_3 - 2R_1 \\ 3R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 3 & 0 & 1 \\ 0 & -1 & -15 & -31 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_4 \\ \\ R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & -1 & -15 & -31 \\ 0 & 0 & -12 & -30 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 + R_2$$

Here, $\rho(A) = 3 < 4$, the no. of vectors.
$\therefore$ The sys. ① has many non zero sol^{ns} for α_1 & $\alpha_2, \alpha_3, \alpha_4$.

$\Rightarrow$ S is LD.

Q Check whether the set $S = \left\{ \underset{A_1}{\begin{array}{c} \begin{matrix} a_{11} & a_{12} \end{matrix} \\ \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \\ \begin{matrix} a_{21} & a_{22} \end{matrix} \end{array}}, \underset{A_2}{\begin{pmatrix} 3 & 2 \\ -6 & 1 \end{pmatrix}}, \underset{A_3}{\begin{pmatrix} 4 & -1 \\ -5 & 2 \end{pmatrix}}, \underset{A_4}{\begin{pmatrix} 3 & -3 \\ 0 & 0 \end{pmatrix}} \right\}$

is LI in M_{22}

Consider, $\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = \bar{0}$

$$A = \begin{array}{c} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{array} \begin{bmatrix} 1 & 3 & 4 & 3 \\ -2 & 2 & -1 & -3 \\ 0 & -6 & -5 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 8 & 7 & 3 \\ 0 & -6 & -5 & 0 \\ 0 & -2 & -2 & -3 \end{bmatrix} \begin{array}{l} \\ R_2 + 2R_1 \\ \\ R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 8 & 7 & 3 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & -1 & -9 \end{bmatrix} \begin{matrix} \\ \\ 4R_3 + 3R_2 \\ 4R_4 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 8 & 7 & 3 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 + R_3 \end{matrix}$$

Here, $r(A) = 3 < 4$, the no. of vectors.
So, the sys ① has many non zero sol^ns for
$\therefore$ S is <u>LD</u>

V= span B

③

Section - 4.5

§ Basis & Dimension

Let V be the vector space. A basis of the vector space V is a set of vectors 'B' in V satisfying the following properties :-

(i) B is LI.

(ii) B spans V ⇒ every vector of a vector space can be expressed as a L.c of vectors in B

↳ i.e, V = span B.

The no. of vectors in a basis is called DIMENSION of vector space V & is denoted by 'dim V'.

ex①: Consider the vector space

$V = \mathbb{R}^2$, with usual addⁿ & scalar multiplic

Then, the set

$S = \{(1,-1), (2,7)\}$

is LI & spans $\mathbb{R}^2$.

No. of vectors in $S = 2 = \dim(\mathbb{R}^2)$, as S is ~~the~~ a basis for $\mathbb{R}^2$.

ex②: The set $B = \{e_1, e_2\} = \{(1,0), (0,1)\}$ is also a basis for R^2, called as the 'std. basis'.

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ex③ : IIly, $B = \{e_1, e_2, e_3\} = \{(0,0,1)(0,1,0)(1,0,0)\}$ is the std. basis for $\mathbb{R}^3$ &
$\dim(\mathbb{R}^3) = 3$.

ex④ For the vector space $V = P_n$, the set $B = \{1, x, x^2, \ldots, x^n\}$ is the Std. Basis & $\boxed{\dim(P_n) = n+1}$.

ex⑤ For the vector space M_{32}, the std. Basis

$$B = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}}_{e_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e_4}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}}_{e_5}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_6} \right\}$$

So, $\dim(M_{32}) = 6$.

Note: For a vector space M_{mn}, $\dim(M_{mn}) = m.n$.

Note:- If the no. of vectors in a basis of a vector space is finite, then, it is said to be a FINITE DIMENSIONAL VECTOR SPACE. Otherwise, it is said to be INFINITE DIMENSIONAL

Note:- All the above examples define only FINITE dimensional vector spaces.
The vector space P is infinite dimensional with the std basis
$$B = \{1, x, x^2 \ldots\}.$$

* In any finite dimensional vector space, the max. no. of LI vectors = dim(V)

☆ SIMPLIFIED SPAN METHOD

To check whether ~~a~~ the given set S is a ~~for~~ Basis, we proceed as follows:

S1. Form the matrix A by writing the given vectors as ROW VECTORS.

S2. Using elementary row operⁿˢ, reduce the matrix A into an upper Δlar matrix.

S3. Find $r(A)$.

- If $r(A) = n$, the no. of vectors, then, set is LI.
- If $r(A) < n$, the set is LD. Then, it cannot be a basis for the given vector space.

S4. If S is LI, & dim (V) = no. of vectors in S = n,

then, S spans V & is a basis.

The above method is referred to as Simplified Span Method.

Note:- ① If S is not a basis, then, the non zero rows in the reduced matrix or the corresponding vectors in S form a basis for the subspace W = span(S).

☆ ② If V is a finite dimensional vector space, with dim(V) = n, then, the max. no. of LI vectors = n.

PROBLEMS

↳ SECTION 4.4

Q1) Check whether the following sets are LI :-

① $S = \{\sin^2 x, \cos^2 x, \cos 2x\}$.

② $S = \{x, \sin x, \cos x\}$

③ $S = \{x^2 e^x, x e^x, (x^2+x-1)e^x\}$.

④ $S = \{t^2, t, e^t\}$

in the vector space of all real valued f^n 'F'

② Let $u_1 = x$, $u_2 = \sin x$, $u_3 = \cos x$.

Consider $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \bar{0}$.

$\Rightarrow \alpha_1 x + \alpha_2 \sin x + \alpha_3 \cos x = 0 \longrightarrow ①$,

eq^n ① is true $\forall\ x \in \mathbb{R}$.

Let $x = 0$

$\Rightarrow \boxed{\alpha_3 = 0}$.

Let $x = \pi/2$

$\Rightarrow \frac{\pi}{2}\alpha_1 + \alpha_2 = 0 . \longrightarrow ②$

Let $x = \pi$.

$\Rightarrow \pi\alpha_1 = 0 \Rightarrow \boxed{\alpha_1 = 0}$

$\Rightarrow 0 + \alpha_2 = 0 \Rightarrow \boxed{\alpha_2 = 0}$

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$.

So, the set is LI.

① Let $v_1 = \sin^2 x$, $v_2 = \cos^2 x$, $v_3 = \cos 2x$

Consider $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \bar{0}$.

$\Rightarrow \alpha_1 \sin^2 x + \alpha_2 \cos^2 x + \alpha_3 \cos 2x = 0 \rightarrow$ ①

$\Rightarrow$ eqn ① is true ∀ $\mathbb{R}$.

M1 Let $x = 0$

$\Rightarrow \alpha_2 + \alpha_3 = 0 \rightarrow$ ②

Let $x = \pi/2$

$\Rightarrow \alpha_1 \cancel{= 0} - \alpha_3 = 0 \rightarrow$ ③

Let $x = \pi$

$\Rightarrow +\alpha_2 + \alpha_3 = 0$ ~~$\Rightarrow \alpha_2 = \alpha_3 = 0$~~.

~~$\Rightarrow \alpha_1 = 0$~~. & solve.

M2) $\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$

$\Rightarrow \sin^2 x - \cos^2 x = -\cos 2x$

$\Rightarrow v_1 - v_2 = -v_3$

$\Rightarrow v_1 - v_2 + v_3 = 0$

$\Rightarrow 1.v_1 + (-1).v_2 + (1) v_3 = 0$.

∴ ~~∄~~ ∃ a linear trivial combinⁿ for the set of scalars.

So, set is LD.

③ Let $v_1 = x^2 e^x$, $v_2 = x e^x$, $v_3 = (x^2 + x - 1) e^x$.

Consider $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \bar{0}$

So, $\alpha_1 x^2 e^x + \alpha_2 x e^x + \alpha_3 (x^2 + x - 1) e^x = 0 \rightarrow$ ①

eqn ① is true ∀ x.

$\Rightarrow$ Let $x = 0$

So, $-\alpha_3 = 0 \Rightarrow \boxed{\alpha_3 = 0}$

Let $x_E = 1$

$\Rightarrow e\alpha_1 + e\alpha_2 = 0$

$\Rightarrow \alpha_1 + \alpha_2 = 0$

Let $x = -1$

$\Rightarrow \frac{\alpha_1}{e} - \frac{\alpha_2}{e} = 0 \Rightarrow \alpha_1 = \alpha_2$

$\Rightarrow \alpha_1 = \alpha_2 = 0$.

So, the set is LI .

④ Let $v_1 = t^2$, $v_2 = t$, $v_3 = e^t$

Consider $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \bar{0} \rightarrow ①$

It is true $\forall$ ~~x~~ t in $\alpha_1 t^2 + \alpha_2 t + \alpha_3 e^t = 0 \rightarrow ①$

Let $t = 0$

$\alpha_3 = 0$.

Let $t = 1$

$\Rightarrow \alpha_1 + \alpha_2 = 0$

Let $t = -1$

$\Rightarrow \alpha_1 - \alpha_2 = 0$.

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

So, set is LI .

Section - 4.5 (continued).

Q. Check which of the following sets is a basis for R^3.

① $S = \{ (2,1,-1), (1,3,5) \}$
② $S = \{ (1,1,1) (1,-1,0) (0,0,0) \}$
③ $S = \{ (0,0,1) (1,0,1) (1,-1,1) (3,0,1) \}$
④ $S = \{ (1,1,1), (1,2,3), (-1,0,1) \}$

Part 2 → In problem 3, if S is not a basis, identify the vectors which form the basis for $\mathbb{R}^3$, if they exist.

① Here, $\dim(\mathbb{R}^3) = 3$.
∵ Set consists of only 2 vectors, it can't be a basis. (It cannot span $\mathbb{R}^3$).

② Here, one of the ~~elem~~ vectors in S is zero vector $(0,0,0)$.
∴ S is LD.
∴ S is not a basis for R^3.

③ Here, $\dim(\mathbb{R}^3) = 3$.
In R^3, there can be a max. of 3 ~~ve~~ LI vectors. Here, S has 4 vectors. Hence, it is LD. So, S is not a basis for $\mathbb{R}^3$.

Aliter Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

$$\sim \begin{array}{c} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} R_3 \\ \\ R_1 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & \underline{1} & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & \underline{1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 - 3R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 + 2R_3 \end{array}$$

Here, $\rho(A) = 3 < 4$, the no. of vectors.
$\therefore$ S is LD & is not a basis.

Part 2 :—
Tracing back the original vectors in S corresponding to the nonzero rows in the reduced matrix, the vectors in $S_1 = \{(0,0,1)\ (1,0,1), (1,-1,1)\}$ form a basis for R^3.

④ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 + R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$

Here, $r(A) = 2 < 3$, the no. of vectors in
So, its not a basis for R^3 as S is LD.

Q. Which of the following forms a basis for $\mathbb{R}^4$
using simplified span method?

① $S = \{(6, 1, 1, -1), (1, 0, 0, 9), (-2, 3, 2, 4)$
$(2, 2, 5, -5)\}$

② $S = \{(1, 1, 1, 1)\ (1, 1, 1, -1), (1, 1, -1, -1)\ (1, -1, -1, -)$

③ $S = \{(1, 3, 2, 0)\ (-2, 0, 6, 7)\ (0, 6, 10, 7)$
$(2, 10, -3, 1)\}$.

① $A = \begin{bmatrix} 6 & 1 & 1 & -1 \\ 1 & 0 & 0 & 9 \\ -2 & 3 & 2 & 4 \\ 2 & 2 & 5 & -5 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 9 \\ 6 & 1 & 1 & -1 \\ -2 & 3 & 2 & 4 \\ 2 & 2 & 5 & -5 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & -55 \\ 0 & 3 & 2 & 22 \\ 0 & 2 & 5 & -23 \end{bmatrix} \begin{matrix} \\ R_2 - 6R_1 \\ R_3 + 2R_1 \\ R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & -55 \\ 0 & 0 & -1 & +187 \\ 0 & 0 & 3 & 87 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_2 \\ R_4 - 2R_2 \end{matrix}$$

-165
+21
-143

$$\sim \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & -55 \\ 0 & 0 & -1 & 187 \\ 0 & 0 & 3 & 648 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 + 3R_3 \end{matrix}$$

1+0
-23

So, $r(A) = 4 =$ the no. of vectors in S.

So, S is LI.

dim $(\mathbb{R}^4) = 4 =$ no. of vectors in S.

So, S ~~is~~ spans $\mathbb{R}^4$

So, S is a basis for $\mathbb{R}^4$

② $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ R_4 \\ \\ R_2 \end{matrix}$$

Here, $\rho(A) = 4 =$ the no. of vectors in S.
So, S is LI.
Here, dim $(\mathbb{R}^4) = 4$, the no. of vectors.
So S spans $\mathbb{R}^4$.
So, it forms a basis for $\mathbb{R}^4$.

③ $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ -2 & 0 & 6 & 7 \\ 0 & 6 & 10 & 7 \\ 2 & 10 & -3 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & \underline{6} & 10 & 7 \\ 0 & 6 & 10 & 7 \\ 0 & 4 & -7 & 1 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ \\ R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 6 & 10 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -41 & -11 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ 3R_4 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 6 & 10 & 7 \\ 0 & 0 & -41 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_4 \\ R_3 \end{matrix}$$

Here, $\rho(A) = 3 < 4$, the no. of vectors.

So, S is LD.

So, S does not form basis for R^4.

Q. Check whether

$$S = \left\{ \begin{pmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{4} \\ \overset{a_{21}}{2} & \overset{a_{22}}{0} \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 5 & -2 \\ 0 & -3 \end{pmatrix} \right.$$

is ~~to~~ a basis for M_{22}.

$$A = \begin{array}{c} \begin{matrix} a_{11} & a_{12} & a_{21} & a_{22} \end{matrix} \\ \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ -3 & 1 & -1 & 0 \\ 5 & -2 & 0 & -3 \end{bmatrix} \end{array}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & \underline{2} & 1 & 0 \\ 0 & +13 & +5 & 0 \\ 0 & -22 & -10 & -3 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_1 \\ R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & \underline{-3} & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix} \begin{matrix} \\ \\ 2R_3 - 13R_2 \\ R_4 + 11R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -9 \end{bmatrix} \begin{matrix} \\ \\ \\ 3R_4 + R_3 \end{matrix}$$

Here, $r(A) = 4$ = the no. of matrices.

