MATHEMATICS II

FIRST YEAR NOTES

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Mathematics II Complex Number Notes, First Edition

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Part-B

Complex No. Analysis

Chapter - 1

Complex nos.

* A complex no. z is a pt. (x,y) in the ~~x-y~~ X-Y plane (Complex plane or Z-plane) where X & Y axis are referred to as real & imaginary axis resp. & we write $z = (x,y)$.

For any 2 complex nos. $z_1 = (x_1, y_1)$ &

$z_2 = (x_2, y_2)$,

we define the oper^{ns} of add^{n} & multiplic^{n} as,

(i) $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

(ii) $z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2 . y_2)$

$= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$.

With these defin^{ns}, we write

$$z = (x,y) = (x,0) + (0,1)(y,0)$$

Here, $i = (0,1)$ is purely an imaginary no.

& we have

$$i^2 = (0,1)(0,1) = (-1,0)$$

& hence, $i^2 = -1$.

$\Rightarrow i = \pm\sqrt{-1}$.

$\therefore$ we have

$$\boxed{z = x + iy}$$

In this notation,

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2)$$

Also, x & y are the real & imaginary parts of a complex no. $z = x + iy$ & we write

$$x = \mathrm{Re}(z) \ \& \ y = \mathrm{Im}(z).$$

Sections 1.2 & 1.3

PROPERTIES OF COMPLEX NOS.

P1 $z_1 + z_2 = z_2 + z_1$
$z_1 \cdot z_2 = z_2 \cdot z_1$; for any 2 complex nos. z_1 & z_2.

P2 $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
$(z_1 z_2) z_3 = z_1 (z_2 z_3)$, for any 3 complex nos z_1, z_2 & z_3.

P3 $\exists$ a complex no. $0 = 0 + i0$ s.t
$z + 0 = 0 + z = z \ \forall$ complex no. z.
Here, 0 is the additive identity.

For any complex no. z, we have
$z \cdot 1 = 1 \cdot z = z$, where $1 = 1 + i0$; 1: multiplicative identity

P4 We define $(-1)z = -z$, then,

$$z + (-z) = (-z) + (z) = 0$$

Here, $-z$ is the additive inverse which exists $\forall$ complex no. z.

If $\boxed{z \neq 0}$, then, we define

$$\frac{1}{z} = z^{-1} = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right) \text{ s.t.}$$

$$z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1.$$

P5 For any 3 complex nos. z_1, z_2, z_3, we have

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

P6 For complex nos. z_1, z_2, z_3 & z_4 with

$$\boxed{z_3 \neq 0 \text{ \& } z_4 \neq 0}$$

$$\left(\frac{z_1 z_2}{z_3 z_4}\right) = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = (z_1 z_2)(z_3^{-1} \cdot z_4^{-1}).$$

—x—

Section - 1.4

* The modulus of a complex no. $z = x + iy$ is denoted & defined by

$$\boxed{|z| = \sqrt{x^2+y^2}}$$

& this denotes the distance of z from the origin $(0+i0)$.

- If $|z_1| < |z_2|$, then, z_1 is closer to the origin.

* TRIANGLE INEQUALITY

For any 2 complex nos, z_1 & z_2, we have

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$

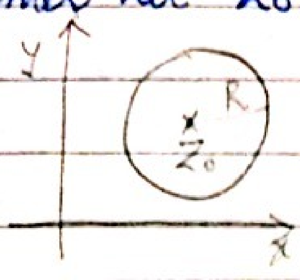
(ii) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

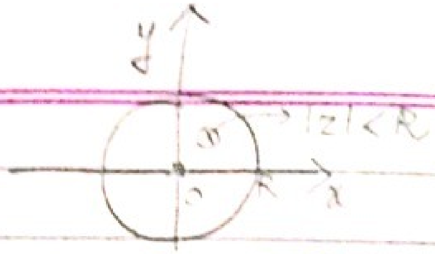
* The lower and upper bounds of $|z_1 \pm z_2|$ is,

$$\boxed{||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|}$$

* The equation of a circle with center at z_0 & radius R units is written as

$$\boxed{|z - z_0| = R}$$





- The equation

$$|z| = R$$

represents the circle with centre at origin & radius R units.

- At any pt. lying inside the above circle, we have,

$$|z| < R.$$

ex Check which of the complex nos. is closer to origin.

① $z_1 = 1+2i$, $z_2 = 3-4i$

$|z_1| = \sqrt{5}$, $|z_2| = 3$.

So, z_1 is closer to origin.

Q Simplify

$$\frac{1}{1+i} - \frac{2+3i}{1-i}$$

$= 1-3i$ ~~$-2+3$~~

Q If z lies on the unit circle $|z| = 1$, about the origin, find the lower & upper bounds for $|z^3 - 2|$.

$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$. ($\Delta$ inequality) → ①

Let $z_1 = z^3$ & $z_2 = 2$.

$\Rightarrow |z_1| = |z^3| = |z|^3$. & $|z_2| = 2$.

Using Δ inequalities ①, we have

$||z^3| - |2|| \leq |z^3 - 2| \leq |z^3| + |2|$

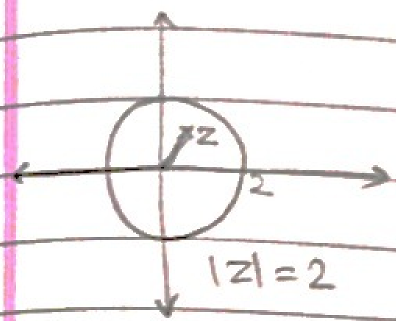
$\Rightarrow ||z|^3 - 2| \leq |z^3 - 2| \leq |z|^3 + 2.$

$\Rightarrow \quad \left| 1^3 - 2 \right| \leq \left| z^3 - 2 \right| \leq 1^3 + 2$

$\Rightarrow \quad \boxed{1 \leq \left| z^3 - 2 \right| \leq 3}$

$\Rightarrow$

Q. If z is a pt. inside the circle, centered at origin, with radius 2, find the upper bound of $\left| z^3 + 3z^2 - 2z + 1 \right|$.



$|z| = 2$

At any interior pt. z, we have $|z| < 2$.

Using Δ inequality, we have

$$\left| z^3 + 3z^2 - 2z + 1 \right| \leq \left| z^3 \right| + 3\left| z^2 \right| + 2|z| + |1|$$
$$\leq |z|^3 + 3|z|^2 + 2|z| + 1$$
$$\leq 2^3 + 3 \cdot 2^2 + 2 \cdot 2 + 1$$
$$\leq 8 + 12 + 4 + 1$$

$$\Rightarrow \left| z^3 + 3z_2 - 2z + 1 \right| \leq 25 \quad \text{Ans}$$

Q. If z lies on the circle $|z| = 2$, show that
$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| \leq \frac{1}{3}.$$

$$\left| z^4 - 4z^2 + 3 \right| \leq |z|^4 - 4|z|^2 + |3|$$

Check

$$\leq 16 - 4(4) + 3$$
$$\leq 3$$

$$\Rightarrow \left| z^4 - 4z^2 + 3 \right| \leq 3$$

$$\Rightarrow \frac{1}{\left| z^4 - 4z^2 + 3 \right|} \geq 3$$

$$\Rightarrow \left| \frac{1}{z^4 - 4z^2 + 3} \right| \leq \frac{1}{3}. \quad \text{Ans.}$$

M2 Consider $z^4 - 4z^2 + 3 = (z^2 - 3)(z^2 - 1)$

$$\therefore |z^4 - 4z^2 + 3| = |z^2 - 3|\,|z^2 - 1|$$

$$|z^2-3|\,|z^2-1| \geq \big|\,|z|^2 - |3|\,\big|\,\big|\,|z|^2 - |1|\,\big|$$
$$\geq |2^2 - 3|\,|2^2 - 1|$$
$$\geq |1|\,|3|$$

$$\Rightarrow |z^4 - 4z^2 + 3| \geq 3.$$

$$\Rightarrow \frac{1}{|z^4 - 4z^2 + 3|} \leq \frac{1}{3}$$

or $\left|\dfrac{1}{z^4 - 4z^2 + 3}\right| \leq \dfrac{1}{3}.$

Section 1.5

CONJUGATE OF A COMPLEX NO. :-

If $z = x + iy$ is a complex no., then its conjugate is denoted & defined by

$$\bar{z} = x - iy.$$

(i) $\overline{(\bar{z})} = z$

(ii) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(iii) $\overline{z_1 z_2} = (\bar{z}_1)(\bar{z}_2)$.

(iv) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \left(\dfrac{\bar{z}_1}{\bar{z}_2}\right)$, if $z_2 \neq 0$.

(v) $x = Re(z) = \dfrac{z + \bar{z}}{2}$.

$y = Im(z) = \dfrac{z - \bar{z}}{2i}$

⑥ $Re(z) \leq |Re(z)| \leq |z|$

$Im(z) \leq |Im(z)| \leq |z|$

⑦ $|z_1 \cdot z_2| = |z_1||z_2|$

⑧ $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $z_2 \neq 0$.

Sections 1.6, 1.7 & 1.8

POLAR FORM, PRODUCT & QUOTIENTS IN POLAR FORM.

Let $z = x + iy$ is a non zero complex no. $(z \neq 0)$.
The polar form of z is given by

$$z = re^{i\theta} \longrightarrow ①$$

, where, r is the modulus of z, $|z|$
distance of z from origin
θ is the angle made by radius vector with the +ve x axis. & is given by

$$\tan\theta = \left(\frac{y}{x}\right) = \left(\frac{Im(z)}{Re(z)}\right)$$

Here, $\boxed{r = |z| > 0}$

- The set of all values of θ is called argument of z & is denoted by $\arg(z)$.

* The principal arg(z) denoted by Arg (z) (or Θ) is the $\boxed{\text{unique}}$ value of θ which lies in the interval $(-\pi, \pi]$

i.e., $\boxed{-\pi < Arg(z) \leq \pi}$

$\therefore$, we have, $\theta = \arg z$
$= Arg(z) + 2n\pi;\ n \in \mathbb{Z}$

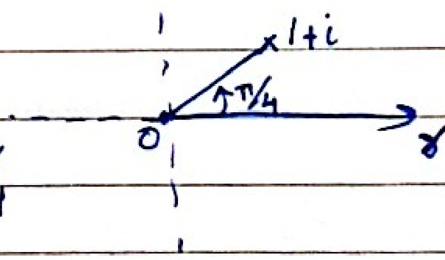
$\therefore$ eqn ① can be written as,

$$z = re^{i(Arg(z)+2n\pi)};\ n = 0, \pm 1, \pm 2, \ldots$$

Q Find the polar form of
① $z = 1+i$
② $z = -1-i$

[Note: The principal arg of a +ve real no. is zero.
The principal arg. of a −ve real no. is π]

① $z = 1+i$

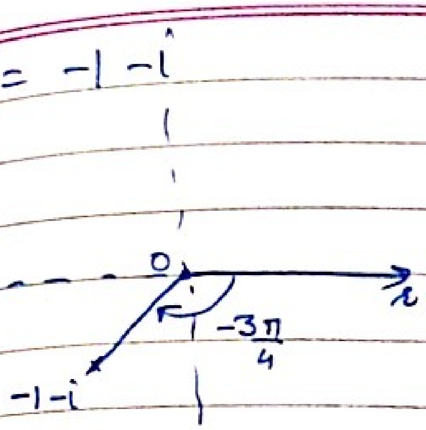


$Arg(z) = \pi/4$

2) $\theta = \arg(z) = Arg(z) + 2n\pi$
$\therefore \theta = \pi/4 + 2n\pi;\ n \in \mathbb{Z}$

$\therefore$ Polar form of $z = re^{i\theta}$
$= \sqrt{2}\, e^{i(\pi/4 + 2n\pi)};\ n \in \mathbb{Z}$

(2) $z = -1 - i$



$$\text{Arg}(z) = -\frac{3\pi}{4}$$

$$\theta = \arg(z) = -\frac{3\pi}{4} + 2n\pi \,;\, n \in \mathbb{Z}$$

So, Polar form of $z = \sqrt{2}\, e^{i\left(-\frac{3\pi}{4} + 2n\pi\right)}$; $n \in \mathbb{Z}$

★ If $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$ are complex nos., then,

(i) $z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right)$
$= r_1 r_2\, e^{i(\theta_1 + \theta_2)}$

(ii) $\dfrac{z_1}{z_2} = \left(\dfrac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$

★ 2 complex nos. z_1 & z_2, defined as above are equal iff.

$$r_1 = r_2$$
$$\& \;\; \theta_1 = \theta_2 + 2n\pi, \; n \in \mathbb{Z}.$$

RESULT:

★ For 2 non zero complex nos. z_1 & z_2, we have

(1) $\arg(z_1 . z_2) = \arg(z_1) + \arg(z_2)$

(2) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Note:- The above results cannot be replaced by the principal arguments. i.e., in general, $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$

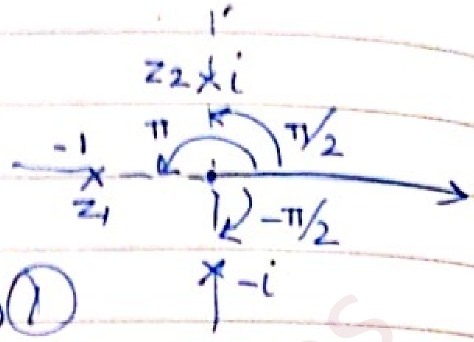
& Illy, $\text{Arg}\left(\dfrac{z_1}{z_2}\right) \neq \text{Arg}(z_1) - \text{Arg}(z_2)$

Q. Check the previous note with the ex.

$z_1 = -1$ & $z_2 = i$

$Arg(z_1) = \pi$

$Arg(z_2) = \pi/2$



$Arg(z_1) + Arg(z_2) = \frac{3\pi}{2} \rightarrow (1)$

$z_1 z_2 = -i$, $Arg(z_1 z_2) = -\frac{\pi}{2} \rightarrow (2)$

Clearly, $(1) \neq (2)$

So, $Arg(z_1 z_2) \neq Arg(z_1) + Arg(z_2)$.