Hence, S is LI.

Here, $\dim(M_{22}) = 4$ = the no. of matrices

So, S spans M_{22} ($\text{Span } S = M_{22}$).

So, S forms a basis for M_{22}.

Q. Which of the following forms a basis for the indicated vector spaces:-

① $S = \{(x^4), x^4 - x^3, x^4 - x^3 + x^2, x^4 - x^3 + x^2 - x, x^3 - 1\}$, $V = P_4$

② $S = \{x - 1, x^2 + x - 1, x^2 - x + 1\}$, $V = P_2$

③ $S = \left\{1, x, \frac{3x^2 - 1}{2}, \frac{5x^3 - 3x}{2}, \frac{35x^4 - 30x^2 + 3}{8}\right\}$

$V = P_4$.

$$A = \begin{array}{c} \begin{array}{ccccc} x^4 & x^3 & x^2 & x & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

The matrix A is a lower Δlar matrix & hence,

~ Here, $r(A) = 5 =$ no. of vectors in S. So, S is LI

Also, $\dim(P_4) = 5 =$ no. of vectors in S.

$\therefore$ span $S = P_4$.

$R_2 - R_1$
$R_3 - R_1$
$R_4 - R_1$

Hence, S is a basis for P_4.

② $$A = \begin{array}{c} \begin{array}{ccc} x^2 & x & c \end{array} \\ \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0. \end{bmatrix} 2R_3 + R_2$$

$r(A) = 2 < 3$, the no. of vectors in S.

So, S is LD.

Hence, S does not form a basis for P_2.

② ~~A =~~

Section - 4.6

FINDING A BASIS ~~FOW~~ FOR W = SPAN(S)
or EXTENSION OF A LI SET INTO A BASIS

Let S be a set of vectors in a vector space V. If S is not a basis for V, we can find a basis for the subspace
W = span S , as follows:

S1) Form the matrix A by writing the given vectors as ROW VECTORS.

S2) Reduce A into an upper Δlar matrix. (OR)

S3) The non zero rows in the reduced matrix the corresponding original vectors in S form a basis for W = span S.

ex:-
Find a basis for W = span S in $\mathbb{R}^5$ s.t

① $S = \{(2, -2, 3, 5, 5), (-1, 1, 4, 14, -8), (4, -4, -2, -14, 18), (3, -3, -1, -9, 13)$

(a) What is the dim (W)?

(b) If S is not a basis for $\mathbb{R}^5$, find a basis for $\mathbb{R}^5$ which includes the vectors in S.

Here,

$$A = \begin{bmatrix} 2 & -2 & 3 & 5 & 5 \\ -1 & +1 & 4 & 14 & -8 \\ 4 & -4 & -2 & -14 & 18 \\ 3 & -3 & -1 & -9 & 13 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccccc} 2 & -2 & 3 & 5 & 5 \\ 0 & 0 & 11 & 33 & -11 \\ 0 & 0 & -8 & -24 & 8 \\ 0 & 0 & -11 & -33 & 11 \end{array}\right] \begin{array}{l} \\ 2R_2 + R_1 \\ R_3 - 2R_1 \\ 2R_4 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccccc} 2 & -2 & 3 & 5 & 5 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right] \begin{array}{l} \\ R_2/11 \\ R_3/-8 \\ R_4/-11 \end{array}$$

$$\sim \left[\begin{array}{ccccc} 2 & -2 & 3 & 5 & 5 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array}$$

Here, $r(A) = 2 < 4$, the no. of vectors in S.
So, S is LD &
Also, it isn't a basis for $\mathbb{R}^5$.

So,
$S_1 = \{(2, -2, 3, 5, 5), (0, 0, 1, 3, -1)\}$ → simplified form
or $S_2 = \{(2, -2, 3, 5, 5), (-1, +1, 4, 14, -8)\}$
form a basis for $W = \text{span } S$.
$\dim W = 2$

Note:
The set S_1 is the simplified form of the basis vectors for $W = \text{span } S$.

Q. Find a basis for $W = \text{span}\, S$ if
$S = \{ x^2 - 3x + 5,\ 3x^3 + 4x - 8,\ 6x^3 - x^2 + 11x - 21,$
$2x^5 - 7x^3 + 5x \}$, $V = P_5$

Also find the simplified form of above basis.

$$A = \begin{array}{c} \begin{array}{cccccc} x^5 & x^4 & x^3 & x^2 & x & c \end{array} \\ \begin{bmatrix} 2 & 0 & -7 & 0 & 5 & 0 \\ 0 & 0 & \underline{6} & -1 & 11 & -21 \\ 0 & 0 & 3 & 0 & 4 & -8 \\ 0 & 0 & 0 & 1 & -3 & 5 \end{bmatrix} \end{array}$$

$$\sim \begin{bmatrix} 2 & 0 & -7 & 0 & 5 & 0 \\ 0 & 0 & 6 & -1 & 11 & -21 \\ 0 & 0 & 0 & \underline{1} & -3 & 5 \\ 0 & 0 & 0 & 1 & -3 & 5 \end{bmatrix} \begin{matrix} \\ \\ 2R_3 - R_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 0 & -7 & 0 & 5 & 0 \\ 0 & 0 & 6 & -1 & 11 & -21 \\ 0 & 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - R_3 \end{matrix}$$

Here, $r(A) = 3 < 4$, the no. of vectors in S
So, S is LD & don't form basis for P_5.

Let
$S_1 = \{ 2x^5 - 7x^3 + 5x,\ 6x^3 - x^2 + 11x - 21,\ x^2 - 3x + 5 \}$
$\hookrightarrow$ forms a basis for $W = \text{span}\, S$ & is in simplified form.

$\dim(W) = 3$

$\Big\{$ or $S_2 = \{ 2x^5 - 7x^3 + 5x,\ 6x^3 - x^2 + 11x - 21,$
$3x^3 + 4x - 8 \}$ is also a basis for $W = \text{span}\, S$ $\Big\}$

Q Find a basis for W = span S using Simplified span method, if

① $S = \{(1,2,-1),(3,6,-3),(4,1,2),(0,0,0),(-1,5,-5)\}$

$V = \mathbb{R}^3$.

② $S = \{(3,2,-1,0,1),(1,-1,0,3,1)(4,1,-1,3,2)(3,7,-2,-9,-1)(-1,-4,1,6,1)\}$ $V = \mathbb{R}^5$.

③ $S = \{(1,1,1,1,1),(1,2,3,4,5)(0,1,2,3,4)(0,0,4,0,-1)\}$, $V = \mathbb{R}^5$.

① $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 4 & 1 & 2 \\ -1 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -7 & 6 \\ 0 & 7 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 4R_1 \\ R_4 + R_1 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & \underline{-7} & 6 \\ 0 & -7 & +6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_3 \\ -R_4 \\ R_2 \\ R_5 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \& \\ R_3 - R_2 \end{matrix}$$

Here, $r(A) = 2 < 5$, the no. of vectors in S
So, S is LD & basis doesn't get formed.

Let $S_1 = \{(1, 2, -1), (0, -7, 6)\}$
or $S_2 = \{(1, 2, -1), (4, 1, 2)\}$
They are basis for $W = \text{span } S$.

② $$A = \begin{bmatrix} 1 & -1 & 0 & 3 & 1 \\ 3 & +2 & -1 & 0 & 1 \\ 4 & 1 & -1 & 3 & 2 \\ 3 & 7 & -2 & -9 & -1 \\ -1 & -4 & 1 & 6 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 & 1 \\ 0 & 5 & -1 & -9 & -2 \\ 0 & 5 & -1 & -9 & -2 \\ 0 & 10 & -2 & -18 & -4 \\ 0 & -5 & 1 & 9 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 4R_1 \\ R_4 - 3R_1 \\ R_5 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 & 1 \\ 0 & 5 & -1 & -9 & -2 \\ 0 & 5 & -1 & -9 & -2 \\ 0 & 5 & -1 & -9 & -2 \\ 0 & 5 & -1 & -9 & -2 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4/2 \\ R_5/-1 \end{matrix}$$

For ex ②, we can extend the basis S_1 or S_2 into a basis for the entire vector space. This can be done by adding std. basis vectors or choosing non-zero vectors in the reduced matrix, keeping the matrix as upper Δ lar. $\longrightarrow$ $S_1 = \{(3,2,-1,0,1)(0,+5,-1,-9,-2),(0,0,1,0,0)(0,0,0,1,0),(0,0,0,0,1)\}$

or $S_2 = \{(3,2,-1,0,1)(0,5,-1,-9,-2)(0,0,6,3,10)(0,0,0,5,9)(0,0,0,0,3)\}$ form basis for $\mathbb{R}^5$

$$\sim \left[\begin{array}{ccccc} 1 & -1 & 0 & 3 & 1 \\ 0 & 5 & -1 & -9 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 - R_2 \\ R_5 - R_2 \end{array}$$

Here, $r(A) = 2 < 5$, the no. of vectors in S.
So, S is LD & doesn't form basis for $\mathbb{R}^5$.
~~Let~~ So,

$S_1 = \{(1,-1,0,3,1), (0,5,-1,-9,-2)\}$

or $S_2 = \{(1,-1,0,3,1),(3,2,-1,0,1)\}$.
form basis for $W = \text{span } S$.

③ $$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 4 & 0 & -1 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 4 & 0 & -1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ \textcircled{1} & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & -1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_4 .$$

Here, $r(A) = 3 < 4$, the no. of vectors in S
So, S is LD & doesn't form basis for $\mathbb{R}^5$.
So, $S_1 = \{(1,1,1,1,1)(0,1,2,3,4)(0,0,4,0,-1)\}$

or $S_2 = \{(1,1,1,1,1),(1,2,3,4,5),(0,0,4,0,-1)\}$
form basis for $w = \text{span } S$ -

T.A
V.Imp

④ $S = \{x^3-3x^2+2, 2x^3-7x^2+x-3, 4x^3-13x^2+x+5\}$
$V = P_3$

⑤ $S = \{(1,4,-2)(-2,-8,4),(2,-8,5),(0,-7,2)\}$
$V = \mathbb{R}^3$

⑥ $S = \{-2x^3+x+2, 3x^3-x^2+4x+6, 8x^3+x^2+6x+10,$
$-4x^3-3x^2+3x+4, -3x^3-4x^2+8x+12\}$
$V = P_3$.

④ $$A = \begin{array}{cccc} x^3 & x^2 & x & c \\ \hline 1 & -3 & 0 & 2 \\ 2 & -7 & 1 & -3 \\ 4 & -13 & 1 & 5 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & -1 & 1 & -7 \\ 0 & -1 & 1 & -3 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$$

$$S_1 = \begin{bmatrix} 1 & -3 & 0 & 9 \\ 0 & -1 & 1 & -7 \\ 0 & 0 & \ominus & \ominus \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Enter the new row here

$$\sim \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & -1 & 1 & -7 \\ 0 & 0 & 0 & 4 \end{bmatrix} R_3 - R_2$$

Here, $\rho(A) = 3 =$ no. of vectors in S.
So, S = LI & its spans the subspace W = span S.
Hence, it forms basis for P_3.
Here, $\dim(P_3) = 4$.
$\therefore S_1 = S \cup \{x\}$.
$= S \cup \{$ a linear polynomial $\}$
forms a basis for P_3.

⑤ $$A = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -8 & 4 \\ 2 & -8 & 5 \\ 0 & -7 & 2 \end{bmatrix}$$

10,

$$\sim \begin{bmatrix} 1 & 4 & -2 \\ -1 & -4 & 2 \\ 2 & -8 & 5 \\ 0 & -7 & 2 \end{bmatrix} R_2/2$$

$$\sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & -16 & 9 \\ 0 & -7 & 2 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - 2R_1 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & -7 & 2 \\ 0 & -16 & 9 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_4 \\ \\ R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & -16 & 9 \\ 0 & 0 & -31 \\ 0 & 0 & 0 \end{bmatrix} \quad 16R_3 - 7R_2$$

$\rho(A) = 3 < 4$, the no. of vectors in S.

So, S is LD.

$S_1 = \{(1,4,-2), (0,-16,9), (0,0,-31)\}$

or $S_2 = (1,4,-2), (2,-8,5), (0,-7,2)\}$

forms a basis for $W = \text{span} S$ & $\dim(W) = 3 = \dim(\mathbb{R}^3)$

$\therefore W = \mathbb{R}^3$.

Q ⑥ $A = \begin{array}{c} \begin{array}{cccc} x^3 & x^2 & x & c \end{array} \\ \begin{bmatrix} -2 & 0 & 1 & 2 \\ -1 & 0 & 4 & 6 \\ 8 & 1 & 6 & 10 \end{bmatrix} \end{array}$

Q Enlarge the following sets into a basis for the corresponding vector space.

① $T = \{(1,1,1,1,1), (0,1,1,1,1), (0,0,1,1,1)\}$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Here, $\rho(A) = 3 =$ no. of vectors in T.

$\therefore$ T is LI.

Adding e_4 & e_5 for R^5, we get the req^d basis.

$\Rightarrow S = T \cup \{(0,0,0,1,0), (0,0,0,0,1)\}$ is
is a basis for $\mathbb{R}^5$.

② $T = \{3x-2,\ x^3-6x+4\}$.

Here, $A = \begin{bmatrix} 1 & 0 & -6 & 4 \\ 0 & 0 & 3 & -2 \end{bmatrix}$

Here, $\rho(A) = 2 =$ no. of vectors in T.
So, T is LI.
Adding e_2 & e_4, for P_3, we get

$S = T \cup \{(0,\overset{\times}{1},0,0), (0,\overset{\times}{0},0,1)\}$.
$\{\checkmark x^2,\ \checkmark 1\}$

So, S is LI & spans P_3. & S is basis for P_3.