Note :- In the above ex,

$arg(z_1) = Arg(z_1) + 2n\pi = \pi + 2n\pi$, $n \in \mathbb{Z}$

$arg(z_2) = Arg(z_2) + 2n\pi = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$.

So, $arg(z_1) + arg(z_2) = \frac{3\pi}{2} + 2n\pi$, $n \in \mathbb{Z} \rightarrow (1)$

$arg(z_1 \cdot z_2) = Arg(z_1 z_2) + 2n\pi \rightarrow (2)$

$= -\frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$.

For (1), when $n = 0$ & for (2) when $n = 1$,

$arg(z_1 z_2) = arg(z_1) + arg(z_2)$.

Section

✡ N^{th} ROOTS OF A COMPLEX NO.

To find the n^{th} roots of a complex no. z_0

~~$z_0 = r_0 e^{i(\theta_0 + 2k\pi)}, k \in \mathbb{Z}$~~,

we proceed as follows :-

S1) Find the general polar form of z_0 as,

$$z_0 = r_0 e^{i(\theta_0 + 2k\pi)}, k \in \mathbb{Z}.$$

, where $r_0 = |z_0|$ & $\theta_0 = Arg(z_0)$

S2) The n^{th} roots of z_0 are solns of eqn.

$$z^n = z_0.$$

$\therefore$ The n^{th} roots are given by

$$\therefore C_k = (z_0)^{1/n}$$

$$C_k = (r_0)^{1/n} e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)} \; ; \; \boxed{k = 0, 1, 2, \dots, (n-1)} \quad \text{(I)}$$

Note :- (1) The eqn (I) gives all the DISTINCT n^{th} roots of z_0.

(2) All these roots are equally spaced & placed on

$$|z| = (r_0)^{1/n} = \sqrt[n]{|z_0|}$$

with the spacing $\left(\frac{2\pi}{n}\right)^c$

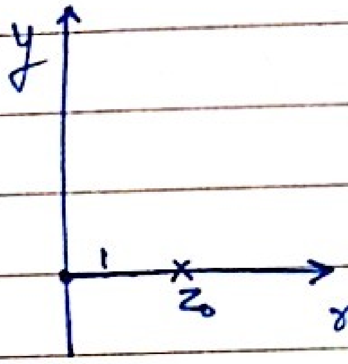
(3) All the remaining roots will be repetitions of these n roots.

Find the cube roots of unity.

Let $z_0 = 1$.

$r_0 = |z_0| = \sqrt{1^2} = 1$, $\theta_0 = \operatorname{Arg}(z_0) = 0$.

$\therefore \theta = \arg(z_0) = \operatorname{Arg}(z_0) + 2k\pi$
$= 0 + 2k\pi$
$= 2k\pi, k \in \mathbb{Z}$



$$z_0 = 1 = r_0 e^{i(\theta_0 + 2k\pi)}$$
$$= 1 e^{i(2k\pi)}$$
$$\Rightarrow z_0 = e^{i(2k\pi)}; \; k \in \mathbb{Z}.$$

$\therefore$ The cube roots of z_0 are given by

$$c_k = (z_0)^{1/3}$$
$$= \left(e^{i(2k\pi)}\right)^{1/3}$$
$$= e^{i\frac{2k\pi}{3}}; \; k = 0, 1, 2.$$

When $k = 0$

$\Rightarrow c_0 = e^0 = 1$.

When $k = 1$

$$c_1 = e^{i2\pi/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$\Rightarrow c_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega$$

When $k = 2$.

$$c_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2.$$

Find cube roots of $-8i$

$z_0 = -8i$

$|z_0| = r_0 = 8$.

$\theta_0 = \operatorname{Arg}(z_0) = -\frac{\pi}{2}$.

$\Rightarrow z_0 = -8i = r_0 e^{i(\theta_0 + 2k\pi)}$

$\Rightarrow z_0 = 8 e^{i(-\frac{\pi}{2} + 2k\pi)}$; $k \in \mathbb{Z}$

Cube roots of z_0 are given by,

$$c_k = (z_0)^{1/3}.$$

$$= 8^{1/3} e^{i(-\frac{\pi}{2} + 2k\pi)^{1/3}}$$

$$c_k = 2e^{i(-\frac{\pi}{6} + \frac{2k\pi}{3})}.$$

When $k = 0$:

$\Rightarrow c_0 = 2e^{i(-\frac{\pi}{6})} = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{3} - i$

$k = 1$:

$c_1 = 2e^{i(-\frac{\pi}{6} + \frac{2\pi}{3})} = 2(+i) = 2i$

$k = 2$: $c_2 = 2e^{i(-\frac{\pi}{6} + \frac{4\pi}{3})} = 2\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -\sqrt{3} - i$

<u>HW</u> Find the sq. root of $\sqrt{3} + i$

$2, \frac{\pi}{6}$ $\quad \sqrt{2}\left[\frac{\sqrt{3}}{2} + \frac{i}{2}\right]$, $\sqrt{2}\left[-\frac{\sqrt{3}}{2} + \frac{i}{2}\right]$

Section

REGIONS IN THE COMPLEX PLANE

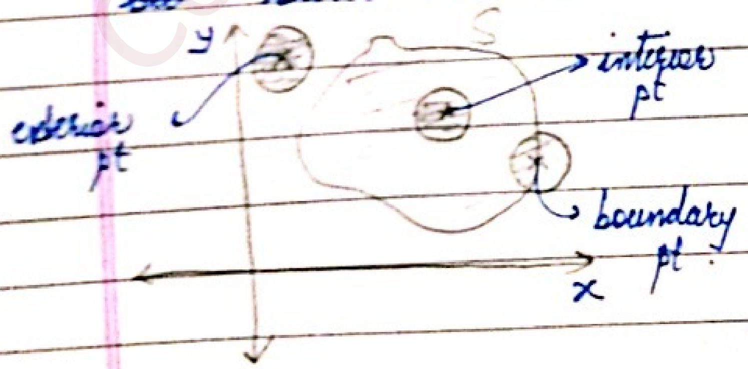
- An ϵ-neighbourhood of a pt. z_0 in the complex plane is a circular disk, centered at z_0 with radius ϵ.

  In the above ϵ-neighbourhood, the pt. z_0 is deleted, then, neighbourhood is referred to as a DELETED neighbourhood of z_0.

- A pt. z_0 is said to be an interior pt. of a set S, in the complex plane, if $\exists$ a neighbourhood of z_0 which completely lies within S.

  It is said to be an exterior pt. of S, if $\exists$ a neighbourhood of z_0 which lies completely outside S.

  It is ~~sed~~ said to be a boundary pt. if every neighbourhood of z_0 contains pts. of S & pts. ~~ou~~ outside S.



- A set S is said to be OPEN if it contains ALL the interior pts.

  It is said to be CLOSED if it contains ALL the boundary pts.

* Deleted neighbourhood of a pt. z_0 : Refers to the ϵ neighbourhood (of $|z-z_0|=\epsilon$) s.t pt. z_0 is deleted. If this deleted neighbourhood contains atleast 1 pt. of S, then z_0 (deleted pt) is called ACCUMUL^n pt.

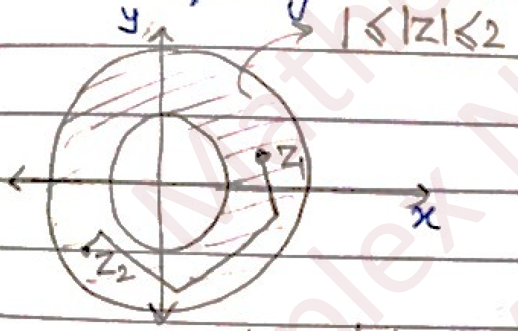
ex. ① The sets $|z|<1$ & $1<|z|<2$ } : open

② The sets $|z|\leq 1$ & $1\leq|z|\leq 2$ } : Closed

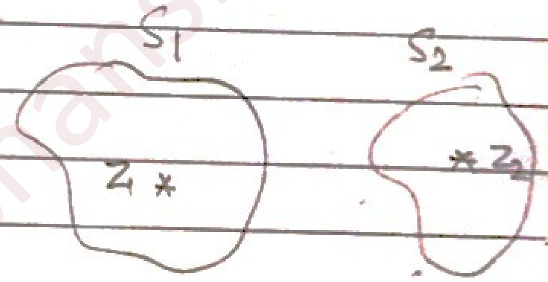
③ The sets $\{|z|\leq 1\} \sim \{0, 1\}$, $1\leq|z|<2$ } : Neither open nor closed.

④ The z plane : Both open & closed.

* A set S is said to be CONNECTED if a POLYGONAL line connecting ANY two pts. z_1 & z_2 in S lies completely within S.



Connected



$S = S_1 \cup S_2$
not connected.

* An OPEN CONNECTED set is called a DOMAIN.
eg: The sets $|z|<1$ & $1<|z|<2$ are domains.

* A domain with some or none or all its boundary pts is called a REGION.
eg: All the above examples are regions.

* ACCUMULATION POINT: A pt. z_0 is said to be an accumul^n pt. of a set S if $\exists$ a EVERY DELETED neighbourhood of z_0 which contains atleast 1 pt. of S.

Note: All the interior & boundary pts. are accumul^n pts.

$\therefore$ If a set contains all its accumul^n pts., then, its a <u>closed set</u>.

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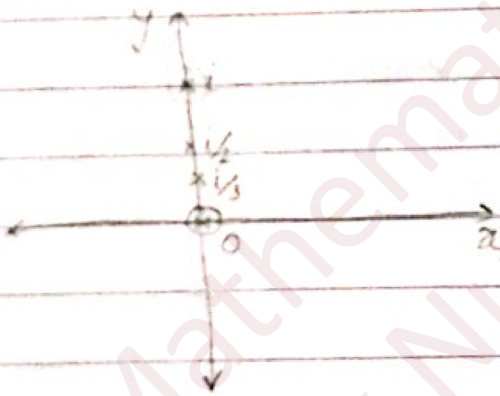
eg: Let $S = \{2+i, 3-i, 2i\}$. Then, S contains no accumul^n pt.

eg②: Let $S = \{\frac{i}{n} \mid n \in Z^+\}$.



$z = 0$ is the accumul^n pt. of S & it doesn't belong to S.

—x—

Chapter - 2

FUNCTIONS OF A COMPLEX VARIABLE

Let S be the set of complex nos. A f^n 'f' on 'S' is a rule that assigns a complex no. 'w' for each complex no. z in S & we write
$$w = f(z)$$

* If, for each z $\exists$ only one 'w', then, w is said to be a single valued f^n of z.
* If $\exists$ 2 or more values of w for a given z, then, w is said to be a many valued f^n.

ex:- $w = z^2$, $\sin z$ are single valued f^{ns}.
eg (2):- $w = \sqrt{z}$ is a two valued f^n
eg (3):- $w = \ln z$ is a many valued f^n.

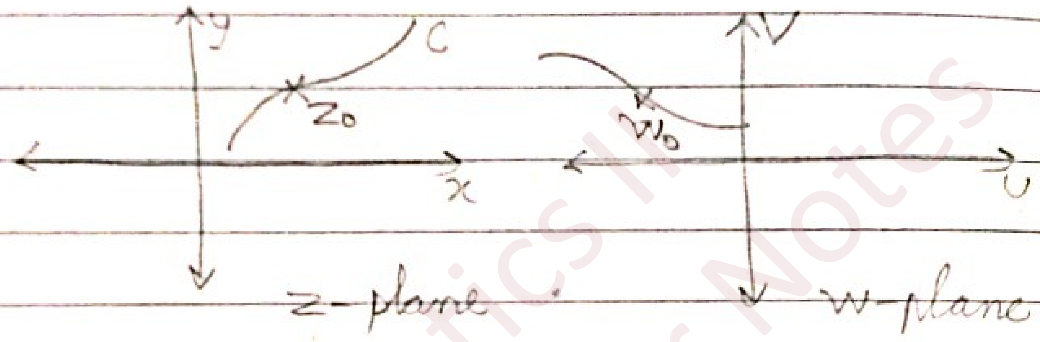
☆ We represent a complex valued f^n as follows:-
(i) Cartesian form: Let $z = x+iy$.
& $w = u+iv$.

Consider $w = f(z)$
$\Rightarrow u+iv = f(x+iy)$
$\Rightarrow u = U(x,y)$
& $v = V(x,y)$ are real valued f^{ns}.

Hence, we represent the complex valued f^n in 2 separate planes, namely:
(i) z-plane, with the x & y axis as real & imaginary axis resp.

(ii) w-plane, with u & v axis as the real & imaginary axis resp.

If z traces a curve C in z-plane, then, correspondingly w will trace another curve C' in w-plane.



(ii) Polar form : Let $z = re^{i\theta}$

& $w = u + iv$.

Consider $w = f(z)$

$\Rightarrow u + iv = f(re^{i\theta})$

$\Rightarrow u = u(r, \theta)$

$v = v(r, \theta)$.

Q Express the f^n $w = z^2$ using cartesian form & Polar

Cartesian	Polar
Let $z = x + iy$	Let $z = re^{i\theta}$
$w = u + iv$	$w = u + iv$
$w = z^2$	$\Rightarrow w = z^2$
$\Rightarrow u + iv = (x + iy)^2$	$\Rightarrow u + iv = (re^{i\theta})^2$
	$= r^2(\cos 2\theta + i\sin 2\theta)$
$\Rightarrow u = x^2 - y^2$	
& $v = 2xy$.	$\Rightarrow u = r^2\cos 2\theta$
	$v = r^2 \sin 2\theta$

Section 15

✱ LIMITS

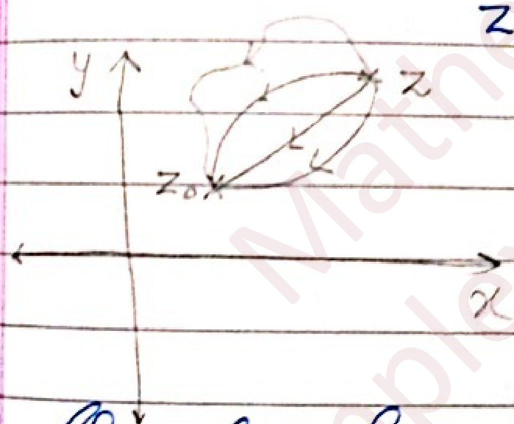
Let $w = f(z)$ be a complex valued f^n, defined at all pts. in a neighbourhood of z_0, possibly not at z_0.

We say that w has a limit 'L' when z approaches z_0 $(z \to z_0)$, if following cond^{ns} are satisfied :-

(i) $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ s.t,

$$0 < |z - z_0| < \delta$$
$$\Rightarrow |f(z) - L| < \epsilon.$$

& , we write, $\lim\limits_{z \to z_0} f(z) = L.$



Here, z approaches z_0 in an infinite no. of ways.

ex Show that $\lim\limits_{z \to 1} f(z) = \frac{i}{2}$; if $f(z) = \frac{iz}{2}$.

Consider $|f(z) - L| < \epsilon$ for given $\epsilon > 0$; $z_0 = 1$

$L = \frac{i}{2}$

$$\Rightarrow \left|\frac{iz}{2} - \frac{i}{2}\right| < \epsilon$$

$$\Rightarrow \left|\frac{i}{2}(z-1)\right| < \epsilon.$$

$$\Rightarrow \left|\frac{i}{2}\right| |(z-1)| < \epsilon$$

$$\Rightarrow \frac{1}{2} |(z-1)| < \epsilon \quad \text{or} \quad |z-1| < 2\epsilon.$$

$\therefore$ Choosing $\delta = 2\epsilon$, we have

$$0 < |z-1| < \delta \Rightarrow \left| f(z) - \frac{i}{2} \right| < \epsilon.$$

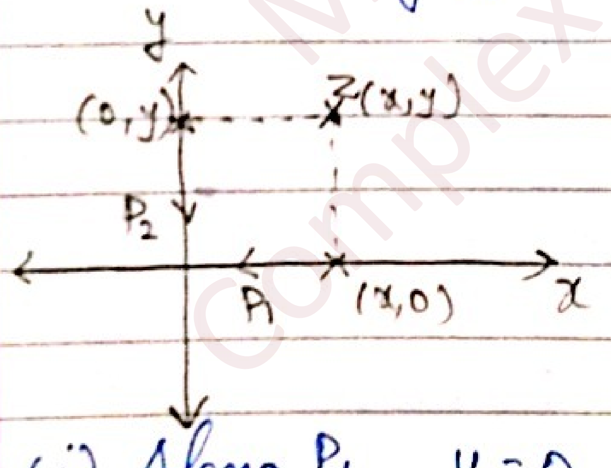
$$\Rightarrow \lim_{z \to 1} f(z) = \frac{i}{2}.$$

Show that $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

(Note: To show that a lim doesn't exist, we find the limits along 2 diff^t paths & show that they are not the same.)

Let $z = x+iy$, $\Rightarrow \bar{z} = x - iy$

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x+iy}{x-iy} \right) \longrightarrow ①$$



We shall find the lim ① along the paths P_1 & P_2.

(i) Along P_1 $\quad y = 0$ & $x \to 0$

$\Rightarrow$ ① becomes

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{\substack{x \to 0 \\ y = 0}} \frac{x+iy}{x-iy} = \lim_{x \to 0} \left(\frac{x}{x} \right)$$

$$= \lim_{x \to 0} 1 = 1.$$

$\hookrightarrow$ (A)

(ii) Along P_2 :- $x=0$; $y \to 0$

So, $\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{\substack{y\to 0\\ x=0}} \frac{x+iy}{x-iy} = \lim_{y\to 0}\left(\frac{+iy}{-iy}\right)$

$= \lim_{y\to 0} (-1) = -1$ → (B)

∴ From eq^{ns} (A) & (B),
the limit (1) doesn't exist.

✡ CONTINUITY of a COMPLEX VALUED FUNCTION

* Let $w = f(z)$ be a complex valued f^n, defined at all pts. in some neighbourhood of z_0. Then, f is said to be cts at z_0 if the following cond^{ns} are satisfied:-
  (i) $f(z_0)$ exists.
  (ii) $\lim_{z\to z_0} f(z)$ exists
  (iii) $\lim_{z\to z_0} f(z) = f(z_0)$, in whatever manner $z \to z_0$

Note: All polynomial, exponential & circular f^{ns} are cts. in their DOMAIN OF DEFINITION.

Note:-
- The sum, product, difference of cts. f^{ns} are cts.
- $\frac{f(z)}{g(z)}$ is cts if $g(z) \neq 0$.
- Compos^n of cts f^{ns} is cts.

Note :- The following are TRUE in case of limits:

1. If
$$\lim_{z \to z_0} f(z_0) = w_0 = u_0 + i v_0$$
& $f(z) = w$
$= u + iv.$

$$\Longleftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$$
$$\& \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

2. If $\lim_{z\to z_0} f(z) = w_0$ & $\lim_{z\to z_0} g(z) = L_0$

then (i) $\lim_{z\to z_0} [f(z) \pm g(z)] = w_0 \pm L_0$

(ii) $\lim_{z\to z_0} [f(z) \cdot g(z)] = w_0 L_0$

(iii) $\lim_{z\to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{w_0}{L_0}$; if $L_0 \neq 0$

(iv) $\lim_{z\to z_0} c = c$; c : complex constt.

(v) $\lim_{z\to z_0} (z)^n = z_0^n$.

(vi) $\lim_{z\to z_0} P(z) = P(z_0)$; P : Polynomial.

* DERIVATIVE of a COMPLEX VALUED FUNCTION

Let $w = f(z)$ be a complex valued f^n. Then, its derivative is denoted & defined by,

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z},$$

($f(z+\Delta z) - f(z)$ → increment in W)

provided, the lim. exists in WHATEVER MANNER $\Delta z \to 0$.

$$= \lim_{\Delta z \to 0} \left(\frac{\Delta W}{\Delta z}\right).$$

<u>ex</u> :- Show $f(z) = z^2$ is dfb at all pts. in the complex plane. Hence, find its derivative.

Let $w = f(z) = z^2$

Then,

$$f(z+\Delta z) = (z+\Delta z)^2 = z^2 + (\Delta z)^2 + 2z(\Delta z)$$

$$\therefore \Delta w = f(z+\Delta z) - f(z) = (\Delta z)^2 + 2z(\Delta z)$$

$$\therefore \frac{\Delta w}{\Delta z} = (\Delta z) + 2z.$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\cancel{z^2} + (\Delta z)^2 + 2z\Delta z - \cancel{z^2}}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\Delta z(\Delta z + 2z)}{\Delta z} = 2z$$

By defin^n :- $\dfrac{dw}{dz} = \lim_{\Delta z \to 0} \dfrac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} (2z) + (\Delta z)$

$$= \lim_{\Delta z \to 0} (2z) + \lim_{\Delta z \to 0} (\Delta z)$$

$$= 2z + 0 = \underline{\underline{2z}}$$

, in whatever manner $\Delta z \to 0$

$\therefore f'(z)$ exists everywhere in the complex plane

& $f'(z) = 2z$

Q. Show that, $f(z) = |z|^2$ is dfb only at origin & nowhere else in the complex plane.

Given :

$f(z) = |z^2| = |z|^2$

$= z \cdot \bar{z}$

$f(z) = |z|^2$

$= |z^2|$

$f(z+\Delta z) = |(z+\Delta z)^2|$

$= |z^2 + (\Delta z)^2 + 2z(\Delta z)|$

$\leq |z|^2 + |\Delta z|^2 + 2|z||\Delta z|$

$f(z+\Delta z) = (z+\Delta z)(\overline{z+\Delta z})$

$= (z+\Delta z)(\bar{z} + \overline{\Delta z})$

$= z.\bar{z} + (\Delta z)(\bar{z}) + z(\overline{\Delta z}) + (\Delta z)(\overline{\Delta z})$

$f(z) = z^2$

$\Rightarrow f(z+\Delta z) - f(z) = (\Delta z)\bar{z} + z(\overline{\Delta z}) + \Delta z(\overline{\Delta z})$

$f(z+\Delta z) - f(z) \leq \frac{|\Delta z|^2 + 2|z||\Delta z|}{\Delta z}$

$\Rightarrow \frac{\Delta w}{\Delta z} = \bar{z} + z\left(\frac{\overline{\Delta z}}{\Delta z}\right) + (\overline{\Delta z}) \leq \frac{|\Delta z|[|\Delta z| + 2|z|]}{\Delta z}$

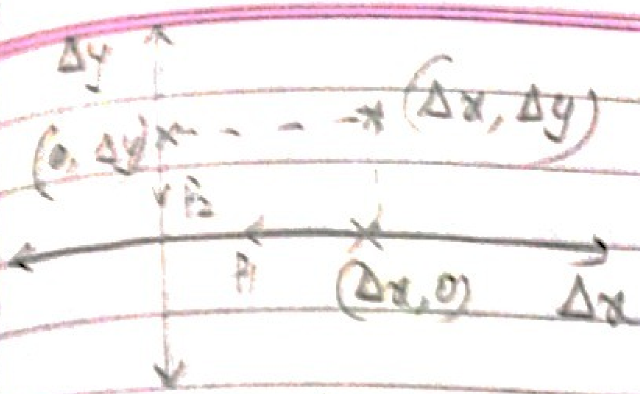
$\begin{cases} = [|\Delta z| + 2|z|] \\ = -[|\Delta z| + 2|z|] \end{cases}$

$\Rightarrow \frac{dw}{dz} = \lim_{\Delta z \to 0}\left(\frac{\Delta w}{\Delta z}\right)$

$= \lim_{\Delta z \to 0}\left[\bar{z} + \left(\frac{\overline{\Delta z}}{\Delta z}\right)z + (\overline{\Delta z})\right]$

$= \bar{z} + \lim_{\Delta z \to 0}\left[z\left(\frac{\overline{\Delta z}}{\Delta z}\right) + (\overline{\Delta z})\right] \longrightarrow$ I

$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}\left[\bar{z} + \left(\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}\right)z + (\Delta x - i\Delta y)\right]$



We shall find limits along P_1 & P_2.

Along P_1 $\quad \Delta y = 0$ & $\Delta x \to 0$

$$\Rightarrow \frac{dw}{dz} = \lim_{\Delta x \to 0}\left(\bar{z} + \left(\frac{\Delta x}{\Delta x}\right) z + \Delta x\right)$$

$$= \lim_{\Delta x \to 0} \bar{z} + z + 0$$

$$= \bar{z} + z \longrightarrow (A)$$

Along P_2 $\quad \Delta x = 0$; $\Delta y \to 0$

$$\Rightarrow \frac{dw}{dz} = \lim_{\Delta y \to 0}\left[\bar{z} + \left(\frac{-i\Delta y}{i\Delta y}\right) z - i\Delta y\right]$$

$$= \lim_{\Delta y \to 0} \bar{z} - z \longrightarrow (B)$$

Clearly, (A) $\neq$ (B),

$\therefore$ If limits exists, then, from (A) & (B), we have

$$\bar{z} + z = \bar{z} - z$$

$$\Rightarrow 2z = 0 \Rightarrow \boxed{z = 0}$$

$\therefore$ f is diff only at origin. & is zero at origin.