Q. Find a basis for the subspaces
① $W = \{(a, b+2a, 3b) \mid a, b \in \mathbb{R}\}$; $V = \mathbb{R}^3$
② $W =$ the set of all 3 vectors whose second coordinate is zero.
③ $S =$ the set of all polynomials with coeff. of x^2 = coeff. of x^3 ; $V = P_3$
④ $W = \{(a+2b, b-2c, c+a) \mid a,b,c \in \mathbb{R}^3\}$

① Consider
$(a, b+2a, 3b) = a(1,2,0) + b(0,1,3)$.
Let $S = \{(1,2,0), (0,1,3)\}$.
Here, $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$
$\Rightarrow S$ is LI & spans W
$\therefore S$ is basis for W (dim $W = 2$)

④ Consider
$(a+2b, b-2c, c+a) = a(1,0,1)+b(2,1,0)+c(0,-2,$
Here.
$S = \{(1,0,1),(2,1,0),(0,-2,1)\}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix} R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -3 \end{bmatrix} R_3 + 2R_2$$

$r(A) = 3 =$ no. of vectors in S.
$\Rightarrow$ S is LI & spans W.
$\therefore$ S is basis for W.

② $W = \{(a,0,b) \mid a,b \in R\}$
$= a(1,0,0) + b(0,0,1)$
$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $(S = \{(1,0,0),(0,0,1)\}$

$r(A) = 2 =$ no. of vectors in S.
$\Rightarrow$ S is LI & spans W.
$\Rightarrow$ S is basis for W.

③ $W = \{ax^3 + bx^2 + cx + d \mid a = b; a, b, c, d \in$
$= \{ax^3 + ax^2 + cx + d \mid , a, c, d \in R\}$
$= \{a(x^3+x^2) + cx + d \mid \quad " \quad \}$

Here,

$S = \{x^3 + x^2, x, 1\}$

$$A = \left\{ \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\}$$

$r(A) = 3 =$ no. of vectors in S.

So, S spans W & is LI.

& S is basis for W.

* * *

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Section - 4.7

§ COORDINATISATION

Or Coordinate vector of an ordered basis

Let V be a finite dimensional vector space with $\dim(V) = n$.

Let $B = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for V.

Let $v \in V$ be any vector.

Then, $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$.

where, $\alpha_1, \alpha_2, \ldots \alpha_n$ are scalars.

Then, the coordinate vector of ~~a~~ v w.r.t basis B ~~v~~ is denoted & defined by

$[v]_B = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$.

This procedure is called COORDINATISATION of v w.r.t the basis B.

eg. Let $B = \{\overset{v_1}{(1,2)}, \overset{v_2}{(0,-1)}\}$ be an ordered basis of vector space $V = \mathbb{R}^2$. Find the coordinate vector of $v = (4,-6)$ w.r.t B.

Let $v = \alpha_1 v_1 + \alpha_2 v_2$

$\Rightarrow (4,-6) = \alpha_1(1,2) \cancel{e} + \alpha_2(0,-1)$

$= (\alpha_1, 2\alpha_1 - \alpha_2)$

$$\Rightarrow \left.\begin{array}{l} \alpha_1 = 4 \\ 2\alpha_1 - \alpha_2 = -6 \end{array}\right\} \rightarrow ①$$

$\Rightarrow \alpha_2 = 2\alpha_1 + 6 = 14$.

∴ The coordinate vector of v w.r.t B is

$[v]_B = [\alpha_1, \alpha_2]$
$= (4, 14).$

★ Procedure to find the coordinate vector of v w.r.t an ordered basis B.

Let V be a vector space with ordered basis
$B = \{v_1, v_2, - - - v_n\}$

Let $v \in V$.

S1) Form the augmented matrix $[A : v]$ by ~~a~~ writing the given vectors of B as COLUMN vectors.

So, $[A : v] = [v_1\ v_2\ v_3 \ldots v_n \mid v]$

S2) Reduce $[A : v]$ into RREF

$[A : v] \sim [I \mid v']$.

S3) Deleting the zero rows, if any, we get, the coordinate vector of v as $[v]_B = (v')$.

eg Find $[-23, 30, -7, -1, -7]_B$, where $V = \text{span } B$ & $B = \{(-4, 5, -1, 0, -1), (1, -3, 2, 2, 5), (1, -2, 1, 1, 3)\}$

Here, the augmented matrix is given by

$$[A : v] = \left[\begin{array}{ccc|c} -4 & 1 & 1 & -23 \\ 5 & -3 & -2 & 30 \\ -1 & 2 & 1 & -7 \\ 0 & 2 & 1 & -1 \\ -1 & 5 & 3 & -7 \end{array}\right]$$

120

223
-7
16

23
5
115

$$\sim \left[\begin{array}{ccc|c} -4 & 1 & 1 & -23 \\ 0 & -7 & -3 & 5 \\ 0 & 7 & 3 & -5 \\ 0 & 2 & 1 & -1 \\ 0 & 19 & 11 & -5 \end{array}\right] \begin{array}{l} \\ 4R_2 + 5R_1 \\ 4R_3 - R_1 \\ \\ 4R_5 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -28 & 0 & 4 & -156 \\ 0 & -7 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 20 & 60 \end{array}\right] \begin{array}{l} 7R_1+R_2 \\ \\ R_3+R_2 \\ 7R_4+2R_2 \\ 7R_5+19R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -7 & 0 & 1 & -39 \\ 0 & -7 & -3 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1/4 \\ \\ R_4 \\ R_5/20 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -7 & 0 & 0 & -42 \\ 0 & -7 & 0 & 14 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1-R_3 \\ R_2+3R_3 \\ \\ R_4-R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1/-7 \\ R_2/-7 \\ \\ \\ \\ \end{array}$$

∴ The coordinate vector of v w.r.t B is $[v]_B = (6, -2, 3)$.

Q. Let B represent an ordered basis of $\mathbb{R}^n$ or P_n or $M_{m\times n}$. Find $[v]_B$ for a given $v \in V$

① $B = \{(1, -4, 1), (5, -7, 2), (0, 4, -1)\}$, $v = (2, -1, 0$

② $B = \{(4, 6, 0, 1), (5, 1, -1, 0) (0, 15, 1, 3)$
$(1, 5, 0, 1)\}$ $V = (0, -9, 1, -2$

③ $B = \{3x^2 - x + 2,\ x^2 + 2x - 3,\ 2x^2 + 3x - 1\}$
$V = 13x^2 - 5x + 20$

①. Here,

$$[A : v] = \left[\begin{array}{ccc|c} 1 & 5 & 0 & 2 \\ -4 & -7 & 4 & -1 \\ 1 & 2 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & 0 & 2 \\ 0 & 13 & 4 & 7 \\ 0 & -3 & -1 & -2 \end{array}\right] \begin{array}{l} \\ R_2 + 4R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 13 & 0 & -20 & -9 \\ 0 & 13 & 4 & 7 \\ 0 & 0 & -1 & -5 \end{array}\right] \begin{array}{l} 13R_1 - 5R_2 \\ \\ 13R_3 + 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 13 & 0 & 0 & +91 \\ 0 & 13 & 0 & -13 \\ 0 & 0 & -1 & -5 \end{array}\right] \begin{array}{l} R_1 - 20R_3 \\ R_2 + 4R_3 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \cancel{-9/13} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} R_1/13 \\ R_2/13 \\ R_3/-1 \end{array}$$

$$\therefore [V]_B = \left(7 - \frac{\cancel{9}}{\cancel{13}}, -1, 5\right)$$

② $[A : v] = \left[\begin{array}{cccc|c} 4 & 5 & 0 & 1 & 0 \\ 6 & 1 & 15 & 5 & -9 \\ 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 1 & -2 \end{array}\right]$

$$\sim \left[\begin{array}{cccc|c} 4 & 5 & 0 & 1 & 0 \\ 0 & -13 & 30 & 7 & -18 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & -5 & 12 & 3 & -8 \end{array}\right] \begin{array}{l} \\ 2R_2-3R_1 \\ \\ 4R_4-R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 52 & 0 & 150 & 48 & -90 \\ 0 & -13 & 30 & 7 & -18 \\ 0 & 0 & -17 & -7 & 31 \\ 0 & 0 & 6 & 4 & -14 \end{array}\right] \begin{array}{l} 13R_1+5R_2 \\ \\ 13R_3-R_2 \\ 13R_4-5R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 884 & 0 & 0 & -234 & 2970 \\ & & & & \\ 0 & 0 & -17 & -7 & 31 \\ & & & & \end{array}\right] \begin{array}{l} 17R_1+150R_3 \\ 17R_2+30R_3 \\ \\ 17R_4+6R_3 \end{array}$$

③ $[A : V] = \begin{array}{c} x \\ x \\ c \end{array} \left[\begin{array}{ccc|c} 3 & 1 & 2 & 13 \\ -1 & 2 & 3 & -5 \\ 2 & -3 & -1 & 20 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 3 & 1 & 2 & 13 \\ 0 & 7 & 11 & -2 \\ 0 & -11 & -7 & 34 \end{array}\right] \begin{array}{l} \\ 3R_2+R_1 \\ 3R_3-2R_1 \end{array}$

$\sim \left[\begin{array}{ccc|c} 21 & 0 & 3 & 93 \\ 0 & 7 & 11 & -2 \\ 0 & 0 & 72 & 216 \end{array}\right] \begin{array}{l} 7R_1-R_2 \\ \\ 7R_3+11R_2 \end{array}$

$\sim \left[\begin{array}{ccc|c} 7 & 0 & 1 & 31 \\ 0 & 7 & 11 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1/3 \\ \\ R_3/72 \end{array}$

$\sim \left[\begin{array}{ccc|c} 7 & 0 & 0 & 28 \\ 0 & 7 & 0 & -35 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1-R_3 \\ R_2-11R_3 \\ \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1/7 \\ R_2/7 \\ \end{array}$

∴ The coordinate vector of V w.r.t B is given

$[v]_B = (4, -5, 3)$

④ $B = \{4x^2 + 3x + 1, 2x^2 - x + 4, x^2 - 2x + 3\}$

$v = \cancel{\{} -5x^2 - 17x + 20$

⑤ ~~A~~ $B = \left\{ \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 2 & 1 \end{bmatrix} \right\}$

$v = \begin{bmatrix} -8 & 35 \\ -14 & 8 \end{bmatrix}$

④ $[A : v] = \left[\begin{array}{ccc|c} 4 & 2 & 1 & -5 \\ 3 & -1 & -2 & -17 \\ 1 & 4 & 3 & 20 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 4 & 2 & 1 & -5 \\ 0 & -10 & -11 & -53 \\ 0 & 14 & 11 & 85 \end{array}\right] \begin{array}{l} \\ 4R_2 - 3R_1 \\ 4R_3 - R_1 \end{array}$

$\sim \left[\begin{array}{ccc|c} 20 & 0 & -6 & -78 \\ 0 & -10 & -11 & -53 \\ 0 & 0 & -22 & 54 \end{array}\right] \begin{array}{l} 5R_1 + R_2 \\ \\ 5R_3 + 7R_2 \end{array}$

17
64

68
-15
(53)

285
425
371
(54)

⑤ $[A : v] = \left[\begin{array}{ccc|c} -2 & 1 & 0 & -8 \\ 3 & 1 & -3 & 35 \\ 0 & -1 & 2 & -14 \\ 2 & 2 & 1 & 8 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} -2 & 1 & 0 & -8 \\ 0 & 5 & -6 & 46 \\ 0 & -1 & 2 & -14 \\ 0 & 3 & 1 & 0 \end{array}\right] \begin{array}{l} \\ 2R_2 + 3R_1 \\ \\ R_4 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -10 & 0 & 6 & -86 \\ 0 & 5 & -6 & 46 \\ 0 & 0 & 4 & -24 \\ 0 & 0 & +23 & -138 \end{array}\right] \begin{array}{l} 5R_1 - R_2 \\ \\ 5R_3 + R_2 \\ 5R_4 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc} 5 & 0 & -3 & -43 \\ 0 & 5 & -6 & 46 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 1 & 6 \end{array}\right] \begin{array}{l} \cancel{2R_1 - 3R_3} \; R_1/2 \\ \cancel{2R_2 + 3R_3} \\ R_3/4 \\ \cancel{4R_4 - 23R_3} \; R_4/23 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -5 & 0 & 0 & -25 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - 3R_3 \\ R_2 + 6R_3 \\ \\ R_4 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1/-5 \\ R_2/5 \\ \\ \\ \end{array}$$

∴ The coordinate vector of
$[V]_B = (5, 2, -6)$

✱ Transition matrix.

Let V be a finite dimensional vector space, with $\dim(V) = n$. Let $B = \{b_1, b_2 \ldots, b_n\}$, $C = \{c_1, c_2, ---, c_n\}$ be ordered basis of V. Then, the trans^n matrix, T, from B to C is a matrix of order n, whose i^{th} column vector is $[b_i]_c$, $i = 1, 2, ---, n$.

* Procedure to find the trans^n matrise from B to C.

S1) Form the augmented matrix $[C \mid B]$ by writing the column vectors of C & B resp.

S2) Using ero's, reduce C into RREF
(i.e $[C \mid B] \sim [I \mid T]$).

S3) The matrix B get transformed to the req^d trans^n

matrix from B to C.