* STANDARD RESULTS

If f & g are complex valued f^{ns} having derivative in some region in the complex plane, then,

① $\frac{d}{dz}(f(z) \pm g(z)) = \frac{df}{dz} \pm \frac{d}{dz}g$

② $\frac{d}{dz}(f(z).g(z)) = f(z)\frac{d}{dz}g + g(z).\frac{d}{dz}f$

③ $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g\frac{df}{dz} - f(z)\frac{d}{dz}g}{[g(z)]^2}$; $g(z) \neq 0$

④ $\frac{d}{dz}[f(g(z))] = f'(g(z))\, g'(z)$

⑤ $\frac{dw}{dt} = \frac{dw}{dz} \cdot \frac{dz}{dt}$

⑥ $\frac{d}{dz}(c) = 0$; c: constt.

⑦ $\frac{d}{dz}(z^n) = nz^{n-1}$.

⑧ $\frac{d}{dz}(P(z)) = P'(z)$; $P(z)$: Polynomial in z.

~~Q. If $P(z)$ is a polynomial~~

Q. Show that $f'(z)$ doesn't exist, if

① $f(z) = \bar{z}$

② $f(z) = Re(z)$

③ $f(z) = Im(z)$, at any pt. in complex plane

~~③ $f(z)$~~

① $f(z) = \bar{z}$

Let $z = x + iy \Rightarrow \bar{z} = x - iy$.

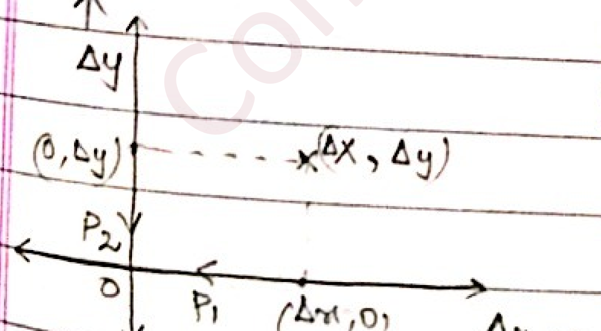
Given $f(z) = \bar{z}$

$$\Rightarrow f(z + \Delta z) = \overline{z + \Delta z}$$
$$= \bar{z} + \overline{\Delta z}$$
$$= \cancel{x - iy + \Delta x - i(\Delta y)}$$
$$= (x + \Delta x) - i(y + \Delta y).$$

$$\Delta w = f(z + \Delta z) - f(z)$$
$$= (\bar{z} + \overline{\Delta z}) - \bar{z}$$
$$= \overline{\Delta z}$$

$$\therefore \frac{\Delta w}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$f'(z) = \lim_{\Delta z \to 0} \left(\frac{\Delta w}{\Delta z}\right)$$
$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}\right) \longrightarrow ①.$$



We shall find limit ① along 2 diff't paths, namely P1 & P2 as shown.

(i) Along P_1,

$\Delta y = 0,\ \Delta x \to 0 \Rightarrow$ ① becomes

$$\Rightarrow \frac{dw}{dz} = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \left(\frac{\Delta x}{\Delta x}\right) = \lim_{\Delta x \to 0} (1) = 1. \longrightarrow Ⓐ$$

(ii) Along P_2
$\Delta y \to 0,\ \Delta x = 0$

$\Rightarrow$ (1) becomes

$$\frac{dw}{dz} = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \left(\frac{0 - i\Delta y}{0 + i\Delta y} \right) = \lim_{\Delta y \to 0} (-1) = (-1) \longrightarrow (B)$$

From the eq^n (A) & (B), limits are not the same along 2 diff^t paths. Hence, f' $\nexists$ anywhere in the complex plane.

(2) Let $z = x + iy$

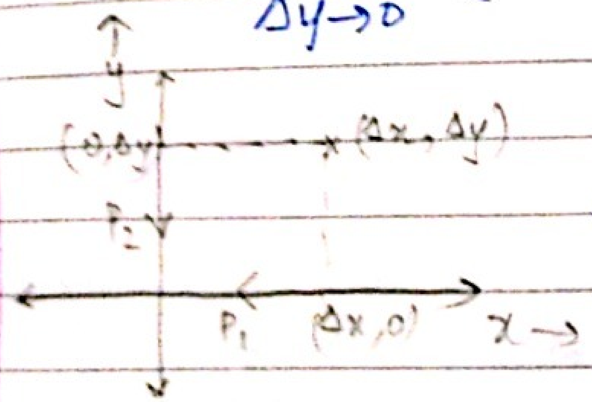
Given :- $f(z) = x$

$$\Rightarrow f(z + \Delta z) = x + \Delta x$$

$$\Delta w = f(z + \Delta z) - f(z)$$
$$= (x + \Delta x) - (x)$$
$$= \Delta x.$$

$$\Rightarrow \frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta x + i\Delta y}.$$

$$f'(z) = \lim_{\Delta z \to 0} \left(\frac{\Delta w}{\Delta z} \right)$$
$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\Delta x}{\Delta x + i\Delta y} \right) \longrightarrow (1).$$



We would find limit (1) along 2 diff^t paths, namely, P_1 & P_2.

(i) Along P_1, $\Delta y = 0$, $\Delta x \to 0$

$$\Rightarrow \frac{dw}{dz} = \lim_{\substack{\Delta y = 0 \\ \Delta x \to 0}} \left(\frac{\Delta x}{\Delta x}\right) = \lim_{\Delta x \to 0} (1) = \underline{1} \longrightarrow (A)$$

(ii) Along P_2, $\Delta x = 0$, $\Delta y \to 0$

$$\frac{dw}{dz} = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \left(\frac{0}{i\Delta y}\right) = 0 \longrightarrow (B)$$

From eqⁿ (A) & (B), limits are not the same along 2 diffᵗ paths. Hence, $f' \nexists$ anywhere in the complex plane.

(3) Let $z = x + iy$

$\Rightarrow f(z) = y$

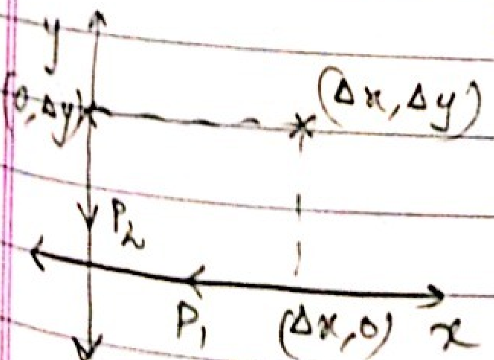
So, $f(z + \Delta z) = y + \Delta y$

$$\Delta w = f(z + \Delta z) - f(z)$$
$$= y + \Delta y - y = \Delta y.$$

$$\frac{\Delta w}{\Delta z} = \frac{\Delta y}{\Delta x + i\Delta y}$$

So, $$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \left(\frac{\Delta w}{\Delta z}\right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\Delta y}{\Delta x + i\Delta y}\right) \longrightarrow (I).$$



We now find limit (I) along 2 diffᵗ paths (P_1 & P_2) as shown.

(i) Along P_1, $\Delta y = 0$, $\Delta x \to 0$

$$\frac{dw}{dz} = \lim_{\substack{\Delta y = 0 \\ \Delta x \to 0}} \left(\frac{0}{\Delta x}\right) = 0 \longrightarrow (A)$$

(ii) Along P_2, $\Delta x = 0$, $\Delta y \to 0$

$$\frac{dw}{dz} = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \left(\frac{\Delta y}{i\Delta y}\right) = \lim_{\Delta y \to 0}\left(\frac{1}{i}\right)$$

$$= \lim_{\Delta y \to 0} (i^3) = (-i) \longrightarrow (B)$$

From eqn (A) & (B), since limits are not equal, so, f' $\nexists$ anywhere in complex plane.

Imp Q. Let f denote the f^n whose values are

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & ; \; z \neq 0 \\ 0 & ; \; z = 0 \end{cases}$$

(a) Show that, if $z = 0$, then, $\frac{\Delta w}{\Delta z} = 1$ at each non zero pt. on the real & imaginary axis in the Δz-plane. (Δx-Δy plane).

(b) then, show that $\frac{\Delta w}{\Delta z} = -1$ at each non zero pt on the line $\Delta y = \Delta x$ in that plane. Conclude from the above observns that $f'(0)$ $\nexists$ (doesn't exist).

$$\frac{\Delta w}{\Delta z} = \frac{\dfrac{(\overline{z+\Delta z})^2}{z+\Delta z} - \dfrac{(\bar z)^2}{z}}{\Delta z}$$

$$= \frac{\dfrac{(\bar z)^2 + (\Delta \bar z)^2 + 2\bar z(\Delta \bar z)}{z+\Delta z} - \dfrac{\bar z^2}{z}}{\Delta z}$$

$$= \frac{\cancel{z(\bar z)^2} + z(\Delta \bar z)^2 + 2z\bar z(\Delta \bar z) - \cancel{z\bar z^2} - \Delta z\,\bar z^2}{\Delta z(z+\Delta z)}$$

$$= \frac{z(\Delta \bar z)^2 - \Delta z(\bar z)^2 + 2z\bar z(\Delta \bar z)}{\Delta z(z+\Delta z)}$$

$$= z(\Delta \bar z) - (\bar z)^2 + 2z\bar z / (z+\Delta z)$$

$$\Rightarrow \frac{\Delta w}{\Delta z} = \frac{2z\bar z - (\bar z)^2 + z(\Delta \bar z)}{(z+\Delta z)}$$

$=$

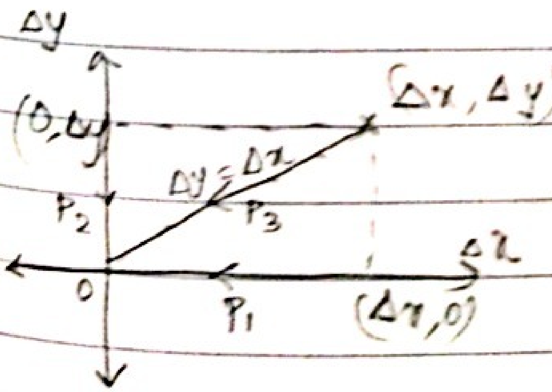
By defin^n, $\Delta w = f(z+\Delta z) - f(z)$

when $z=0$,

$$\Delta w = f(0+\Delta z) - f(0)$$
$$= f(\Delta z)$$

$$\therefore \Delta w = \frac{(\Delta \bar z)^2}{\Delta z}$$

$$\Rightarrow \frac{\Delta w}{\Delta z} = \frac{(\Delta \bar z)^2}{(\Delta z)^2} = \frac{(\Delta x - i\Delta y)^2}{(\Delta x + i\Delta y)^2}$$



Along P_1 $\Delta y = 0$

$$\therefore \frac{\Delta w}{\Delta z} = \frac{(\Delta x)^2}{(\Delta x)^2} = 1 \longrightarrow (A)$$

Along P_2, $\Delta x = 0$.

$\therefore$ (1) becomes.

$$\left(\frac{\Delta w}{\Delta z}\right) = \frac{(-i\Delta y)^2}{(i\Delta y)^2} = (-1)^2 = 1 \longrightarrow (B)$$

Along P_3, $\Delta y = \Delta x$.

$$\Rightarrow \left(\frac{\Delta w}{\Delta z}\right) = \frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)^2} = \left(\frac{1-i}{1+i}\right)^2$$

$$= \left(\frac{(1-i)(1-i)}{2}\right)^2$$

$$= (-i)^2 = -1 \longrightarrow (C)$$

By defin^n,

$$f'(0) = \lim_{\Delta z \to 0} \left(\frac{\Delta w}{\Delta z}\right) \text{ at } z = 0.$$

$$= \begin{cases} 1 & , P_1 \,\&\, P_2 \\ -1 & , \text{along } P_3 \end{cases}$$

$\therefore$ The limit takes diff^t values along diff^t paths. $\therefore f'(0) \nexists$.

Section - 21

§ CAUCHY-RIEMANN EQUATIONS
(CR equations in cartesian form).

Let
$$f(z) = u + iv \; (= u(x,y) + i(v(x,y)))$$
has a derivative
$f'(z)$ at every pt. in ~~&~~ some nbd of a pt. z_0. Then, the first order partial derivatives,
$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \& \frac{\partial v}{\partial y} \text{ exist and}$$
satisfy the eq^{ns}
$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \text{ at } z_0(x_0, y_0) \longrightarrow (1)$$

The eq^{ns} (1) are called the CR eq^{ns} in cartesian form & we also write them as:-

$$\boxed{u_x = v_y \;, \; u_y = -v_x} \; *$$

Imp:- Note:- (1) The CR equations are only necessary cond^{ns} for a f^n to have a derivative at a pt. They are NOT SUFFICIENT for a f^n to have a derivative.

★ **Sufficient Cond^{ns} for a f^n to have a derivative:**

Let $f(z) = u + iv$ be defined at all pts in some nbd of z_0. Then, $f'(z_0)$ exists iff the following cond^{ns} are true:

(i) u_x, u_y, v_x & v_y exist and are cts.

(ii) The CR eq^{ns} $u_x = v_y$ & $u_y = -v_x$ are satisfied at z_0.