Q. Find the trans^n matrix P, from B to C, given

① $B = \{(1,0,0), (0,1,0)\ (0,0,1)\}$
$C = \{(1,5,1), (1,6,-6), (1,3,14)\}$

② $B = \{(1,0,-1), (10,5,4), (2,1,1)\}$
$C = \{(1,0,2), (5,2,5), (2,1,2)\}$

③ $B = \{2x^2+3x-1,\ 8x^2+x+1,\ x^2+6\}$
$C = \{x^2+3x+1,\ 3x^2+4x+1,\ 10x^2+17x+5\}$

② $[C \mid B] = \left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 10 & 2 \\ 0 & 2 & 1 & 0 & 5 & 1 \\ 2 & 5 & 2 & -1 & 4 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 10 & 2 \\ 0 & 2 & 1 & 0 & 5 & 1 \\ 0 & -5 & -2 & -3 & -16 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & -3 & 2 & -5 & -1 \\ 0 & 2 & 1 & 0 & 5 & 1 \\ 0 & 0 & 1 & -6 & -7 & -1 \end{array}\right] \begin{array}{l} 2R_1 - 5R_2 \\ \\ 2R_3 + 5R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -16 & -26 & -4 \\ 0 & 2 & 0 & 6 & 12 & 2 \\ 0 & 0 & 1 & -6 & -7 & -1 \end{array}\right] \begin{array}{l} R_1 + 3R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & -13 & -2 \\ 0 & 1 & 0 & 3 & 6 & 1 \\ 0 & 0 & 1 & -6 & -7 & -1 \end{array}\right] \begin{array}{l} R_1/2 \\ R_2/2 \\ \end{array}$$

∴ The trans^n matrix P from B to C is

$$P = \begin{bmatrix} -2 & -6 & -1 \\ 3 & 6 & 1 \\ -6 & -7 & -1 \end{bmatrix}$$

③ $R[C|B] = \left[\begin{array}{ccc|ccc} 1 & 3 & 10 & 2 & 8 & 1 \\ 3 & 4 & 17 & 3 & 1 & 0 \\ 1 & 1 & 5 & -1 & 1 & 6 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 10 & 2 & 8 & 1 \\ 0 & \underline{-5} & -13 & -3 & -23 & -3 \\ 0 & -2 & -5 & -3 & -7 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 5 & 0 & 11 & 1 & -29 & -4 \\ 0 & -5 & -13 & -3 & -23 & -3 \\ 0 & 0 & \underline{1} & -9 & 11 & 31 \end{array}\right] \begin{array}{l} 5R_1+3R_2 \\ \\ 5R_3-2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 100 & -150 & -345 \\ 0 & \cancel{0}-5 & 0 & -120 & 120 & 400 \\ 0 & 0 & 1 & -9 & 11 & 31 \end{array}\right] \begin{array}{l} R_1-11R_3 \\ R_2+13R_3 \\ \end{array}$$

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×13

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 20 & -30 & -69 \\ 0 & 1 & 0 & 24 & -24 & -80 \\ 0 & 0 & 1 & -9 & 11 & 31 \end{array}\right] \begin{array}{l} R_1/5 \\ R_2/-5 \\ \end{array}$$

$\therefore$ The trans^n matrix P from B to C is

$$P = \begin{bmatrix} 20 & -30 & -09 \\ 24 & -24 & -80 \\ -9 & 11 & 31 \end{bmatrix}$$

—x—

Q. Find the trans^n matrix P from B to C, if

$$B = \left\{ \begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix} \right\}$$

{The given sets form a basis for the vector space $U_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\} \subseteq M_{22}$}.

$$(C \mid B) = \left[\begin{array}{ccc|ccc} \underline{22} & 12 & 33 & 7 & 1 & 1 \\ 7 & 4 & 12 & 3 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & -1 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} \underline{22} & 12 & 33 & 7 & 1 & 1 \\ 7 & 4 & 12 & 3 & 2 & -1 \\ 2 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \cancel{22R_2 - 7R_1} \\ R_{34} \\ R_3 \cancel{- R_4} \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 22 & 12 & 33 & 7 & 1 & 1 \\ 0 & \underline{4} & 33 & 17 & 37 & -29 \\ 0 & -1 & -11 & -7 & -12 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ 22R_2 - 7R_1 \\ 11R_3 - R_1 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 22 & 0 & -66 & 44 & -110 & 188 \\ 0 & 4 & 33 & 17 & 37 & -29 \\ 0 & 0 & \underline{-11} & -11 & -11 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \\ \\ 4R_3 + R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 22 & 0 & 0 & 22 & \overset{-44}{-55} & 22 \\ 0 & 4 & 0 & -16 & 4 & 4 \\ 0 & 0 & -11 & -11 & -11 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - 6R_3 \\ R_2 + 3R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \overset{-2}{-5/2} & 1 \\ 0 & 1 & 0 & -4 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1/22 \\ R_2/4 \\ R_3/-11 \\ \\ \end{array}$$

∴ The trans^n matrix from B to C is

$$P = \begin{bmatrix} 1 & \overset{-2}{-5/2} & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

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Chapter 5

LINEAR TRANSFORM^n

Section - 5.1

* Introduction

Let U & W be vector space over the same field of scalars. Then, a map 'L' from U to W ($L: U \rightarrow W$) is said to be a linear transform^n, if the following cond^{ns} are satisfied -

(i) $L(U_1 + U_2) = L(U_1) + L(U_2)$

(ii) $L(\alpha U) = \alpha L(U) \quad \forall \; U_1, U_2, U \in U$, α: scalar.

(L is a linear transform^n, if it preserves the oper^{ns} — vector add^n & scalar multiplic^n).

eg: Let $L: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, given by $L[(x, y, z)] = (x+y, y-2z)$.

Check whether L is a linear transform^n.

Let $U, V \in \mathbb{R}^3$, with $U = (x_1, y_1, z_1)$ & $V = (x_2, y_2, z_2)$

~~Show~~ & $\Rightarrow U + V = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$.

Consider,

$$L(U+V) = L[(x_1 + x_2, y_1 + y_2, z_1 + z_2)]$$

$$= [(x_1 + x_2) + (y_1 + y_2), (y_1 + y_2) - 2(z_1 + z_2)] \longrightarrow ①$$

Now, $L(U) = L[(x_1, y_1, z_1)]$

$$= [x_1 + y_1, y_1 - 2z_1] \longrightarrow ②$$

$L(v) = L[(x_2, y_2, z_2)]$
$= (x_2 + y_2, y_2 - 2z_2) \longrightarrow ③$

Now ② + ③
$\Rightarrow L(u) + L(v) = [(x_1 + x_2) + (y_1 + y_2), (y_1 + y_2) - 2(z_1$

↳ same as eqⁿ ①.

So, $L(u + v) = L(u) + L(v)$

Now, Let α be any scalar, $u = (x, y, z) \in \mathbb{R}^3$.

Now, $\alpha u = (\alpha x, \alpha y, \alpha z)$

$L(\alpha u) = L[(\alpha x, \alpha y, \alpha z)]$
$= [\alpha(x + y), \alpha(y - 2z)]$
$= \alpha(x + y, y - 2z)$

$\Rightarrow L(\alpha u) = \alpha L(u)$

So, L is a linear transformⁿ.

☆ PROPERTIES OF A Linear Transformⁿ

Let $L : U \to W$ be a linear transformⁿ. Then,

$P(i)$ $L(0_U) = 0_W$

$P(ii)$ $L(-u) = -L(u)$, $\forall u \in U$

★ $P(iii)$ $L(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) = \alpha_1 L(u_1) + \alpha_2 L(u_2) + \dots + \dots \alpha_n L(u_n)$.

↳ $\forall$ scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ & vectors $u_1, u_2, \dots, u_n \in U$.

eg Check whether $L : \mathbb{R}^3 \to \mathbb{R}^2$, given by $L[(x, y, z)] = (x + y, y - 2z + 3)$ is a linear transformⁿ.

(M1) $\bar{O}_v = (0,0,0)$

$L(\bar{O}_v) = L(0,0,0)$

$= (0+0, 0-2(0)+3)$

$= (0, 3)$

$\neq O_w$

$\therefore L(\bar{O}_w) \neq O_w$

So, L is not linear transform^n.

(M2) Let $u = (1,2,4)$

$v = (2,-1,2)$

$u+v = (3,1,6)$

$= (3+1, 1-12+3)$

$= (4, -8)$

$L(u) = L(1,2,4) = (3, 2-8+3) = (3,-3)$

$L(v) = L(2,-1,2) = (1, -1-4+3) = (1,-2)$

Q. Check whether the following are linear transform^{ns}:

① $L : \mathbb{R} \rightarrow R^3$ by $L(x) = (x, 2x, 3x)$

② $L : \mathbb{R} \rightarrow \mathbb{R}^3$ by $L(x) = (1, x, x^2)$

③ $L : R^3 - \mathbb{R}^3$ by $L(x,y,z) = (x,y,0)$

④ $L : \mathbb{R}^3 \rightarrow R^2$ by $L(x,y,z) = (x,y,4)$

→ ⑤ $L : P_2 \rightarrow \mathbb{R}$ by $L(ax^2+bx+c) = a+b+c$

⑥ $L : P_2 \rightarrow \mathbb{R}$ by $L(ax^2+bx+c) = abc$

⑦ $L : P_2 \rightarrow R$ by $L(ax^2+bx+c) = a+b+c+$①

⑧ $L : \mathbb{R} \rightarrow \mathbb{R}^3$ by $L(x) = (x, x^2, x^3)$.

① ~~Here~~, $L(x) = (x, 2x, 3x)$

Here, $L : R \rightarrow \mathbb{R}^3$

Let $U = \mathbb{R}$ & $W = \mathbb{R}^3$.

So, $O_U = 0$

$\therefore L(O_U) = L(0) = (0, 0, 0)$
$= O_W$.

Let $U, V \in \mathbb{R}$

Consider $L(U+V) = (U+V, 2(U+V), 3(U+V))$
$= (U+V, 2U+2V, 3U+3V)$
$= (U, 2U, 3U) + (V, 2V, 3V)$
$= L(U) + L(V)$

$\Rightarrow L(U+V) = L(U) + L(V)$

Let α be a scalar & $U \in \mathbb{R}$.

$L(\alpha U) = (\alpha U, 2\alpha U, 3\alpha U)$
$= \alpha(U, 2U, 3U)$
$= \alpha(L(U))$

$\Rightarrow$ ~~α~~ $L(\alpha U) = \alpha L(U)$.

$\therefore$ L is a ~~l~~ linear transform^n.

② $L(x) = (1, x, x^2)$

~~Of~~ Here, $U = \mathbb{R}$, $W = \mathbb{R}^3$

$O_U = 0$, $L(O_U) = L(0) = (1, 0, 0) \neq O_W$.

So, $L(O_U) \neq O_W$.

So, L is not a linear transform^n.

Aliter

Let $x, y \in \mathbb{R}$.

$L(x) = (1, x, x^2)$, $L(y) = (1, y, y^2)$.

$\therefore L(x) + L(y) = (2, (x+y), x^2+y^2)$

$L(x+y) = (1, (x+y), (x+y)^2)$

Clearly, $L(x+y) \neq L(x) + L(y)$.
So, its not a linear transformⁿ.

⑧ $L(x) = (x, x^2, x^3)$

$$L : \underset{U}{\mathbb{R}} \longrightarrow \underset{W}{\mathbb{R}^3}$$

$O_U = 0$ $\therefore$ $L(O_U) = L(0) = (0,0,0) = O_W$

$\Rightarrow L(O_U) = O_W$.

Let $1, 2 \in \mathbb{R}$.

So, $L(1) = (1, 1^2, 1^3) = (1, 1, 1)$

$L(2) = (2, 2^2, 2^3) = (2, 4, 8)$

$L(1+2) = L(3) = (3, 3^2, 3^3) = (3, 9, 27)$

$L(1) + L(2) = (3, 5, 9)$.

Clearly,

$L(1+2) \neq L(1) + L(2)$

So, its not a linear transformⁿ.

③ $$L : \underset{U}{\mathbb{R}^3} \longrightarrow \underset{W}{\mathbb{R}^3}$$

$L(x, y, z) = (x, y, 0)$.

$O_U = (0,0,0)$.

So, $L(O_U) = L(0,0,0) = (0,0,0) = O_W$.

So, $L(O_U) = O_W$.

Let $u, v \in U$ with $u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2)$

Now, $L(u) = L(x_1, y_1, z_1) = (x_1, y_1, 0)$.

$L(v) = L(x_2, y_2, z_2) = (x_2, y_2, 0)$.

$L(u) + L(v) = ((x_1 + x_2), (y_1 + y_2), (0))$

Now, $L(u+v) = (x_1 + x_2, y_1 + y_2, 0)$

So, $L(u) + L(v) = L(u+v)$.

Let α be a scalar & $u = (x, y, z) \in U$.
So, $\alpha u = (\alpha x, \alpha y, \alpha z)$.

$$\Rightarrow L(\alpha u) = L(\alpha x, \alpha y, \alpha z) = (\alpha x, \alpha y, 0)$$
$$= \alpha(x, y, 0)$$
~~$\neq \alpha u$~~
$$= \alpha . L(u)$$

$$\Rightarrow L(\alpha u) = \alpha L(u)$$
So, its a linear transform^n.

④ $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$\qquad U \qquad W$.

$L(x, y, z) = (x, y, 4)$
Here, $0_U = (0, 0, 0)$.
$L(0_U) = L(0, 0, 0) = (0, 0, 4) \neq 0_W$.
So, $L(0_U) \neq 0_W$.
So, its not a linear transform^n.

⑤ $L : P_2 \longrightarrow \mathbb{R}$
$\qquad U \qquad W$.

$L(ax^2 + bx + c) = a + b + c$.
Here, $0_U = 0x^2 + 0x + 0 = 0$.
$\therefore L(0_U) = L(0) = 0 = 0_W$
Let $u, v \in P_2$ with $u = a_1x^2 + b_1x + c_1$
$v = a_2x^2 + b_2x + c_2$.

$\Rightarrow L(u) = L(a_1x^2 + b_1x + c_1) = a_1 + b_1 + c_1$
$L(v) = L(a_2x^2 + b_2x + c_2) = a_2 + b_2 + c_2$
$L(u) + L(v) = (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2)$
$= (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2)$
$= L((a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2))$
$= L(u + v)$

So, $L(U) + L(V) = L(U+V)$.

Let α be a scalar & $U = ax^2 + bx + c$.

So, ~~b.~~ $\alpha U = \alpha a x^2 + \alpha b x + \alpha c$

$\Rightarrow L(\alpha U) = L(\alpha a \cancel{x}^2 + \alpha b x + \alpha c)$

$= \alpha a + \alpha b + \alpha c$

$= \alpha(a + b + c)$

$= \alpha\, L(ax^2 + bx + c)$

$= \alpha\, L(U)$

$\Rightarrow$ ~~B~~ $L(\alpha U) = \alpha L(U)$.

So, It is a linear transformn.

⑥ $L : \underset{U}{P_2} \longrightarrow \underset{W}{\mathbb{R}}$.

$L(ax^2 + bx + c) = abc$.

$L(\alpha x) = \underline{\underline{\alpha 3abc}}$

$= L$

$\alpha\, L(U) + L(V)$

Let $U, V \in U$ s.t $U = (a_1 x^2 + b_1 x + c_1)$

& $V = (a_2 x^2 + b_2 x + c_2)$.

Now, $L(U) = L(a_1 x^2 + b_1 x + c_1)$

$\Rightarrow L(U) = a_1 b_1 c_1$

Now, $L(V) = L(a_2 x^2 + b_2 x + c_2)$

$\Rightarrow L(V) = a_2 b_2 c_2$

So, $L(U) + L(V) = a_1 b_1 c_1 + a_2 b_2 c_2 \rightarrow ①$

Now, $L(U+V) = L((a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2))$

$= (a_1+a_2)(b_1+b_2)(c_1+c_2) \rightarrow ②$

Clearly ① $\neq$ ②.

So $L(U) + L(V) \neq L(U+V)$

So, its not a linear transformn.

⑦ $L : \underset{U}{P_2} \longrightarrow \underset{W}{\mathbb{R}}$

$L(ax^2+bx+c) = a+b+c+1$

Here, $O_U = 0x^2+0x+0 = 0$

& $L(O_U) = L(0) = 0+0+0\underline{+1} = 1 \neq O_W$

So, $L(O_U) = O_W$.

So, its not a linear transform^n.

Q. Check whether the following mappings are LT.

① $L : M_{23} \longrightarrow M_{22}$ by

$$L\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

② $L : M_{mn} \longrightarrow M_{nm}$ by $L(A) = A^T$

linear operator

③ $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $L[(a_1, a_2, a_3)] = (a_1, a_2, -a_3)$

④ $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $L[(x,y)] = (3x+4y, -x+2y)$

⑤ $L : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ by

$L[(x_1, x_2, x_3, x_4)] = (x_1+2, x_2-1, x_3, -3)$

① Let $U = M_{23}$ & $W = M_{22}$.

Consider $O_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore L(O_U) = L\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_W$.

$\therefore L(O_U) = O_W$,

Now, consider $u, v \in U$ s.t

$u = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix}$ & $v = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix}$

So, $L(U) = L\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix}\right) = \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}$

$L(V) = L\left(\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix}\right) = \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}$.

Now,

$U + V = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix}$.

$= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & c_1 + c_2 \\ d_1 + d_2 & e_1 + e_2 & f_1 + f_2 \end{bmatrix}$

So, $L(U+V) = L\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ d_1+d_2 & e_1+e_2 & f_1+f_2 \end{bmatrix}\right)$

$= L\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & 0 \end{bmatrix}\right)$.

$= L(U) + L(V)$

$\Rightarrow L(U+V) = L(U) + L(V)$

Let α be a scalar & $U = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.

So, $\alpha U = \begin{bmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \end{bmatrix}$

So, $L(\alpha U) = \begin{bmatrix} \alpha a & \alpha b \\ 0 & 0 \end{bmatrix}$

Now, $\alpha L(U) = \alpha \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ 0 & 0 \end{bmatrix}$.

So, $L(\alpha U) = \alpha L(U)$.

So, its a LT.

~~③ $L : \mathbb{R}^3 \to \mathbb{R}^3$~~
~~$U \quad W$~~
~~$L[(a_1, a_2, a_3)]$~~

Note :-
A linear transformⁿ $L : U \to U$ is called a LINEAR OPERATOR.

④ $L : \underset{U}{\mathbb{R}^2} \to \underset{W}{\mathbb{R}^2}$

$L[(x,y)] = (3x+4y, -x+2y)$

Here, $O_U = (0,0)$.

$L(O_U) = L(0,0) = (0+0, -0+0)$
$= (0,0) = O_W$.

So, $L(O_U) = O_W$

Let $u, v \in U$ s.t $u = (x_1, x_2)$
$v = (y_1, y_2)$.

$L(u) = (3x_1 + 4y_1, -x_1 + 2y_1)$

$L(v) = (3x_2 + 4y_2, -x_2 + 2y_2)$

$L(u) + L(v) = [\cancel{6x_1}\ 3(x_1+x_2) + 4(y_1+y_2), -(x_1+x_2) + 2(y_1+y_2)]$

$L(u+v) = [3(x_1+x_2) + 4(y_1+y_2), -(x_1+x_2) + 2(y_1+y_2)]$.

So, $L(u) + L(v) = L(u+v)$.

Let α be a scalar.

So, $L(\alpha u) = (3\alpha x_1 + 4\alpha y_1, -\alpha x_1 + 2\alpha y_1)$

$\alpha L(u) = \alpha(3x_1 + 4y_1, -x_1 + 2y_1)$
$= (3\alpha x_1 + 4\alpha y_1, -\alpha x_1 + 2\alpha y_1)$

So, $L(\alpha u) = \alpha(L(u))$

So, its a LT.

⑤ $L : R^4 \to \mathbb{R}^4$.

$L(x_1, x_2, x_3, x_4) = (x_1+2, x_2-1, x_3, -3)$

Here, $O_V = (0, 0, 0, 0)$

$L(O_V) = L(0, 0, 0, 0) = (0+2, 0-1, 0, -3)$

$= (2, -1, 0, -3)$

$\neq (0, 0, 0, 0)$ $[O_W$ ~~(O_U)~~$]$

So, $L(O_U) \neq O_W$.

So, its not ~~an~~ a LT.

$L(O_W) = 0$ 3

Q Check whether L is a LT.

× ① $L : P \to P$ by $L(p) = p^2 + p$.

✓ ② $L : P \to P$ by $L(P(x)) = x\,p(x) + p(1)$.

✓ ③ $L : M_{22} \to M_{22}$ by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-2c+d & 3b-c \\ -4a & b+c-3d \end{bmatrix}$

× ④ $L : M_{22} \to \mathbb{R}$ by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

✓ ⑤ $f : P_3 \to R$ by $f(ax^3 + bx^2 + cx + d) = a + b + c + d$

× ⑥ $L : \underset{U}{\mathbb{R}^2} \to \underset{W}{\mathbb{R}}$ by $L(x, y) = \sqrt{x^2 + y^2}$

⑥ $O_U = (0, 0)$

$L(O_U) = L(0, 0) = \sqrt{0^2 + 0^2} = 0 = O_W$.

So, $L(O_U) = O_W$.

Let $u, v \in \mathbb{R}^2$ with $u = (x_1, y_1)$

$v = (x_2, y_2)$

So, $L(u) + L(v) = L(x_1, y_1) + L(x_2, y_2)$

$= \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} \longrightarrow ①$

$L(u+v) = L(x_1 + x_2, y_1 + y_2)$

$= \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \longrightarrow ②$.

$\because ① \neq ②$, so, L is not a LT.

① $L : \underset{U}{P} \rightarrow \underset{W}{P}$ by $L(P) = p^2 + p$.

$O_v = 0$

$L(O_v) = L(0) = 0 = O_w$

So, $L(O_v) = O_w$.

Let α be a scalar and ~~$v = \alpha v$~~ $(p \in P)$

~~αv~~

$L(\alpha p) = L(\alpha p) = (\alpha p)^2 + \alpha p$

$= \alpha^2 p^2 + \alpha p$.

$\alpha L(p) = \alpha(p^2 + p) = \alpha p^2 + \alpha p$.

So, $L(\alpha p) \neq \alpha L(p)$.

So, its not a LT .

② $L : \underset{U}{P} \rightarrow \underset{W}{P}$ by $L(P(x)) = x P(x) + P(1)$

$\bar{O}_u(x) = 0$, for any x [In particular, $\bar{O}_v(1) = 0$]

$L(\bar{O}_v(x)) = x \bar{O}(x) + \bar{O}(1)$

$= 0 + 0 = 0 = \bar{O}_W$

$\Rightarrow L(\bar{O}_v(x)) = \bar{O}_w(x)$.

Let $p, q \in P$. then,

$L(P(x)) = x P(x) + P(1)$

& $L(Q(x)) = x Q(x) + Q(1)$

Now, $L((P+Q)(x)) =$ ~~$L(P(x) + Q(x))$~~

$= x (P+Q)(x) + (P+Q)(1)$

$= x [P(x) + Q(x)] + P(1) + Q(1)$

$= x P(x) + P(1) + x Q(x) + Q(1)$

$\Rightarrow L((P+Q)(x)) = L(P(x)) + L(Q(x))$

So.

Let α be a scalar & $p \in P$.

So, $L((\alpha p)(x)) = x(\alpha p)(x) + \alpha p(1)$

$= x[\alpha\, p(x)] + [\alpha . p(1)]$

$= \alpha\, [\, x p(x) + p(1)]$

$= \alpha\ \ L(P(x))]$

$\Rightarrow L((\alpha p)(x)) = \alpha\ \ L(P(x))$

So, L is a LT.

④ $L : \underset{U}{M_{22}} \rightarrow \underset{W}{R}$ by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

$O_U = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$L(O_U) = 0 \cdot 0 - 0 \cdot 0 = 0 = O_W$

So, $L(O_U) = O_W$

Consider 2 matrices $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$.

So, $L(A) + L(B) = L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

$= a_1 d_1 - b_1 c_1 + a_2 d_2 - b_2 c_2 \rightarrow ①$

Now.

$L(A+B) = L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

$= L\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right)$

$= (a_1 + a_2)(d_1 + d_2) - (b_1 + b_2)(c_1 + c_2) \rightarrow ②$

Clearly, ① ≠ ②. So, its not a LT.

✧ FINDING A LINEAR TRANSFORMATIO

Let $L : U \to W$ be a LT.

Let $U_1, U_2, ---U_n$ be ~~to~~ BASIS vectors for U,

with $\dim(U) = n$.

S1) Let $u \in U$ be any vector.

; express $U = \alpha U_1 + \alpha_2 U_2 + ---- + \alpha_n U_n. \rightarrow ①$

[CHECK WHETHER THE GIVEN VECTORS FORM

✧ A BASIS. FOR U]

S2) Taking the LT on both sides of ①, we get

$L(U) = L(\alpha_1 U_1 + \alpha_2 U_2 + ---- + \alpha_n U_n)$

$\Rightarrow L(U) = \alpha_1 L(U_1) + \alpha_2 L(U_2) + ... + \alpha_n L(U_n)$

This is the required LT.

Q. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator,

given by $L(1,0,0) = (-2,1,0)$

$L(0,1,0) = (3,-2,1)$

$L(0,0,1) = (0,-1,3)$

what is $L(-3,2,4)$? Also find the LT.

Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

Then, S is a basis for $\mathbb{R}^3 = U$.

{ Check whether S is a basis. If not, find the LI vectors & extend S into a basis. }

Let $U = (x,y,z) \in \mathbb{R}^3$ be any vector in $\mathbb{R}^3$.

Then, $U = (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$ ①

Taking LT, L, on both sides of ①, we get

$L(U) = L(x,y,z) = L[x(1,0,0) + y(0,1,0) + z(0,0,1)]$

$= x\, L[(1,0,0)] + y\, L[(0,1,0)] + z\, L[(0,0,1)]$

$= x(-2,1,0) + y(3,-2,1) + z(0,-1,3)$

$\Rightarrow L(v) = (-2x + 3y,\ x - 2y - z,\ y + 3z)$

$\Rightarrow L(x,y,z) = (-2x+3y,\ x-2y-z,\ y+3z)$

It is the req^d LT.

$L(-3,2,4) = (6+6,\ -3-4-4,\ 2+12)$

$\Rightarrow L(-3,2,4) = (12, -11, 14)$

Q. Find the LT $L: P_3 \rightarrow P_3$, given by

V. imp ⇒

$L(1+x) = 1+x$

$L(2+x) = x+3x^2$

$L(x^2) = 0$

ax^3+bx^2+cx+d

$L(0,0,1,1) = (0,0,1,1)$

$L(0,0,1,2) = (0,3,1,0)$

$L(0,1,0,0) = (0,0,0,0)$

Let

$S = \{1+x,\ 2\cancel{x}+x,\ x^2\}$

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc} & x^2 & x & c \\ \end{array}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$1\ 0\ 0\ 0$ ↗ is missing

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} R_3 - R_2$$

Here, $r(A) = 3 =$ ~~rk~~ no. of vectors in S.

So, S is LI..

Also, $\dim(U) = \dim(P_3) = 4$.

So, by adding the std. basis vectors ~~x^2~~ x^3, we define

$S_1 = \{x^3,\ 1+x,\ 2+x,\ x^2\}$, as the basis for P_3.

*Define $L(x^3) = 0$

Let $v \in P_3$ with $v = ax^3 + bx^2 + cx + d$.

Express u in terms of vectors in S_1.

$\Rightarrow$ Let $v = \alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 (1+x) + \alpha_4 (2+x)$

$$[B : v] = \begin{array}{c} x^3 \\ x^2 \\ x \\ c \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 1 & 2 & d \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 1 & d-c \end{array}\right] R_4 - R_3$$

$\Rightarrow \boxed{\alpha_1 = a}, \boxed{\alpha_2 = b}, \alpha_3 + \alpha_4 = c$

$\boxed{\alpha_4 = d-c}$

$\Rightarrow \boxed{\alpha_3 = 2c-d}$

Here, $r(B) = r(B:v) = 4$, the no. of vectors. So,

$v = ax^3 + bx^2 + cx + d$

$v = ax^3 + bx^2 + (2c-d)\overset{(1+x)}{\cancel{x}} + (d-c)(2+x)$

$\therefore$ ~~If~~ Taking L on both sides

$\Rightarrow L(ax^3 + bx^2 + cx + d) = a\,L(x^3) + b\,L(x^2)$
$+ (2c-d)L(1+x) + (d-c)L(2+x)$

$= a.0 + b.0 + (2c-d)(1+x)$
$+ (d-c)(x + 3x^2)$

$L(ax^3 + bx^2 + cx + d) = 3(d-c)x^2 + \cancel{2}(c \cancel{+d})x + \cancel{d}(2c-d$

is the req^d LT.