Note:- We find $f'(z)$ in cartesian form as follows:

$$f'(z) = u_x + i v_x .$$

Q. Show that $f(z) = e^z$ is dfb at all pts in the complex plane & hence, find its derivative.

Let $z = x + iy$ & $f(z) = u + iv$

Given, $f(z) = e^z$

$$\Rightarrow u + iv = e^{x+iy}$$
$$= e^x \cdot e^{iy}$$
$$= e^x [\cos y + i \sin y]$$
$$\Rightarrow u = e^x \cos y$$
$$\& \ v = e^x \sin y .$$

$u_x = e^x \cos y$. $\quad v_x = e^x \sin y$

$u_y = -e^x \sin y$. $\quad v_y = e^x \cos y$.

As per CR eq^{ns},

- $u_x = v_y$ & $u_y = -v_x$ are true at all pts in the z-plane.
- Also, the partial derivatives: u_x, u_y, v_x & v_y are cts everywhere in z-plane.

$\therefore f'(z)$ exists at all pts in z plane.

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$$\therefore f'(z) = u_x + i v_x$$
$$= e^x \cos y + i(e^x \sin y)$$
$$= e^x(\cos y + i \sin y)$$
$$= e^x \cdot e^{iy}$$
$$= e^{(x+iy)}$$
$$\Rightarrow f'(z) = e^z.$$

☆ CR EQUATIONS IN POLAR FORM

Let $f(z) = u + iv$ be defined at all pts in some nbd of a non zero pt. $z_0 = r_0 e^{i\theta_0}$.

Then, $f'(z_0)$ exists iff, the following cond^{ns} are satisfied:-

(i) u_r, u_θ, v_r & v_θ exist and are cts at z_0.

& (ii) $\boxed{u_r = \frac{v_\theta}{r}, \quad v_r = -\frac{u_\theta}{r}}$ are satisfied at z_0. $\rightarrow$ (1)

Note (1): Eq^{ns} (1) are referred to as CR eq^{ns} in POLAR FORM.

Note (2): In polar form,
$$f'(z) = e^{-i\theta}(u_r + i v_r).$$

Q. Show that $f'(z) \not\exists$ at any pt. if

(1) $f(z) = \bar{z}$

(2) $f(z) = z - \bar{z}$

(3) $f(z) = 2x + ixy^2$

(4) $f(z) = e^{x-iy}$

(3) Let $z = x+iy$ & $f(z) = u+iv$.
$$f(z) = 2x + ixy^2.$$
$$\Rightarrow u + iv = 2x + ixy^2.$$

$u = 2x$ & $v = xy^2$

$u_x = 2$ $\quad$ $v_x = y^2$

$u_y = 0$ $\quad$ $v_y = 2xy$.

Clearly, CR eq^ns not satisfied simultaneously $u_x \neq v_y$ & $u_y \neq -v_x$ at any pt. in Z plane.

$\therefore f'(z) \nexists$ anywhere in the complex plane.

(4) Let $z = x+iy$ & $f(z) = u+iv$.

$f(z) = e^{x-iy}$

$\Rightarrow u+iv = e^{x-iy}$

$= e^x \cdot e^{-iy}$

$= e^x(\cos y - i\sin y)$

$\Rightarrow u = e^x \cos y$ $\quad$ $v = -e^x \sin y$

$u_x = e^x \cos y$ $\quad$ $v_x = -e^x \sin y$

$u_y = -e^x \sin y$ $\quad$ $v_y = -e^x \cos y$.

Clearly, CR eq^ns $u_x = v_y$ & $u_y = -v_x$ are not satisfied at any pt. in Z plane.

$\therefore f'(z) \nexists$ anywhere in complex plane.

(1) Let $z = x+iy$ & $f(z) = u+iv$

$f(z) = \bar{z}$

$\Rightarrow u + iv = \bar{z} = x - iy$

$\Rightarrow u = x$, $\quad$ $v = -y$

$u_x = 1$ $\quad$ $v_x = 0$

$u_y = 0$ $\quad$ $v_y = -1$

Clearly, $u_x = v_y$ & $u_y = -v_x$ are not satisfied simultaneously $\forall$ pts in Z plane.

$\therefore f'(z) \nexists$ anywhere in z plane.

② Let $z = x+iy$, $f(z) = u+iv$.

$f(z) = z - \bar{z}$

$\Rightarrow u + iv = (x+iy) - (x-iy)$

$= x - x + iy + iy$

$= 2iy$

$\Rightarrow u = 0$ $\qquad\qquad v = 2y$.

$\Rightarrow u_x = 0$ $\qquad\qquad v_x = 0$

$u_y = 0$ $\qquad\qquad v_y = 2$

Clearly, the CR eq^ns ($u_x = v_y$ & $u_y = -v_x$) are not satisfied simultaneously $\forall$ pts. in z plane.

So, $f'(z) \nexists$ anywhere in complex plane.

Q. Show that $f'(z)$ & $f''(z)$ exist everywhere in the z plane & find $f''(z)$, if

① $f(z) = e^{-x-iy}$

② $f(z) = z^3$

③ $f(z) = \cos x \cosh y - i \sin x \sinh y$.

② Let $z = x+iy$ & $f(z) = u+iv$

$f(z) = z^3$

$\Rightarrow u + iv = (x+iy)^3$

$= x^3 + (iy)^3 + 3x^2(iy) + 3x(iy)^2$

$= x^3 - iy^3 + 3ix^2y - 3xy^2$

$\Rightarrow u + iv = x^3 - 3xy^2 + i(3x^2y - y^3)$

$\Rightarrow u = x^3 - 3xy^2$ $\qquad\qquad v = 3x^2y - y^3$

$\Rightarrow u_x = 3x^2 - 3y^2$. $\qquad\qquad v_x = 6xy$.

$u_y = -6xy$ $\qquad\qquad v_y = 3x^2 - 3y^2$.

Clearly, the CR eq^{ns} are true ($U_x = V_y$ & $U_y = -V_x$)
~~everywhere in z plane.~~
~~So, $f'(z)$ exists everywhere in complex plane.~~
Also, these partial derivatives are cts everywhere in z plane.
So, $f'(z)$ ~~exists~~ exists at all pts. in the z plane & is given by

$$f'(z) = U_x + iV_x$$

$$= (3x^2 - 3y^2) + i(6xy)$$

$$= 3(x^2 - y^2 + i2xy)$$

$$= 3(x+iy)^2$$

$$\Rightarrow f'(z) = 3z^2.$$

Let $f'(z) = p + iq = 3x^2 - 3y^2 + i(6xy)$

$\Rightarrow p = 3x^2 - 3y^2$, $q = 6xy$.

$\Rightarrow p_x = 6x$. $q_x = 6y$

$p_y = -6y$. $q_y = 6x$.

Clearly, $p_x = q_y$ & $p_y = -q_x$.
So, CR eq^{ns} are true.
Also, the partial derivatives are cts everywhere in z plane.
So, $f''(z)$ exists everywhere in z plane.

$$f''(z) = p_x + iq_x$$

$$= 6x + i6y$$

$$= 6(x+iy)$$

$$\Rightarrow f''(z) = 6z.$$

Ans

② The Hyperbolic sine f^n of x, denoted and defined by

$$\sinh x = \sinh(x) = \frac{e^x - e^{-x}}{2}$$

Ily, we define $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Relationship b/w hyperbolic & circular f^{ns}

$\sin(ix) = i\sinh x$
$\cos(ix) = \cosh x$
$\tan(ix) = i\tanh x$.

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$.

$\dfrac{d}{dx}(\sinh x) = \cosh x$ $\quad$ $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$.

$\dfrac{d}{dx}(\cosh x) = \sinh x$

③ $f(z) = \cos x \cosh y - i \sin x \sinh y$.
$= \cos x \cos(iy) - \sin x \sin(iy)$
$= \cos(x+iy)$
$= \cos z$.

let $z = x+iy$ & $f(z) = u+iv$
Given, $f(z) = \cos z$.
$\Rightarrow u = \cos x \cosh y$, $v = -\sin x \sinh y$
$u_x = (\cosh y)(-\sin x)$ $\quad$ $v_x = -\cos x \sinh y$

$u_y = (\sinh y)\cos x$ $\quad$ $v_y = (\cosh y)(-\sin x)$

Clearly, $U_x = V_y$ & $-V_x = U_y$.

So, the CR eq^{ns} are true everywhere in the complex plane & also, the derivatives U_x, U_y, V_x & V_y are cts everywhere in z plane.

$\therefore$ $f'(z)$ exists at all pts. in z-plane & is given by

$$\begin{aligned}
f'(z) &= \cancel{V_x}\, U_x + iV_x \\
&= -\sin x(\cosh y) - i\cos x(\sinh y) \\
&= -[\sin x(\cos iy) + \cos x \sin(iy)] \\
&= -[\sin(x+iy)] \\
&= -\sin z.
\end{aligned}$$

$\Rightarrow$ $p + iq = -\sin x(\cosh y) - i\cos x(\sinh y)$

$p = -\sin x\cosh y$ & $q = -\cos x\sinh y$.

$p_x = -\cos x\cosh y$. $\quad q_x = \sin x\sinh y$

$p_y = -\sin x\sinh y$. $\quad q_y = -\cos x\cosh y$.

Clearly, $p_x = q_y$ & $p_y = -q_x$

So, CR eq^{ns} are true at all pts in z plane.

& p_x, p_y, q_x & q_y are cts at all pts in z plane.

So, $f''(z)$ exists at all pts on the complex plane

$$\begin{aligned}
f''(z) &= p_x + iq_x \\
&= (-\cos x\cosh y) + i\sin x\sinh y \\
&= -[\cos x\cos(iy) + \sin x\sin(iy)] \\
&= -[\cos(x+iy)]
\end{aligned}$$

$\Rightarrow f''(z) = -\cos z$.

Q. Determine, where $f'(z)$ exists & find its derivative when

(i) $f(z) = x^2 + iy^2$

(ii) $f(z) = z\, Im(z)$.

(i) Let $z = x + iy$ & $f(z) = u + iv$

$f(z) = x^2 + iy^2$

$\Rightarrow u + iv = x^2 + iy^2$

$\Rightarrow u = x^2$ $\qquad$ $v = y^2$

$\Rightarrow u_x = 2x$ $\qquad$ $v_x = 0$

$u_y = 0$ $\qquad$ $v_y = 2y$

Clearly,

For CR eq^ns to be true

$u_x = v_y$ & $u_y = -v_x$

Now $\qquad \Rightarrow 0 = 0$ ✓ (for every pt. in z plane)

$2x = 2y$ should be true at every pt. in z plane

$\Rightarrow \boxed{x = y}$.

∴ CR eq^ns are true ~~iff~~ at all pts in z plane where $x = y$.

Also the partial derivatives are cts when $x = y$.

∴ f' exists at all pts on the line $y = x$ {or the pt. when $z = x(1+i)$ }

& is given by

$f'(z) = u_x + iv_x \Big|_{x=y}$

$= 2x + 0$.

$\Rightarrow f'(z) = 2x$.

(ii) $z = x+iy$ & $f(z) = u+iv$

$$f(z) = u+iv = z(\mathrm{Im}(z))$$
$$= (x+iy)y$$
$$= xy + iy^2$$

$\therefore u = xy, \quad v = y^2.$

$u_x = y \qquad v_x = 0$

$u_y = x \qquad v_y = 2y$

$\Rightarrow u_y = -v_x \Rightarrow x = 0$

$u_x = v_y \Rightarrow y = 2y \Rightarrow y = 0$

CR eq^{ns} are true only at origin & above partial derivatives are cts only at origin.

$\therefore f'$ exists only at origin & nowhere else in complex plane

$$f'(0) = u_x + iv_x \Big|_{z=0}$$
$$= y + i0 \Big|_{x=0,\, y=0}$$
$$= y \Big|_{x=0,\, y=0}$$
$$\Rightarrow f'(0) = 0,$$

Q. Show that when $f(z) = x^3 + i(1-y)^3$, it is legitimate to write

$$f'(z) = 3x^2, \text{ only if } z = i$$

Let $z = x+iy \qquad f(z) = u+iv.$

$\Rightarrow f(z) = x^3 + i(1-y)^3$

$\Rightarrow u = x^3, \quad v = (1-y)^3.$

$U_x = 3x^2$ $\qquad V_x = 0$

$U_y = 0$ $\qquad V_y = -3(1-y)^2$

For CR eq^{ns} to be true,

$U_x = V_y$ & $V_x = -V_y$

$V_x = -V_y = 0$ is true $\forall$ pts. in z plane.

Now, $U_x = V_y$

$\Rightarrow \cancel{3}x^2 = -\cancel{3}(1-y)^2$

$\Rightarrow x^2 = -(1-y^2)^2$

$x^2 + (1-y)^2 = 0 \longrightarrow (1)$

eq^n (1) is true iff

$x^2 = 0$ & $(1-y)^2 = 0$

$\Rightarrow x = 0$ & $1-y = 0$

$\Rightarrow x = 0, y = 1 \longrightarrow (2)$

So, at $(x=0, y=1)$, CR eq^{ns} would be true simultaneously,

So, $f'(z)\Big|_{x=0,y=1} = U_x + iV_x\Big|_{x=0,y=1}$

$= 3x^2 + i(0)\Big|_{x=0,y=1}$

$\Rightarrow f'(z) = 3x^2\Big|_{\text{only when } x=0,\, y=1}$

$\longrightarrow (3)$

Now, from (2), $z = 0 + 1i = i$.