HW Q. Find the LT $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, defined by $T(1, 2) = (3, 0)$
& $T(2, 1) = (1, 2)$

$S = \{(1, 2)(2, 1)\}$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} R_2 - 2R_1$$

Here, $r(A) = 2 =$ no. of vectors in S.

So, S is LI.

Also, $\dim(U) = \dim(\mathbb{R}^2) = 2$.

Let $v = (x, y) \in \mathbb{R}^2$ be any vector in $\mathbb{R}^2$.

Then, $v = (x, y) = x(1, 2) + y(2, 1)$

$= (x + 2y, 2x + y)$

Taking LT on both sides.

$\Rightarrow L(v) = L(x, y) = x\, L(1, 2) + y\, L(2, 1)$

$= x(3, 0) + y(1, 2)$

$\Rightarrow L(v) = (3x + y, 2y)$

This is the req^d LT.

Date 1/4/12

Section - 5.3

(section 5.2 after 5.4)

✻ The DIMENSION THEOREM

(Kernel & Range of a LT).

Let $L : U \rightarrow W$ be a LT. Then,

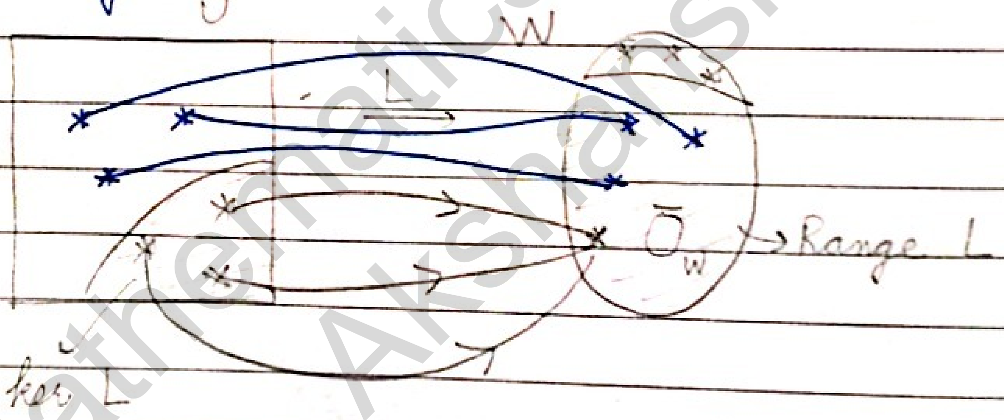
① The kernel or null space of L is denoted & defined by

$$\ker L = N(L) = \{ u \in U \mid L(u) = \bar{0}_W \}.$$

null space

② The range space of L is denoted & defined by

$$\text{range } L = R(L) = \{ L(v) \mid v \in U \}.$$



✻ Note 1:

The kernel of a LT, L, from U to W

($\ker L : U \rightarrow W$)

is a subspace of U.

✻ Note 2:

The range of L is a subspace of W.

✻ Result :- The DIMENSION THEOREM or RANK-NULLITY THEOREM.

The rank of the LT, L, is defined as,

rank of L = $\varrho(L)$

= dim {range L}

nullity of L = $n(L)$

= dim {ker L}.

The dimension thm. is given by:

rank + nullity = dim (U)

i.e dim {range L} + dim {ker L} = dim U.

——o——

eg. find the kernel & range of

$$L : \underset{U}{\mathbb{R}^3} \longrightarrow \overset{W}{\mathbb{R}^2}$$

$$L[(x_1, x_2, x_3)] = (x_1 - x_2, x_1 + x_3)$$

By defin^n, ker(L) = $\{ v \in U \mid L(v) = \bar{0}_W \}$.

Let $L[(x_1, x_2, x_3)] = \bar{0}_W = (0,0)$

$\Rightarrow (x_1 - x_2, x_1 + x_3) = (0,0)$.

$\Rightarrow x_1 - x_2 = 0$

$x_1 + x_3 = 0$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} R_2 - R_1$$

$\varrho(A) = 2 < 3$, the no. of unknowns.

So, sys is LD & has many nonzero sol^{ns}

Let ~~x_3~~ $x_3 = k$, parameter

$\Rightarrow x_2 + x_3 = 0 \Rightarrow \boxed{x_2 = -k}$

Also, $\boxed{x_1 = k}$

So, ker L = $\{(-k, -k, k) \mid k \in \mathbb{R}\}$

$= \{k(-1, -1, 1) \mid k \in \mathbb{R}\}$

$\Rightarrow$ null space of L = ker L = span $\{(-1,-1,1)\}$.

$\therefore$ nullity = dim(ker L)
$= 1$.

To find the range :-

By defin :-

$$L[(x_1, x_2, x_3)] = (x_1 - x_2, x_1 + x_3)$$
$$= x_1(1,1) + x_2(-1,0) + x_3(0,1)$$

Let $S = \{(1,1), (-1,0), (0,1)\}$.

/* Checking for LI */

Consider

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} R_2 + R_1$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0. \end{bmatrix} R_3 - R_2$$

$\rho(B) = 2 < 3$, no. of vectors in S.

$\therefore$ S is LD.

$S_1 = \{(1,1), (0,1)\}$ is LI & spans range L.

$\therefore$ Range L = span S

rank = dim (range L) = 2.

Consider rank + nullity = 2 + 1 = 3 = dim U.

$\therefore$ The rank nullity thm. is ~~justified~~ verified.

Q Find the kernel & range of the LT.

$L : \underset{V}{\mathbb{R}^4} \longrightarrow \overset{W}{\mathbb{R}^3}$ by

$L[(x_1, x_2, x_3, x_4)] = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$

Also, verify the dimention thm.

To find ker L :

Let $L[(x_1, x_2, x_3, x_4)] = \bar{O}_W = (0,0,0)$

$\Rightarrow (x_1 - x_4, x_2 + x_3, x_3 - x_4) = (0,0,0)$

$\Rightarrow$

$x_1 - x_4 = 0 \Rightarrow x_1 = x_4$

||ly, $x_2 = -x_3$

$x_3 = x_4$

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Here, $r(A) = 3 < 4$, the no. of unknowns.

So, $\exists$ many solⁿ (non zero).

Let $x_3 = k$

$\Rightarrow x_3 = x_4 = x_1 = k$

$x_2 = -k$

So, $\ker L = \{(k, -k, k, k) \mid k \in \mathbb{R}\}$

$= \{k(1,-1,1,1) \mid k \in \mathbb{R}\}$

So, $n(L) = \ker(L) = \text{span}\{(1,-1,1,1)\}$.

$\therefore$ nullity $= \dim(\ker L) = \underline{1}$.

To find range L :

By defin^n :-

$$L[(x_1, x_2, x_3, x_4)] = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$$

$$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,1,1) + x_4(-1,0,-1)$$

Let $S = \{(1,0,0), (0,1,0), (0,1,1), (-1,0,-1)\}$

Consider

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ \cancel{R_3 - R_2} \\ R_4 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 + R_3 \end{matrix}$$

$r(A) = 3 < 4$, no. of vectors in S.

$\therefore$ S is LD.

$\therefore S_1 = \{(1,0,0),$ ~~(0,1)~~ $(0,1,0), (0,0,1)\}$

$= \{e_1, e_2, e_3\}$ is LI & spans range

∴ Range L = span S_1.

Consider rank + nullity $= 3 + 1 = 4$
$= \dim(V)$.

So, rank nullity thm. is verified.

Q Find the kernel & range space of L.

$L : P_2 \rightarrow P_2$ by $L(at^2+bt+c) = \left[(a+2b)t + (b+c)\right]$

Q. Check whether $(-4t^2 + 2t - 2) \in \ker L$.

$t^2 + 2t + 1$ is in range L

Consider

$L[-4t^2+2t-2]$ $\quad a = -4, b = 2, c = -2$

$= (-4+4)t + (2-2) = 0 = \bar{0}_w$

So, $-4t^2 + 2t - 2 \in \ker L$.

Let $L(at^2+bt+c) = t^2+2t+1$

$\Rightarrow (a+2b)t + (b+c) = t^2 + 2t + 1$

Coeff of t^2 :- $0 = 1$.

t :- $a+2b = 2$

constt :- $b+c = 1$

The first eq^n $(0 = 1)$, being absurd eq^n, we cannot solve the given sys.

∴ t^2+2t+1 doesn't have a pre-image in U & hence,

$t^2+2t+1 \notin$ range L.

To find ker L :-

Let $L(at^2+bt+c) = \bar{0}_W$.

$\Rightarrow (a+2b)t + (b+c) = 0$.

$\Rightarrow a+2b = 0, \; b = -c$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\rho(A) = 2 < 3$, no. of unknowns.

∴ The sys. is LD & has many nonzero solns.

Let $c = k$

So, $b = -k$

$a = 2k$

So, $\ker L = \{ 2k.t^2 - kt + k \mid k \in \mathbb{R} \}$

$\Rightarrow \ker L = \{ k(2t^2 - t + 1) \mid k \in \mathbb{R} \}$.

∴ $\ker L = \text{span}\{2t^2 - t + 1\}$.

$\dim(\ker L) = 1 =$ nullity.

To find range :-

By defin :-

$L(at^2 + bt + c) = (a + 2b)t + (b + c)$

$= a(t) + b(2t + 1) + c(1)$

Let $S = \{t, 2t + 1, 1\}$

Consider

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0. \end{bmatrix} R_3 - R_2$$

$\rho(B) = 2 < 3$, no. of vectors in S.

So, its LD.

So, $S_1 = \{t, 1\}$ is LI & spans Range L.
So, Range L = span S_1

Now,
Consider rank + nullity $= 1 + 2 = 3 = \dim(U)$.
So, rank nullity thm is verified.

Section - 5.4

we know $r(L) = \dim(\text{range } L)$
So, find rank(L). If
$r(L)\,[= \dim(\text{range } L)] = \dim(W)$
Then, L is onto

§ One-one & onto LT.

$\ker(L) = 0_u$

one-one (i) $L : U \to W$ be a LT. (say). Then, L is said to be one to one if distinct vectors in U are mapped to different vectors in W.

i.e., $L(U_1) = L(U_2)$
$\Rightarrow U_1 = U_2 \;\forall\; U_1, U_2 \in U$.

(ii) onto :
If every vector in W has a pre-image in U,
i.e., $\forall\; w \in W \;\exists\; u \in U$ s.t $L(u) = w$.

Note: ① ~~A~~ LT $(L : U \to W)$ ~~a~~ is ONE-ONE
iff $\ker L = \{\bar{0}_U\}$.
② $L : U \to W$ is ONTO
iff range $L = W$

Q. Let $L : R^3 \to \mathbb{R}^3$ be a LT s given by.

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 5 & 1 & -1 \\ -3 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Does $(1,-2,3) \in \ker L$?
Does $(-16,12,-8) \in$ range L.
Also, find $\ker L$ and range L.

$L[(x_1, x_2, x_3)] = (5x_1 + x_2 - x_3, -3x_1 + x_3, x_1 - x_2 - x_3$

Consider

$$L[(1,-2,3)] = (5-2-3, -3+3, 1+2-3)$$
$$= (0,0,0)$$
$$= \bar{0}_W$$

So, $(1,-2,3) \in \ker L$.

Let $L[(x_1, x_2, x_3)] = (-16, 12, -8)$

$\Rightarrow (5x_1 + x_2 - x_3, -3x_1 + x_3, x_1 - x_2 - x_3) = (-16, 12, -8)$

$$\Rightarrow \begin{aligned} 5x_1 + x_2 - x_3 &= -16 \\ -3x_1 \quad + x_3 &= 12 \\ x_1 - x_2 - x_3 &= -8 \end{aligned}$$

$\Rightarrow x_3 = 12 + 3x_1 \Rightarrow x_1 - x_2 - (12 + 3x_1) = -8$
$\Rightarrow x_1 - x_2 - 12 - 3x_1 = -8$
$\Rightarrow -2x_1 - x_2 = 4$

Also, $5x_1 + x_2 - (12 + 3x_1) = -16$
$\Rightarrow 2x_1 + x_2 = -4$
$-2x_1 - x_2 = 4$

$$A:B = \left[\begin{array}{ccc|c} 1 & -1 & -1 & -8 \\ 5 & 1 & -1 & -16 \\ -3 & 0 & 1 & 12 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & -8 \\ 0 & \underline{6} & 4 & 24 \\ 0 & -3 & -2 & -12 \end{array}\right] \begin{array}{l} \\ R_2 - 5R_1 \\ R_3 + 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & -8 \\ 0 & 6 & 4 & 24 \\ 0 & 0 & 0 & 0 \end{array}\right] 2R_3+R_2 .$$

$\rho(A) = \rho(A:B)$.

So, sys. has a sol^n.

∴ $(16, 12, -8) \in$ range L.

Now, find ker L & range L.

Q. Let $L : \underset{U}{P_3} \longrightarrow \underset{W}{P_3}$ be a LT given by

$$L(ax^3+bx^2+cx+d) = 2cx^3+(a+b)x+(d+c)$$

(i) does $4x^3-4x^2 \in$ ker L?

(ii) does $4x^3-3x^2+7 \in$ range L?

Consider $L(4x^3-4x^2) = 2(0)x^3+(4-4)x+(0+0)$

$a=4, b=-4$ $= 0 . = \bar{0}_W$.

∴ $4x^3-4x^2 \in$ ker L.

Now.

Let $L(ax^3+bx^2+cx+d) = 4x^3-3x^2+7$

$\Rightarrow 2cx^3+(a+b)x+(d+c) = 4x^3-3x^2+7$

coeff. of x^3: $2c = 4$.

x: $a+b = 0$.

c: $d+c = 7$

x^2: $0 = -3$.

~~So~~ ∵ coeff. of x^2 ~~co~~ on comparing gives absurd eq^n. So, sys. doesn't have a sol^n. Hence,

$4x^3-3x^2+7 \notin$ range L.