Hence, from (3),

It is legitimate to write $f'(z) = 3x^2$ only when $x = 0$ & $y = 1$ i.e $z = 0 + 1i \Rightarrow z = i$.

Ans

Q. Show that each of the following fns is diffb in the indicated domain of definn & hence, find $f'(z)$.

① $f(z) = \frac{1}{z^4}$; $z \neq 0$

HW ② $f(z) = \sqrt{r}\, e^{i\theta/2}$; $r > 0, -\pi < \theta < \pi$

③ $f(z) = e^{-\theta}\cos(\ln r) + i e^{-\theta}\sin(\ln r)$; $r > 0, 0 < \theta < 2\pi$.

① We shall use polar coordinates and we let

$z = re^{i\theta}$, $f(z) = u + iv$

$$f(z) = \frac{1}{z^4}.$$

$$= \frac{1}{(re^{i\theta})^4}$$

$$= \frac{1}{r^4 e^{4i\theta}}$$

$$= r^{-4}(e^{-4i\theta})$$

$\Rightarrow u + iv = r^{-4}(\cos 4\theta - i\sin 4\theta)$

$\Rightarrow u = \frac{1}{r^4}\cos 4\theta$, $v = -\frac{1}{r^4}\sin 4\theta$.

$U_r = \frac{-4\cos 4\theta}{r^5}$ $\quad V_r = \frac{4\sin 4\theta}{r^5}$

$U_\theta = \frac{-4\sin 4\theta}{r^4}$ $\quad V_\theta = \frac{-4\cos 4\theta}{r^4}$

Clearly, $U_r = \frac{V_\theta}{r}$ & $V_r = -\frac{U_\theta}{r}$.

So, CR eqns are true ∀ pts. in z plane ($z \neq$

Also, the partial derivatives, u_r, u_θ, v_r & v_θ are cts when $z \neq 0$

$\therefore$ f has a derivative at all pts. in the complex plane except when $z=0$ & its derivative is given by:

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta}\left[\frac{-4}{r^5}\cos 4\theta + i\left(\frac{4}{r^5}\sin 4\theta\right)\right]$$

$$= e^{-i\theta}\left[\frac{-4}{r^5}\right]e^{-i4\theta}$$

$$= \frac{-4e^{-5i\theta}}{r^5}$$

$$= \frac{-4}{r^5 e^{5i\theta}}$$

$$= \frac{-4}{(re^{i\theta})^5}$$

$$\Rightarrow f'(z) = \frac{-4}{z^5}, \quad z \neq 0.$$

(2) $\cancel{\text{Let } z = re^{i\theta} \text{ \& } f(z) = u+iv}$

$\Rightarrow f(z) = \sqrt{r}\, e^{i\theta/2}$

$=$

(3) Let $z = re^{i\theta}$ & $f(z) = u + iv$.

Now, $f(z) = e^{-\theta}(\cos(\ln r) + i\sin(\ln r))$

$\Rightarrow u = e^{-\theta}\cos(\ln r)$, $v = e^{-\theta}\sin(\ln r)$

$u_r = \frac{-\sin(\ln r)}{r}e^{-\theta}$, $v_r = \frac{\cos(\ln r)}{r}e^{-\theta}$

$u_\theta = -e^{-\theta}\cos(\ln r)$, $v_\theta = -e^{-\theta}\sin(\ln r)$

Clearly $u_r = \frac{v_\theta}{r}$ and $v_r = -\frac{u_\theta}{r}$

So, the CR eq^ns are true ∀ pts. in z plane & u_r, v_r, u_θ & v_θ are cts $(r>0, \theta \in (0, 2\pi))$

∴ $f'(z)$ exists at all pts in z plane's domain

& $f'(z) = e^{-\theta}(u_r + i v_r)$

$$= e^{-\theta}\left(-\frac{\sin(\ln r)}{r}e^{-\theta} + i\frac{\cos(\ln r)}{r}e^{-\theta}\right)$$

$$= -\frac{e^{-2\theta}}{r}\left(\sin(\ln r) - i\cos(\ln r)\right)$$

$$f'(z) = \frac{ie^{-2\theta}}{r}\left(\cos(\ln r) + i\sin(\ln r)\right)$$

Q. Let u & v denote the real & imaginary parts of the f^n

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Verify that the CR eq^ns are satisfied at the origin, but $f'(0)$ ∄. → derived in previous sections

let $z = x+iy$, $f(z) = u+iv$

$\bar{z} = x - iy$

$$f(z) = \frac{(\bar{z})^2}{z}$$

$$= \frac{(x-iy)^2}{(x+iy)}\frac{(x-iy)}{(x-iy)}$$

$$= \frac{(x^2-y^2-2xyi)(x-iy)}{x^2+y^2}$$

$$\Rightarrow f(z) = \frac{(x^3 - xy^2 + 2xy^2) + i(-x^2y + y^3 - 2x^2y)}{x^2+y^2}$$

$$= \frac{(x^3 - 3xy^2) + i(y^3 - 3x^2y)}{x^2+y^2}$$

$$\Rightarrow f(z) = \frac{x^3 - 3xy^2}{x^2+y^2} + i\left(\frac{y^3 - 3x^2y}{x^2+y^2}\right)$$

$$\Rightarrow u(x,y) = \frac{x^3 - 3xy^2}{x^2+y^2}$$

$$v(x,y) = \frac{y^3 - 3x^2y}{x^2+y^2}$$

We shall use the following defin^ns to find the partial derivatives at a pt. (x_0, y_0).

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h}$$

, provided, the limit exists.

IIly,

$$\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{u(x_0, y_0+h) - u(x_0, y_0)}{h}$$

, provided, the limit exists.

~~Given,~~ $\cancel{f(z) = \frac{(\bar{z})^2}{z}}$;

Given, $f(0) = 0 \Rightarrow u(0,0) = 0$ & $v(0,0) = 0$.

$\therefore$, at the origin, we have.

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{h - 0}{h} = 1$$

$$U_y(0,0) = \lim_{h \to 0} \frac{U(0,h) - U(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$V_x(0,0) = \lim_{h \to 0} \frac{V(h,0) - V(0,0)}{h}$$

$$\Rightarrow V_{x(0,0)} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

$$V_{y(0,0)} = \lim_{h \to 0} \frac{V(0,h) - V(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = 1.$$

Clearly, $U_x = V_y$ & $U_y = -V_x = 0$. (At $(0,0)$)

So, the CR eq^{ns} are true ~~in~~ at the ~~z plane~~ origin.

(Find $f'(z)|_{(0,0)}$ ~~at~~ & show $\nexists$ (done before))

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Section

Analytic Functions

Let f be a f^n defined at all pts. in a neighbourhood of a pt. z_0. Then, f is said to be analytic at z_0, if, following cond^{ns} are satisfied.
(i) $f'(z_0)$ exists
& (ii) f' exists at all pts. in some nbd of z_0.

- f is said to be Analytic on an OPEN SET S, if, it has a ~~de~~ derivative at all pts. in S.
- f is Analytic on a CLOSED SET S, if, it is analytic on every open set contained in S.

* If a f^n is analytic at all pts. in the finite complex plane (z plane, without the pts at ∞), then, it is said to be an ENTIRE FUNCTION

* Note :
An analytic f^n is also referred to as,
REGULAR or HOLOMORPHIC FUNCTION.

ex. ① The f^n $f(z) = e^z + \sin(z)$ is analytic everywhere in z plane. Hence, its an entire f^n.
② The f^n $f(z) = 1/z$ is analytic everywhere in the complex plane except origin. Hence, its not an entire f^n.

☆ **Singular points:**

A pt. z_0 is said to be a singular pt. of a f^n f if the following cond^{ns} are satisfied:

(i) f is not analytic at z_0

(ii) It's analytic at some (atleast 1) pt. in EVERY nbd of z_0.

ex: The f^n $f(z) = z^2+5$ ~~is no~~ doesn't have a singular pt.

The f^n $f(z) = \frac{1}{z}$ has a singular pt $\rightarrow$ $\underline{\underline{z=0}}$

ex: The singular pts. of $f(z) = \dfrac{z}{(z^2+9)(z^2-4)}$

are given by denominator eq^n $(z^2+9)(z^2-4)=0$

$\Rightarrow z^2+9=0$ or $z^2-4=0$.

$\Rightarrow z = \pm 3i, \pm 2$ are singular pt.

ex: The f^n $f(z) = |z|^2$ is NOWHERE analytic in the z plane, though, it has a derivative at the origin. Also, it has no singular pt. in the z plane.

☆☆ To show that a f^n f is analytic at a pt., we use the following fact:

The f^n $f(z) = u+iv$ is analytic at z_0 if ~~if~~ the partial derivates u_x, u_y, v_x, v_y, ~~and~~ exists, are cts, & satisfy the CR eq^{ns}:

$u_x = v_y$ & $u_y = -v_x$.

The above idea can be extended to polar form too

Q. Show that, the following fns are entire fns.

(1) $f(z) = (3x+y) + i(3y-x)$
(2) $f(z) = \sin x \cosh y + i \cos x \sinh y$
(3) $f(z) = e^{-y}\sin x - i e^{-y}\cos x$
(4) $f(z) = (z^2-2)e^{-x}e^{-y}$

(4) Let $z = x+iy$

Then, $f(z) = \cancel{e^{-y}\sin x - ie^{-y}\cos x}$, $f(z) = u+iv$

$= (z^2-2)e^{-x}e^{-iy}$

$= (z^2-2)e^{-z}$

$= e^{-x}[((x+iy)^2-2)][e^{-iy}]$

$= e^{-x}[(x^2-y^2-2)+i(2xy)][e^{-iy}]$

$= e^{-x}[(x^2-y^2-2)+i(2xy)] \times [\cos y - i\sin y]$

$= e^{-x}\Big[[(x^2-y^2-2)\cos y + 2xy\sin y] + i\left(-(x^2-y^2-2)\sin y + 2xy\cos y\right)\Big]$

$\Rightarrow u = e^{-x}[(x^2-y^2-2)\cos y + 2xy\sin y]$

$v = e^{-x}[2xy\cos y - (x^2-y^2-2)\sin y]$

$u_x = -e^{-x}[(x^2-y^2-2)\cos y + 2xy\sin y] + e^{-x}[2x\cos y + 2y\sin y]$.

$u_y = e^{-x}[-2y\cos y - (x^2-y^2-2)\sin y + 2x\sin y + 2xy\cos y]$

$v_x = -e^{-x}[2xy\cos y - (x^2-y^2-2)\sin y] + e^{-x}[2y\cos y - (2x)\sin y]$.

$\Rightarrow v_x = e^{-x}[2y\cos y + (x^2-y^2-2)\sin y - 2x\sin y - 2xy\cos y]$

$V_y = e^{-x}[2x\cos y - 2xy\sin y + 2y\sin y - (x^2-y^2-2)\cos y]$

Here, $U_x = V_y$ & $U_y = -V_x$.

$\therefore$ The C.R eq^{ns} are satisfied at all pts. in the complex plane &, these partial derivatives are cts in that plane.

$\therefore$ f has a derivative throughout the finite complex plane. #

Hence, its analytic throughout the z plane.

$\therefore$ f is an entire f^n.

Write them pt. by pt.

(3) Let $z = x+iy$, $f(z) = U+iV$

Now, $f(z) = e^{-y}\sin x - ie^{-y}\cos x$

$= -ie^{-y}(\cos x + i\sin x)$

$\cancel{= -ie^{-y}}$

$U = e^{-y}\sin x$ $\qquad V = -e^{-y}\cos x$

$U_x = e^{-y}\cos x$ $\qquad V_x = e^{-y}\sin x$

$U_y = -e^{-y}\sin x$ $\qquad V_y = e^{-y}\cos x$

Here, $U_x = V_y$ & $V_y = -V_x$

$\therefore$ The C.R eq^{ns} are satisfied $\forall$ pts. in z plane & partial derivatives are cts. in that plane

$\therefore$ f has a derivative throughout z plane.

Hence, it's analytic throughout z plane

$\therefore$ f is an Entire f^n.

Q. Show that f is Nowhere analytic if

(1) $f(z) = 2xy + i(x^2-y^2)$

(2) $f(z) = e^y e^{ix}$

① Let $z = x+iy$

$$f(z) = 2xy + i(x^2-y^2), \quad f(z) = u+iv.$$

$\Rightarrow u = 2xy$, $\qquad v = x^2-y^2$.

$u_x = 2y \qquad v_x = 2x$

$u_y = 2x \qquad v_y = -2y$.

When $u_x = v_y \Rightarrow 2y = -2y \Rightarrow y=0$

||ly $u_y = -v_x \Rightarrow 2x = -2x \Rightarrow x=0$.

$\therefore$ The C.R. eq^{ns} are true only at the origin & the partial derivatives u_x, u_y, v_x & v_y are cts at origin. $\therefore$ f' exists only at ORIGIN.

$\therefore$ f is nowhere analytic in z plane.

② Let $z = x+iy \qquad f(z) = u+iv$.

$$f(z) = e^y \cdot e^{ix} = e^y(\cos x + i\sin x)$$

$\Rightarrow u = e^y \cos x \qquad v = e^y \sin x$

$u_x = -e^y \sin x \qquad v_x = e^y \cos x$

$u_y = e^y \cos x \qquad v_y = e^y \sin x$

When $u_x = v_y \Rightarrow -e^y \sin x = e^y \sin x \Rightarrow \cancel{x = 0 + 2n\pi, n \in \mathbb{Z}}$

$\Rightarrow e^y \sin x = 0 \longrightarrow ①$

$u_y = -v_x \Rightarrow e^y \cos x = -e^y \cos x \Rightarrow \cancel{\cos x = 0 + (2n+1)\frac{\pi}{2}}$, $n \in \mathbb{Z}$

$\Rightarrow e^y \cos x = 0 \longrightarrow ②$

The eq^{ns} ① & ② don't have a sol^n in the finite complex plane. $\therefore$ f' $\nexists$ anywhere in z plane.

$\therefore$ It's nowhere analytic.

Note: If f and g are analytic f^{ns} at all pts. in a domain D, then,

(i) $f(z) \pm g(z)$ is analytic in D.

(ii) $f(z).g(z)$ is analytic in D.

(iii) $c.f(z)$ is analytic in D, c: complex const.

(iv) $\dfrac{f(z)}{g(z)}$ is analytic at all pts. where $g(z) \neq 0$

(v) $(g \circ f)(z) = g(f(z))$ is also analytic if defined.

Q. Find the singular pts. & state why the f^n is analytic everywhere except these pts.

① $f(z) = \dfrac{2z+1}{z(z^2+1)}$ $\rightarrow 0, i, -i$

② $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$ $\rightarrow -2, -1 \pm i$

③ $f(z) = \cot z$ $\rightarrow 2n\pi, n \in Z$

① $f(z)$, being a rational fraction of polynomials, which are analytic throughout the complex plane, except at those pts, where denominator is zero. Hence, the singular pts. are given by

$$z(z^2+1) = 0$$
$$\Rightarrow z = 0 \text{ or } z^2+1 = 0$$
$$\Rightarrow z = 0 \text{ or } z = i, -i \text{ are singular pts.}$$

② Singular pts. are given by

$$(z+2)(z^2+2z+2) = 0$$
$$\Rightarrow z = -2 \text{ or } (z+1)^2 = 0$$
$$\Rightarrow z = -2 \text{ or } z = -1 \pm i \text{ are singular pts.}$$

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(3) The singular pts are given by
$\sin z = 0$
$\Rightarrow z = 2n\pi,\ n \in \mathbb{Z}$ are singular pts.

☆ Harmonic Functions

A real valued f^n $\phi = \phi(x,y)$, having its partial derivatives & satisfying the Laplace eq^n
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \left(\text{or } \phi_{xx} + \phi_{yy} = 0 \text{ or } \nabla^2 = 0 \right)$$
is called a harmonic f^n.

If u & v are real valued f^{ns} of x & y, which are harmonic & satisfying the C.R. eq^{ns}, then, 'v' is said to be a ~~ha~~ HARMONIC CONJUGATE of u.

* Result: If $f(z) = u + iv$ is an analytic f^n then, both u & v are harmonic f^{ns} & 'v' is a harmonic conjugate of u.
Here, u can't be a harmonic conjugate of v. But, if $f(z)$ is a constt, then, both u & v are harmonic conjugates of each other.

Q. Verify that the f^n
$g(z) = \ln r + i\theta \quad (r > 0,\ 0 < \theta < 2\pi)$ is analytic in the indicated domain, with the derivative $g'(z) = \frac{1}{z}$. Hence, show that the composite f^n

$g(z^2+1)$ is analytic in the quadrant $x>0, y>0$; with the derivative $\frac{2z}{z^2+1}$.

Let $z = re^{i\theta}$, $g(z) = u + iv$.

Given $g(z) = \ln r + i\theta$.

$\Rightarrow u + iv = \ln r + i\theta$.

$\Rightarrow u = \ln r \qquad v = \theta$.

$\Rightarrow u_r = \frac{1}{r} \qquad v_r = 0$

$u_\theta = 0 \qquad v_\theta = 1$.

Clearly, $u_r = \frac{v_\theta}{r}$ & $v_r = -\frac{u_\theta}{r}$

So, C.R eq^ns are true $\forall$ z in the given domain & also, these partial derivatives are cts.

$\therefore$ $g'(z)$ exists at all pts. in the given domain.

Hence, it's analytic.

By defin^n,

$$g'(z) = e^{-i\theta}[u_r + iv_r]$$
$$= e^{-i\theta}\left[\frac{1}{r} + i0\right]$$
$$= \frac{1}{r}e^{-i\theta} = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

$\Rightarrow g'(z) = \frac{1}{z}$.

Let $f(z) = z^2+1$ [Show that f is analytic (: HW) at all pts. in the complex plane with $f'(z) = 2z$].

Hence, $(g \circ f)(z) = g(f(z))$ is also analytic in the indicated domain, with the derivative

$U+iV$ $\quad U-iV$

$U_x = V_y$ $\quad U_x = -V_y$

$V_x = -U_y$ $\quad -V_x = U_y$

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$$(g\circ f)'(z) = g'(f(z))\, f'(z)$$

$$= \frac{1}{f(z)}\, f'(z)$$

$$= \frac{2z}{z^2+1}$$

Q. Let a f^n f be analytic in a domain D.
Prove: $f(z)$ must be a constt. in D if
(i) $f(z)$ is real valued $\forall$ z in D.
(ii) $\overline{f(z)}$ is analytic in D.
(iii) $|f(z)|$ is constt. in D.

Sol. Let $f(z) = U + iV$ be an analytic f^n in D.
Then, both U & V satisfy the CR eq^{ns}, namely,
$U_x = V_y$, $V_x = -U_y \longrightarrow$ (1).

(i) When f is real valued f^n in D, then,
$V=0$

$\Rightarrow$ $V_x = 0$ & $V_y = 0$

$\Rightarrow$ $U_x = 0$ & $U_y = 0$ (From (1))

$\Rightarrow$ $U = c$, a constt.

$\Rightarrow$ $f(z) = U + iV = c + i(0) = c$, a constt.

H.P.

(ii) Let $\overline{f(z)} = \overline{(U+iV)} = U - iV = P + iQ$
be analytic in D

$\Rightarrow$ $P = U$, $Q = -V$ satisfy CR eq^{ns}

$\Rightarrow$ $P_x = Q_y$, $P_y = -Q_x$

$\Rightarrow$ $U_x = -V_y$, $U_y = V_x \longrightarrow$ (2)

$\Rightarrow U_x = V_y = -V_y \Rightarrow V_y = 0$

$\Rightarrow U_x = 0$

$\Rightarrow U_y = -V_x = V_x \Rightarrow 2V_x = 0 \Rightarrow V_x = 0$

$\& \; U_y = 0.$

$\Rightarrow U = c_1,\ V = c_2,\ c_1, c_2$ are constt.

$\Rightarrow f(z) = U + iV = c_1 + ic_2$, a constt.

H.P

(iii) M1

$\because |f(z)| =$ constant, we have

$|f(z)| = c$

$\Rightarrow |f(z)|^2 = c^2.$

$\Rightarrow f(z)\overline{f(z)} = c^2$

$\Rightarrow \overline{f(z)} = \dfrac{c^2}{f(z)}. \longrightarrow (3)$

Case ①, If $(f(z)) = 0$.

Then, it's a constt already. So, proof is already done.

Case ② If $(f(z)) \neq 0$.

Then, both numerator & denominator of the eq^n ③ are analytic in D. Hence, $\overline{f(z)}$ is analytic in D. $\therefore$, by part 2 of the above, $f(z)$ must be a constt. in D

M2 $|f(z)|^2 = c^2$

$\Rightarrow U^2 + V^2 = c^2$

$\Rightarrow 2UU_x + 2VV_x = 0.$

$\Rightarrow U.U_x + VV_x = 0 \longrightarrow ① \quad \left(\dfrac{\partial}{\partial x}\right)$

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|||ly, $uu_y + vv_y = 0 \rightarrow$ (2). $\left(\frac{\partial}{\partial y}\right)$

$\Rightarrow uu_x - vu_y = 0.$ $\times u$

$u u_y + v u_x = 0$ $\times v$ $\left[\because u_x = v_y, \; u_y = -v_x\right]$

$\Rightarrow u^2 u_x - uvu_y = 0$

$uvu_y + v^2 u_x = 0$

Add

$\Rightarrow (u^2+v^2) u_x = 0$

$\Rightarrow u_x = 0.$

So, from (1), $v_x = 0$

Now, by CR eqns which are true,

u_y & $v_y = 0$

So, this is possible when u & v are constts.

So, $f(z) = c_1 + ic_2$, constts.

H.P

Q. Show, $u(x,y)$ is harmonic in some domain, &, also, find a harmonic conjugate: $v(x,y)$, when:

(1) $u(x,y) = 2x - x^3 + 3xy^2$

(2) $u(x,y) = \dfrac{y}{x^2+y^2}$

(3) $u(x,y) = y^3 - 3x^2y$.

Hence, find an analytic f^n 'f' s.t. $f(z) = u + iv$

(1) $u(x,y) = 2x - x^3 + 3xy^2$

$u_x = 2 - 3x^2 + 3y^2.$

$u_{xx} = -6x$

$u_y = 6xy$

$u_{yy} = 6x.$

Consider $u_{xx} + u_{yy} = -6x + 6x = 0.$

$\therefore$ u satisfies the Laplace eq^n.

Also, the partial derivatives, u_x, u_y, u_{xx} & u_{yy} are cts at all pts. in the complex plane.

$\therefore$ u is a harmonic f^n.

If v is the harmonic conjugate of u, then u & v satisfy the CR eq^{ns}.

$\Rightarrow u_x = v_y$ & $v_x = -u_y$

$\Rightarrow v_y = 2 - 3x^2 + 3y^2$	$v_x = -u_y = -6xy$
On integr^n.	On integr^n.

$v = \int (2 - 3x^2 + 3y^2)\, dy$ $\qquad$ $v = \int -6xy\, dx$

$= \left[2y - 3x^2 y + \frac{3y^3}{3}\right] + f(x)$ $\qquad$ $= -6y \cdot \frac{x^2}{2} + g(y)$

$v = 2y - 3x^2 y + y^3 + f(x)$, f: arbitrary f^n $\qquad$ $v = -3x^2 y + g(y)$

no need to write (already included)

$\therefore v(x,y) = [2y - 3x^2 y + y^3] + c$, c: real constt.

$\therefore$ The req^d analytic f^n,

$f(z) = u + iv$

$= (2x - x^3 + 3xy^2) + i(2y - 3x^2 y + y^3 + c)$

$= 2(x + iy) - ((x^3 - iy^3) + 3xy(y + ix$ ...

$$\Rightarrow f(z) = 2(x+iy) - (x+iy)^3 + iC.$$

$$\Rightarrow f(z) = 2z - z^3 + iC.$$

(2) $U(x,y) = \dfrac{y}{x^2+y^2}$

$$U_x = \frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$U_{xx} = \frac{(x^2+y^2)^2(-2y) + 2xy(2)(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{[x^2+y^2][-2x^2y - 2y^3 + 8x^2y + 8x^2y^3]}{(x^2+y^2)^4}$$

$$= \frac{2(xy)(4x^3-x) + 2y^3(1+4x^2)}{(x^2+y^2)^3}$$

$$U_{xx} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$U_y = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$U_{yy} = \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)(2)(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2y) - (4y)(x^2-y^2)}{(x^2+y^2)^3}$$

$$= \frac{-2x^2y - 2y^3 - 4x^2y + 4y^3}{(x^2+y^2)^3}$$

$$U_{yy} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

Consider $u_{xx} + u_{yy}$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3} + [-1]\left[\frac{6x^2y-2y^3}{(x^2+y^2)^3}\right] = 0.$$

Also, the ∴, u satisfies the Laplace eqn.
Also the partial derivatives are cts everywhere in the complex plane, except origin.
∴, u is a harmonic fn.
Let v be a harmonic conjugate of u.
Then, u & v satisfy the CR eqns
$U_x = V_y$ & $U_y = -V_x$.

$$\Rightarrow V_y = \frac{-2xy}{(x^2+y^2)^2}$$

On integration

$$v = -x\int \frac{2y\,dy}{(x^2+y^2)^2}$$

$$x^2+y^2 = t$$
$$\Rightarrow 2y\,dy = dt$$

$$= -x\int \frac{dt}{t^2}$$

$$= -x\left(-\frac{1}{t}\right) + f(x)$$

$$\Rightarrow v = \frac{-x}{x^2+y^2} + f(x) \rightarrow (1)$$

$$V_x = \frac{y^2-x^2}{(x^2+y^2)^2} \rightarrow (2)$$

Differentiating (1)

$$\Rightarrow V_x = \frac{(x^2+y^2)(-1) - x(2x)}{(x^2+y^2)^2} + f'(x)$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + f'(x) \rightarrow (3)$$

From (2) & (3),
$f'(x) = 0$
$\Rightarrow f(x) = c$

$$\therefore V(x,y) = \frac{x}{x^2+y^2} + c$$

To find $f(z) = u + iv$ (an analytic fn)

$$f(z) = u + iv$$

$$= \left(\frac{y}{x^2+y^2}\right) + i\left(\frac{x}{x^2+y^2}\right) + ic$$

$$= \frac{y + ix}{x^2+y^2} + ic$$

$$= \frac{i(x - iy)}{(x+iy)(x-iy)} + ic$$

$$= \frac{i}{(x+iy)} + ic$$

$$\Rightarrow f(z) = \frac{i}{z} + ic.$$

Chapter - 3

ELEMENTARY FUNCTIONS.