Q. Find the ker & range of the following LT. Also, verify the dimension thm. Check whether L is one-one, onto.

HW ① $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 5 \\ -2 & 3 & -13 \\ 3 & -3 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

② $L: \overset{U}{\mathbb{R}^3} \longrightarrow \overset{W}{\mathbb{R}^2}$

$$L(x_1, x_2, x_3) = \begin{bmatrix} 3 & 2 & 11 \\ 2 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

③ $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$L(x_1, x_2) = (x_1, x_1 + x_2, x_2).$$

② Let $L([x_1, x_2, x_3]) = \bar{0}_W$

$$\Rightarrow (3x_1 + 2x_2 + 11x_3, 2x_1 + x_2 + 8x_3) = (0,0)$$

$$\Rightarrow \quad 3x_1 + 2x_2 + 11x_3 = 0$$

$$2x_1 + x_2 + 8x_3 = 0$$

$$A = \begin{bmatrix} 3 & 2 & 11 \\ 2 & 1 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 & 11 \\ 0 & -1 & 2 \end{bmatrix} 3R_2 - 2R_1$$

$r(A) = 2 < 3$, no. of unknowns.

∴ The sys. has many non zero sol^{ns}.

Let $x_3 = k$.

$\Rightarrow \quad -x_2 + 2x_3 = 0 \quad \Rightarrow \quad x_2 = 2k$

$3x_1 + 2x_2 + 11x_3 = 0$

$\Rightarrow x_1 = \frac{1}{3}(-4k + (-11)k) = \frac{-15}{3}k.$

$\therefore \ker L = \{(-15k, 2k, k) \mid k \in \mathbb{R}\}$

$= \{k(-5, 2, 1) \mid k \in \mathbb{R}\}$

$= \text{Span}\{(-5, 2, 1)\}.$

Here, $\dim(\ker L) = 1.$

{The set $\{(-5, 2, 1)\}$ is a basis for ker L }

$\because \ker L \neq 0_v,$

$\Rightarrow$ L is not one-one.

To find range L.

By defin^n:- $L([x_1, x_2, x_3]) = (3x_1 + 2x_2 + 11x_3, 2x_1 + x_2 + 8x_3)$

$= x_1(3, 2) + x_2(1, 1) + x_3(11, 8)$

$\therefore S = \{(3,2), (2,1), (11,8)\}.$

$$B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ 11 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 \\ 0 & -1 \\ 0 & 2 \end{bmatrix} \begin{matrix} \\ 3R_2 - 2R_1 \\ 3R_3 - 11R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 2R_2 \end{matrix}$$

Here, $r(B) = 2 < 3$, no. of vectors in S.

$\therefore$ S is LD.

$\therefore S_1 = \{(3, 2), (0, -1)\}$ or $S_2 = \{(3, 2), (2, 1)\}$ is is LI & spans range L.

$\therefore$ S_1 is a basis for range L.

$\dim(\text{range } L) = 2$

$= \dim(\mathbb{R}^2) = \dim(W)$.

$\Rightarrow$ range $L = W = \mathbb{R}^2$.

$\therefore$ L is onto.

Now,

dim(range L) $\qquad$ = dim(ker L)

rank + nullity $= 2 + 1 = 3 =$ ~~B~~ $\dim(U)$.

So, dimension of thm. is verified.

③ $L : U \longrightarrow W$.

$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$.

$L[(x_1, x_2)] = (x_1, x_1 + x_2, x_2)$.

To find ker L.

$L[(x_1, x_2)] = \bar{0}_W$

$\Rightarrow (x_1, x_1 + x_2, x_2) = (0, 0, 0)$

$\Rightarrow x_1 = 0, \quad x_2 = 0, \; x_1 + x_2 = 0$.

So, $\ker L = \{0, 0\} = \{\bar{0}_U\}$.

$\therefore$ L is one-one.

Here, $\dim(\ker L) = 0$.*

To find range L.

By defin^n:-

$L[(x_1, x_2)] = (x_1, x_1 + x_2, x_2)$

$= x_1(1, 1, 0) + x_2(0, 1, 1)$

$\therefore S = \{(1, 0, 0), (0, 1, 1)\}$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

rank(B) = 2, no. of vectors in S.

So, S is LI.

Q. Let $L: \overset{U}{P_2} \longrightarrow \overset{W}{P_3}$ be a LT given by
$$L(ax^2+bx+c) = cx^3+bx^2+ax.$$
Find ker L & range L

Check whether L is one-one & onto.

To find Ker L

Let $L(ax^2+bx+c) = 0_W$

$\Rightarrow cx^3+bx^2+ax = 0.$

equate coeff. of $x^3 : c=0$

$x^2 : b=0$

$x : a=0,$

So, $\ker L = \{ax^2+bx+c \mid, a=b=c=0\}$

$= \{0\}$

$= \{\bar{0}_U\}$

$\therefore$ L is one-one

/* { Note :- $\dim(\ker L) = 0$ } */.

To find range L :-

$L(ax^2+bx+c) = cx^3+bx^2+ax$

$= c(x^3)+b(x^2)+a(x)$

$\Rightarrow S = \{x^3, x^2, x\}.$

Then, S is LI & spans range L.

$\Rightarrow$ S is a basis for range L.

$\Rightarrow$ range L = span S.

$\dim(\text{range } L) = 3. (\neq \dim(W) = \cancel{3}\, 4)$

$\Rightarrow$ ~~L is~~ range $L \neq P_3$

& L is not onto.

—x—.

OBJECTIVE PROBLEMS

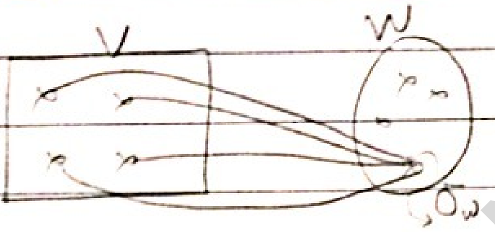
Q. Suppose

$L: V \to W$ is the LT given by

$L(v) = \bar{O}_W \quad (\forall\ v \in V)$

Find $\ker L$ & range L.

Ans :- $\ker(L) = \boxed{V}$ range $\to \bar{O}_W$



L is neither one-one nor onto.

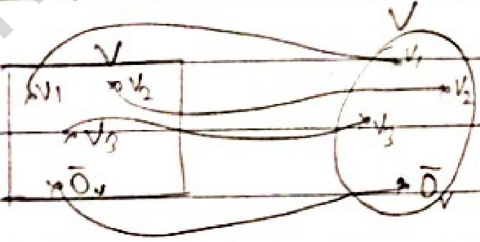
Q. Suppose $L: V \to V$ is a LT given by

$L(v) = v \quad \forall\ v \in V$.

Find $\ker(L)$ & range (L)

$\{\bar{O}_V\}$ $\quad$ V

L is both one-one & onto



Q. Which of the following are one-one or onto?

① $L: \mathbb{R}^3 \to \mathbb{R}^2$ by $L([x, y, z]) = (x+y, y+z)$

② $L: \mathbb{R}^3 \to \mathbb{R}^3$ by $L([x, y, z]) = (2x, x+y+z, -y)$

③ $L: \underset{V}{P_2} \to \underset{W}{P_2}$ by $L(ax^2+bx+c) = (a+b)x^2 + (b+c)x + (c+a)$

④ $L: P_2 \to M_{22}$ by $L(ax^2+bx+c) = \begin{bmatrix} a+c & 0 \\ b-c & -3a \end{bmatrix}$

⑤ $L: \mathbb{R}^2 \to \mathbb{R}^3$

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -3 & 4 \\ -6 & 9 \\ 7 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -3x_1 + 4x_2 \\ -6x_1 + 9x_2 \\ 7x_1 - 8x_2 \end{bmatrix}$$

To find ker L :

Let $L(ax^2+bx+c) = \bar{O}_W$

$$\begin{bmatrix} a+c & 0 \\ b-c & -3a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\Rightarrow$ $a+c = 0$

$b-c = 0$

$-3a = 0 \quad \Rightarrow a = 0 = c \;,\; b = c \Rightarrow a=b=c=0$

$\therefore \ker L = \{ ax^2+bx+c \mid a=0=b=c \}$.

$= \{0x^2+0x+0\}$

$= \{\bar{O}_V\}$

$\therefore$ L is one-one

/* dim (ker L) = 0 */

To find range L

By defin^n; $L(ax^2+bx+c) = \begin{bmatrix} a+c & 0 \\ b-c & -3a \end{bmatrix}$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right\}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 3. \end{bmatrix} R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} R_3 + R_2$$

$r(B) = 3$ = no. of vectors in S.

So, S is LI. & spans range L.

∴ S is a basis for range L.

∴ dim(range L) = 3 ($\neq$ dim(W) = 4).

∴ range L $\neq$ W.

So, its not onto.

③ To find ~~ker~~ range (L):

Let $L(ax^2+bx+c) =$ ~~Dw~~ $(a+b)x^2 + (b+c)x + (c+a)$

$= a(x^2+1) + b(x^2+x) + c(x+1)$

$\Rightarrow S = \{x^2+1, x^2+x, x+1\}$.

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} R_3 - R_2$$

$r(B) = 3$ = no. of vectors in S.

So, S is ~~a~~ LI & spans range L.

∴ S is a basis for range L.

∴ dim(range L) = 3 ($\neq$ ~~dom~~ = dim(W) = P_2 = 4.

∴ range L = W. So, its onto.

To find ker(L).

let $ax^2+bx+c = \bar{O}_W$

$\Rightarrow (a+b).0 + (b+c).0 + (c+a) = 0$

$\Rightarrow$ Equating coeff. of x^2 : $a+b=0$

$x : c+b=0$

$c : c+a=0$.

$\Rightarrow a=b=0=c$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} R_3 + R_2$$

$\Rightarrow r(A) = 3 =$ no. of unknowns.

$\Rightarrow$ Sys. has only the zero solⁿ.

$\Rightarrow a = 0 = b = c$

$\therefore \ker L = \{ax^2+bx+c \mid a=b=c=0\}$

$= \{0 = \bar{O}_W\}$

* L is one-one & $\dim(\ker(L)) = 0$

② To find ker(L)

$L[(x,y,z)] = (2x, x+y+z, -y)$

Let $L[(x,y,z)] = \bar{O}_W$.

$\Rightarrow (2x, x+y+z, -y) = (0,0,0)$.

$\Rightarrow 2x = 0. \quad \Rightarrow x=0$

$x+y+z=0. \quad \Rightarrow z=0$

$-y = 0 \quad \Rightarrow y=0$

So, $\ker L = \{(x,y,z) \mid x = y = z = 0\}$
$= \{(0,0,0) = \overline{0}_U\}$

$\Rightarrow$ L is one-one. & $\dim(\ker L) = 0$

<u>To find range L</u> :-

Let $L(x,y,z) = (2x, x+y+z, -y)$

$= x(2,1,0) + y(0,1,-1) + z(0,1,0)$

$\Rightarrow S = \{(2,1,0), (0,1,-1), (0,1,0)\}$

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} R_3 - R_2$$

$$\sim \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} R_3 + R$$

So, rank(B) = 3 = no. of vectors in S.

So, S is LI & is basis for range L.

Also, dim(range L) = 3 = dim(W)

So, L is onto. (range L = W)

Puffin
Date 8/4/12
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Section - 5.2

MATRIX OF A LT.

Let $L : U \to W$ be a LT & $\dim(U) = n$,
$\dim(W) = m$.

Let $S = \{u_1, u_2, \ldots, u_n\}$ &
$T = \{w_1, w_2, \ldots, w_m\}$ be ordered basis of U & W resp. Then, we have

$$L(u_j) = \cancel{\alpha_{1j}}\ \alpha_{1j} w_1 + \alpha_{2j} w_2 + \ldots + \alpha_{mj} w_m, \quad j = 1, 2, \ldots n.$$

∴ The coordinate vector of $L(u_j)$ w.r.t the basis T is given by

$$[L(u_j)]_T = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{bmatrix}; \quad j = 1, 2, \ldots, n.$$

∴ The matrix of LT (L) w.r.t basis S & T ~~A =~~ is denoted & defined by

$$A_{ST} = (L : S, T) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \ldots & \alpha_{1j} & \ldots \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2j} & \ldots \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mj} & \ldots \alpha_{mn} \end{bmatrix}$$

ex :- Find the matrix of the LT.

$L: \mathbb{R}^3 \to \mathbb{R}^2$, given by

$L[(x,y,z)] = (x+y, y-z)$ w.r.t the basis

$S = \{\overset{v_1}{(1,0,1)}, \overset{v_2}{(0,1,1)}, \overset{v_3}{(1,1,1)}\}$

& $T = \{\underset{w_1}{(1,2)} \underset{w_2}{(-1,1)}\}$.

$L(v_1) = L(1,0,1) = (1,-1)$.

$L(v_2) = L(0,1,1) = (1,0)$.

$L(v_3) = L(1,1,1) = (2,0)$

Find solns. Let $L(v_1) = (1,-1) = \alpha_1 \overset{w_1}{(1,2)} + \alpha_2 \overset{w_2}{(-1,1)}$

$= (\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$.

$\Rightarrow \alpha_1 - \alpha_2 = 1$

& $2\alpha_1 + \alpha_2 = -1$

$\Rightarrow 3\alpha_1 = 0$

$\Rightarrow \alpha_1 = 0 \quad \Rightarrow \alpha_2 = -1$

Let $L(v_2) = (1,0) = \beta_1(1,2) + \beta_2(-1,1)$

$= \beta_1 - \beta_2 = 1 \Rightarrow \beta_1 = \beta_2 + 1$

$2\beta_1 + \beta_2 = 0 \Rightarrow 3\beta_1 = 0 - 2/3$

So, ~~$\beta_1 = 0 = \beta_2$~~.

$\Rightarrow \beta_1 = 1/3$ & $\beta_2 = -\frac{2}{3}$,

Let $L(v_3) = \gamma_1(1,2) + \gamma_2(-1,1)$

$\Rightarrow (2,0) = (\gamma_1 - \gamma_2, 2\gamma_1 + \gamma_2)$

$\Rightarrow \gamma_1 - \gamma_2 = 2$ & $2\gamma_1 + \alpha_2 = 0$

$\Rightarrow \gamma_1 = \frac{2}{3}$ & $\gamma_2 = \frac{-4}{3}$.

$$\therefore A_{ST} = (L : S, T) = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}$$

$$\Rightarrow A_{ST} = \begin{bmatrix} 0 & 1/3 & 2/3 \\ -1 & -2/3 & -4/3 \end{bmatrix}$$

$\therefore$ matrix of LT, L w.r.t S & T is given by $A_{ST} = \begin{bmatrix} 0 & 1/3 & 2/3 \\ -1 & -2/3 & -4/3 \end{bmatrix}$

★ PROCEDURE TO FIND MATRIX OF A LT

S1) Find $L(u_1), L(u_2), \ldots, L(u_n)$

S2) Form the augmented matrix $w_1, w_2, \ldots, w_m$.

$$\left[\begin{array}{cccc|cccc} w_1 & w_2 & \ldots & w_m & L(u_1) & L(u_2) & \ldots & L(u_n) \end{array}\right]$$

S3) Reduce the above matrix as

$$\sim [I_m \mid A]$$

Then, A is the req^d ~~m~~ matrix of L.

(For the previous problem,

$$\left[\begin{array}{cc|ccc} 1 & -1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 0 & 0 \end{array}\right] \cancel{R_2 - 2R_1}$$

$$\sim \left[\begin{array}{cc|ccc} 1 & -1 & 1 & 1 & 2 \\ 0 & +3 & -3 & -2 & -4 \end{array}\right] R_2 - 2R_1$$

$$\sim \left[\begin{array}{cc|ccc} 3 & 0 & 0 & 1 & 2 \\ 0 & 3 & -3 & -2 & -4 \end{array}\right] 3R_1 + R_2$$

$$\sim \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1/3 & 2/3 \\ 0 & 1 & -1 & -2/3 & -4/3 \end{array}\right] \begin{array}{l} R_1/3 \\ R_2/3 \end{array}$$

$$= [I_2 \mid A]$$

Q. Find the matrix w.r.t the given basis:

(1) $L : \overset{V}{\mathbb{R}^3} \longrightarrow \overset{W}{\mathbb{R}^2}$ by

$$L[(x,y,z)] = (-2x+3z, x+2y-z)$$

$$B = \{ \underset{v_1}{(1,-3,2)}, \underset{v_2}{(-4,13,-3)}, \underset{v_3}{(2,-3,20)} \}$$

$$C = \{ \underset{w_1}{(-2,-1)} \; \underset{w_2}{(5,3)} \}$$

② $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$L[(x,y,z)] = (-6x+4y-z, -2x+3y-5z, 3x-y+7x).$$

Basis S & T are standard basis $\rightarrow$ $A_{ST} = \begin{bmatrix} -6 & -2 & 3 \\ 4 & 3 & -1 \\ -1 & -5 & 7 \end{bmatrix}$

③ $L : \overset{V}{M_{22}} \longrightarrow \overset{W}{\mathbb{R}^3}$ by

$$L\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = (a-c+2d, 2a+b-d, -2c+d)$$

$$B = \left\{ \overset{v_1}{\begin{bmatrix} 2 & 5 \\ 2 & -1 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}} \right\}$$

$$C = \{ \underset{w_1}{(7,0,-3)}, \underset{w_2}{(2,-1,-2)}, \underset{w_3}{(-2,0,1)} \}$$

① $L(v_1) = L(1,-3,2) = (-2+6, 1-6-2)$
$= (4, -7)$

$L(v_2) = (8-9, -4+26+3) = (-1, 25)$

$L(v_3) = (-4+60, 2-6-20) = (56, -24)$

So,

$$[W : L(V)] = \left[\begin{array}{cc|ccc} -2 & 5 & 4 & -1 & 56 \\ -1 & 3 & -7 & 25 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|ccc} -2 & 5 & 4 & -1 & 56 \\ 0 & 1 & -18 & 51 & -104 \end{array}\right] 2R_2 - R_1$$

$$\sim \left[\begin{array}{cc|ccc} -2 & 0 & 94 & -256 & 576 \\ 0 & 1 & -18 & 51 & -104 \end{array}\right] \begin{array}{l} R_1 - 5R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|ccc} 1 & 0 & -47 & 128 & -288 \\ 0 & 1 & -18 & 51 & -104 \end{array}\right] -R_1/2$$

104
×5
520
+56
576

$$\sim [\, I_2 \mid A \,]$$

So, the req^d matrix of L w.r.t B & C

$$\text{(L:B,C)} = A_{BC} = \begin{bmatrix} -47 & 128 & -288 \\ -18 & 51 & -104 \end{bmatrix}$$

③ $L(u_1) = (-2, 10, -5)$

$L(u_2) = (0, -7, 1)$

$L(u_3) = (1, -6, 1)$

$L(u_4) = (5, -2, 4)$

So, the augmented matrix

$$[W : L(U)] = \left[\begin{array}{ccc|cccc} 7 & 2 & -2 & -2 & 0 & 1 & 5 \\ 0 & -1 & 0 & 10 & -7 & -6 & -2 \\ -3 & -2 & 1 & -5 & 1 & 1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|cccc} 7 & 2 & -2 & -2 & 0 & 1 & 5 \\ 0 & -1 & 0 & 10 & -7 & -6 & -2 \\ 0 & -8 & 1 & -41 & 7 & 10 & 43 \end{array}\right] \begin{array}{l} \\ \\ 7R_3 + 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|cccc} 7 & 0 & -2 & 18 & -14 & -11 & 1 \\ 0 & -1 & 0 & 10 & -7 & -6 & -2 \\ 0 & 0 & 1 & -121 & 63 & 58 & 59 \end{array}\right] \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 8R_2 \end{array}$$

242 18 224 63 2 126 158 2 116 159 2 118

$$\sim \left[\begin{array}{ccc|cccc} 7 & 0 & 0 & -224 & 112 & 105 & 119 \\ 0 & -1 & 0 & 10 & -7 & -6 & -2 \\ 0 & 0 & 1 & -121 & 63 & 58 & 59 \end{array}\right] R_1+2R_3$$

$$\sim \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -32 & 16 & 15 & 17 \\ 0 & 1 & 0 & -10 & 7 & 6 & 2 \\ 0 & 0 & 1 & -121 & 63 & 58 & 59 \end{array}\right] \begin{array}{l} R_1/7 \\ -R_2 \\ \end{array}$$

$$\sim [\, I_3 \mid A \,]$$

So, ~~to~~ the required matrix of L w.r.t B & C is

$$(L : B, C) = A_{BC} = \begin{bmatrix} -32 & 16 & 15 & 17 \\ -10 & 7 & 6 & 2 \\ -121 & 63 & 58 & 59 \end{bmatrix}$$ Ans.

—o—

Section - 5.5

INVERTIBLE LT.

- A LT, $L : U \longrightarrow W$ is said to be invertible iff L is ONE-ONE & ONTO.
  i.e., $L^{-1} : W \rightarrow U$ exists only when L is one-one & onto.
  (An ISOMORPHISM)

Ex Find whether L is NON-SINGULAR (INVERTIBLE) hence, find L^{-1} if exists.

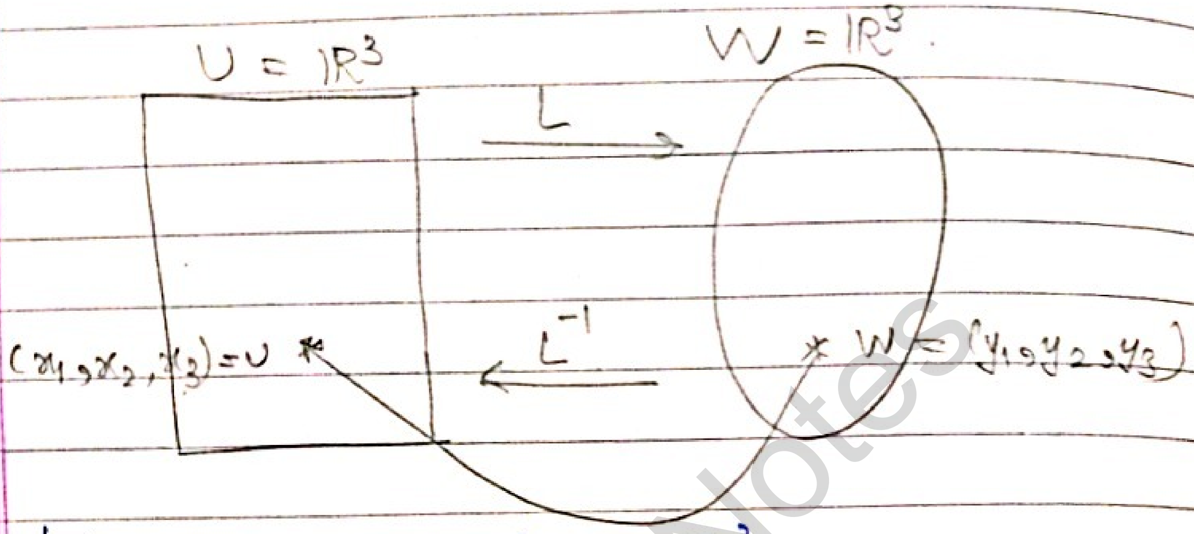
①. $L : P_2 \longrightarrow \mathbb{R}^3$
$L(a+bx+cx^2) = (a, b, c)$

② $L : \underset{U}{\mathbb{R}^3} \longrightarrow \underset{W}{\mathbb{R}^3}$
$L(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_3 + x_2, x_3)$

③ $L : P_2 \longrightarrow P_2$
$L(a + bx + cx^2) = ((a+b) + (b+2c)x + (a+b+3c)x^2)$

② $L : \underset{U}{\mathbb{R}^3} \longrightarrow \underset{W}{\mathbb{R}^3}$ ~~; $L^{-1} : \underset{\mathbb{R}^3}{W} \rightarrow \underset{\mathbb{R}^3}{U}$~~

Show that L is one-one & onto by finding the ker(L) & range(L).
Here, $L^{-1} : \underset{W}{\mathbb{R}^3} \longrightarrow \underset{U}{\mathbb{R}^3}$



Let $w = (y_1, y_2, y_3) \in \mathbb{R}^3$.

Let $L^{-1}(w) = L^{-1}(y_1, y_2, y_3) = (x_1, x_2, x_3)$

$\Rightarrow L(x_1, x_2, x_3) = (y_1, y_2, y_3)$

$\Rightarrow (x_3 + x_2 + x_1, x_3 + x_2, x_3) = (y_1, y_2, y_3)$

$$\Rightarrow \begin{aligned} x_1 + x_2 + x_3 &= y_1 \\ x_2 + x_3 &= y_2 \\ x_3 &= y_3. \end{aligned}$$

$$\Rightarrow \begin{aligned} x_3 &= y_3 \\ x_2 &= y_2 - y_3 \\ x_1 &= y_1 - y_3 - (y_2 - y_3) = y_1 - y_2 \end{aligned}$$

$\Rightarrow L^{-1}[(y_1, y_2, y_3)] = (y_1 - y_2, y_2 - y_3, y_3)$ is the req^d LT.

① $L : \underset{U}{P_2} \to \underset{W}{\mathbb{R}^3}$

Show its one-one & onto & hence, L^{-1} exists.

$\therefore \; \underset{W}{L^{-1}} : \mathbb{R}^3 \longrightarrow \underset{U}{P_2}$

Let $w = (a, b, c) \in \mathbb{R}^3$.

Let $L^{-1}(w) = u = \alpha_0 x^2 + \alpha_1 x + \alpha_2$.

$\Rightarrow W = L(\alpha_0 x^2 + \alpha_1 x + \alpha_2)$.

$\Rightarrow (a, b, c) = (\alpha_2, \alpha_1, \alpha_0)$

$\Rightarrow a = \alpha_2$

$b = \alpha_1$

$c = \alpha_0$.

$\therefore L^{-1}[a, b, c)] = cx^2 + bx + a$

is the reqd inverse LT.

③ $L : \overset{U}{P_2} \longrightarrow \overset{W}{P_2}$

Here, L is one-one & onto & hence L^{-1} exist (to be proved)

$L^{-1} : \underset{W}{P_2} \longrightarrow \underset{U}{P_2}$

Let $L^{-1}\{\alpha_0 x^2 + \alpha_1 x + \alpha_2\} = a + bx + cx^2$

$\Rightarrow \alpha_0 x^2 + \alpha_1 x + \alpha_2 = (a+b) + (b+2c)x + (a+b+x)x^2$

$\Rightarrow \alpha_0 = a + b + 3c \quad \Rightarrow c = \dfrac{\alpha_0 - (\alpha_1 + \alpha_2 - 2c) - (2c - \alpha_1)}{3}$

$\alpha_1 = b + 2c \quad \Rightarrow b = \alpha_1 - 2c$

$\alpha_2 = a + b \quad \Rightarrow b = \alpha_2 - a \Rightarrow a = \alpha_2 - \alpha_1 + 2c$

$\Rightarrow c = \dfrac{\alpha_0 - \cancel{\alpha_1} - \alpha_2 + 2c + 2c + \cancel{\alpha_1}}{3}$

$\Rightarrow \quad c = \dfrac{\alpha_0 - \alpha_2}{3}$

$a = \dfrac{\alpha_2 - 3\alpha_1 + 2\alpha_0}{3}$

$b = \dfrac{3\alpha_1 - 2\alpha_0 + 2\alpha_2}{3}$

$\Rightarrow$ $L^{-1}\{\alpha_0 x^2 + \alpha_1 x + \alpha_2\}$

$$= \left(\frac{\alpha_2 - 3\alpha_1 + 2\alpha_0}{3}\right) + \left(\frac{3\alpha_1 - 2\alpha_0 + 2\alpha_2}{3}\right)x$$

$$+ \frac{1}{3}(\alpha_0 - \alpha_2)x^2$$

is the req^d LT.