Section - 29

The Exponential Function :-

* If z is a complex variable, then, we define, the exponential f^n as

$$* \quad \exp(z) = e^z = e^{x+iy}$$
$$= e^x(\cos y + i\sin y)$$

* RESULTS :

1. Let $e^z = \rho e^{i\phi}$

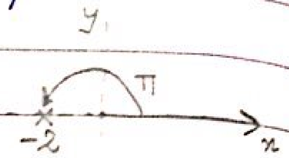
$$\Rightarrow e^{x+iy} = \rho e^{i\phi}$$
$$\Rightarrow e^x \cdot e^{iy} = \rho e^{i\phi}$$
$$\Rightarrow \rho = e^x \quad \Rightarrow |e^z| = e^x$$
$$\phi = \arg(e^z) = y + 2n\pi \ ; \ n \in \mathbb{Z}.$$

2. When x is a real variable, then, e^x can ~~never~~ never be -ve. On the other hand, e^z can take -ve values.
3. e^z is a periodic f^n with a period $2\pi i$.
4. For any 2 complex nos. z_1 & z_2,
   (i) $e^{z_1} e^{z_2} = e^{z_1+z_2}$
   (ii) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$.

ex Find all the values of $e^z = -2$.

We shall first express -2 in polar form.

Let $-2 = re^{i\theta}$.



Then, $r = |-2| = 2$.

$\mathrm{Arg}(-2) = \pi$.

$\therefore \theta = \mathrm{Arg}(-2) + 2n\pi \;;\; n \in \mathbb{Z}$.

$= \pi + 2n\pi$

$= (2n+1)\pi \;;\; n \in \mathbb{Z}$.

Consider $e^z = -2$

$\Rightarrow e^{x+iy} = 2e^{i(2n+1)\pi}$

$\Rightarrow e^x . e^{iy} = 2e^{i(2n+1)\pi} \;;\; n \in \mathbb{Z}$

$\Rightarrow e^x = 2$ & $y = (2n+1)\pi$

$\Rightarrow x = \ln 2 \;,\; y = (2n+1)\pi$.

$\therefore z = x + iy$

$= \ln 2 + i(2n+1)\pi \;;\; n \in \mathbb{Z}$.

ex Find all the values of $e^z = 1+i$.

Let $1+i = re^{i\theta}$

$r = \sqrt{1^2+1^2} = \sqrt{2}$.

$\mathrm{Arg}(1+i) = \frac{\pi}{4}$

$\theta = \mathrm{Arg}(1+i) + 2n\pi = \frac{\pi}{4} + 2n\pi \;;\; n \in \mathbb{Z}$

Consider

$e^z = 1+i$

$\Rightarrow e^{x+iy} = \sqrt{2}\, e^{i(\frac{\pi}{4}+2n\pi)}$

$\Rightarrow e^x . e^{iy} = \sqrt{2}\, e^{i(\pi/4+2n\pi)}$

$\Rightarrow e^x = \sqrt{2} \;,\; y = \frac{\pi}{4} + 2n\pi$

$\Rightarrow x = \ln\sqrt{2} = \frac{1}{2}\ln 2 \;,\; y = \frac{\pi}{4} + 2n\pi \;,\; n \in \mathbb{Z}$.

So, $z = \frac{\ln 2}{2} + i\left(\frac{\pi}{4} + 2n\pi\right), \; n \in \mathbb{Z}$.

* $\sin(iz) = i\sinh(z)$
* $\cos(iz) = \cosh(z)$
* $\cosh^2 y - \sinh^2 y = 1$

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CIRCULAR & HYPERBOLIC FUNCTIONS.

When z is a complex variable, then the hyperbolic $\sin z$ is denoted & defined by–

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2} \qquad \cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$

Circular f^{ns}:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Result. * $\sin z$, $\cos z$ are periodic f^{ns} with period $2\pi i$.

Q * Prove $\sin(z)$ is unbounded f^{n}.

$$\sin z = \sin(x+iy)$$
$$= \sin x \cos iy + \cos x \sin(iy)$$
$$= \sin x \cosh y + i \cos x \sinh y$$

Now, $|\sin z|^2 = (\sin x \cosh y)^2 + (\cos x \cdot \sinh y)^2$

$$= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y.$$
$$= \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y$$
$$(\because \cosh^2 y - \sinh^2 y = 1)$$
$$= \sin^2 x + \sinh^2 y(\sin^2 x + \cos^2 x)$$
$$= \sin^2 x + \underline{\sinh^2 y}.$$

Now, $\because \sinh(y)$ is not bounded, $\sin(z)$ is also not bounded.

HW * $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

Result ★ The zeroes of a f^n of the complex variable z are given by

$$f(z) = 0.$$

- The zeroes of $\sin(z)$ are given by

$$\sin z = 0$$
$$\Rightarrow z = n\pi, \ n \in \mathbb{Z}$$

- Zeroes of $\cos(z)$ are given by

$$\cos z = 0$$
$$\Rightarrow z = (2n+1)\frac{\pi}{2}, \ n \in \mathbb{Z}.$$

Q. Find the zeroes of $\sinh(z)$.

We know, $\sin(iz) = i\sinh(z)$

Now, for $\sinh(z) = 0$

$$\Rightarrow \frac{\sin(iz)}{i} = 0$$
$$\Rightarrow \sin(iz) = 0$$
$$\Rightarrow iz = n\pi$$
$$\Rightarrow z = \frac{n\pi}{i}$$

$$\text{or } z = n\pi i, \ n \in \mathbb{Z}.$$

★ LOGARITHMIC FUNCTIONS

If z is a non zero complex variable, then, we define its logarithm as,

$$\log(z) = \log|z| + i\arg(z) \longrightarrow (1)$$

~~Logs~~

If $z = re^{i\theta}$

Then, $\log z = \ln r + i\theta \longrightarrow (1)$

The principal branch of $\log z$ is denoted & defined by,

$$\text{Log } z = \ln|z| + i\,\text{Arg}(z)$$

$$(r > 0,\ -\pi < \text{Arg}(z) \leq \pi)$$

Result :- * $\log z = \ln|z| + i\arg(z)$

$= \ln|z| + i(\text{Arg}(z) + 2n\pi)$, $n \in \mathbb{Z}$.

$= (\ln|z| + i\,\text{Arg}(z)) + 2n\pi i$.

$\Rightarrow \boxed{\log z = \text{Log } z + 2n\pi i}$, $n \in \mathbb{Z}$.

Note:- $\log(e^z) \neq z$

$= z + 2n\pi i$.

☆ Branch:

Let f be a many valued f^n. A branch of f is a single valued $F(z)$ of f in a domain which is analytic in that domain & F is one of the values of f in that domain.

☆ Branches of $\log z$:

Let α be a real no.

By restricting θ s.t. $\alpha < \theta < \alpha + 2\pi$; then, we have a branch of $\log z$; $\log z = \ln|z| + i\arg(z)$; $|z| > 0$, $\alpha < \arg z < \alpha + 2\pi$

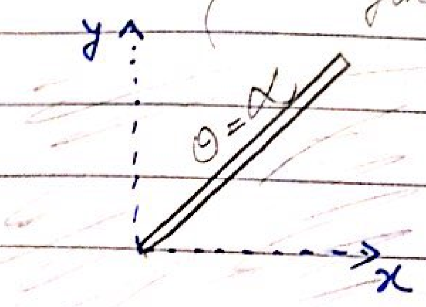
$\log z = \ln|z| + i\arg(z)$.

$\hookrightarrow z > 0,\ \alpha < \arg(z) < \alpha + 2\pi$

or $\log z = \ln r + i\theta$

$\hookrightarrow (r > 0),\ \alpha < \theta < \alpha + 2\pi$

Domain given by these inequalities.

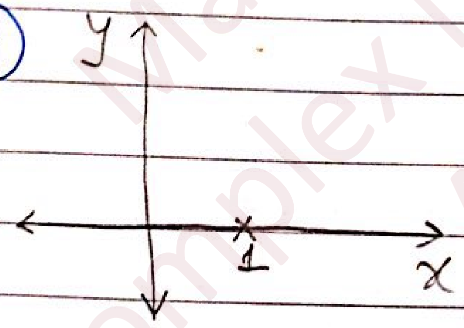


here, $\theta = \alpha$ is called a **Branch cut** & the origin is called **Branch pt.** which is common $\forall$ branch cuts

Q. Find : (1) $\log 1$ (3) $\mathrm{Log}\ 1$

(2) $\log(-1)$ (4) $\mathrm{Log}(-1)$.

(1)



Let $z = 1$

Then, $r = |z| = 1$

$\mathrm{Arg} = \mathrm{Arg}(1) = 0$.

$\arg(z) = \theta = \mathrm{Arg}(z) + 2n\pi,\ n \in \mathbb{Z}$.

$= 0 + 2n\pi$.

$\Rightarrow \theta = 2n\pi;\ n \in \mathbb{Z}$.

$\log z = \ln|z| + i\arg(z)$

$\Rightarrow \log 1 = \ln 1 + i(2n\pi)$

$= 2n\pi i,\ n \in \mathbb{Z}$.

(3) $\mathrm{Log}\ z = \ln|z| + i\,\mathrm{Arg}(z)$

$\Rightarrow \mathrm{Log}\ 1 = \ln(1) + i\,0$.

$= 0 + i0$

$= 0$

Q. Find ① $\log(1-i)$, ② $\text{Log}(1-i)$

$z = 1-i$

$\Rightarrow |z| = \sqrt{2}$

$\text{Arg}(z) = -\frac{\pi}{4}$

$\theta = \arg(z) = -\frac{\pi}{4} + 2n\pi, \; n \in \mathbb{Z}$

$\log z = \ln|z| + i \arg(z)$

$\Rightarrow \log(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2n\pi\right), \; n \in \mathbb{Z}$

② $\text{Log}\, z = \ln|z| + i\,\text{Arg}(z)$

$\Rightarrow \text{Log}\, z = \ln\sqrt{2} - i\frac{\pi}{4} = \frac{1}{2}\ln 2 - i\frac{\pi}{4}$

Q. Express $\log i$ as a single valued f^n in the regions:

(i) $r > 0, \; \frac{\pi}{4} < \theta < \frac{9\pi}{4}$

(ii) $r > 0, \; \frac{2\pi}{3} < \theta < \frac{8\pi}{3}$

Let $z = i$.

$r = |z| = 1$

$\text{Arg}(z) = \frac{\pi}{2}$

$\Rightarrow \theta = \text{Arg}(z) + 2n\pi; \; n \in \mathbb{Z}$

$= \frac{\pi}{2} + 2n\pi, \; n \in \mathbb{Z}$

(i) $r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}$

$\Rightarrow r > 0,\ 45^\circ < \theta < 405^\circ$

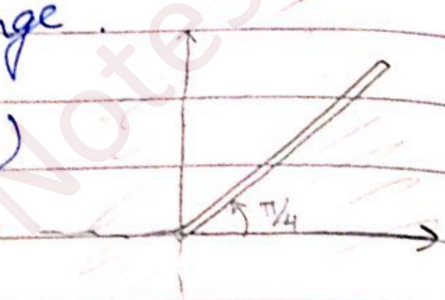
$\theta = \frac{\pi}{2} = 90^\circ \qquad n = 0$

$= \frac{\pi}{2} + 2\pi = 450^\circ, \quad n = 1$

$= \frac{\pi}{2} - 2\pi = -270^\circ, \quad n = -1$

- - - - -

Only, $\theta = \frac{\pi}{2}$ lies in given range.



$\therefore \log z = \log|z| + i\arg(z)$

$\Rightarrow \log i = \ln 1 + i\left(\frac{\pi}{2}\right)$

$= \frac{i\pi}{2},\ r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}.$

(ii) $r > 0,\ \frac{2\pi}{3} < \theta < \frac{8\pi}{3}$

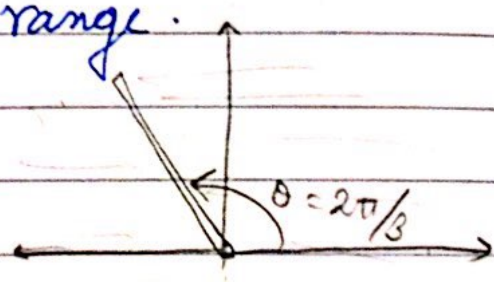
$r > 0,\ 120^\circ < \theta < 480^\circ$

$\theta = \frac{\pi}{2} = 90^\circ \qquad n = 0$

$= \frac{\pi}{2} + 2\pi = 450^\circ \qquad n = 1$

$= \frac{\pi}{2} - 2\pi = -270^\circ \qquad n = -1$

$\therefore$ Only $\theta = \frac{5\pi}{2} (= 450^\circ)$ lies in given range.



$\log z = \ln|z| + i\arg(z)$

$\Rightarrow \log i = \ln 1 + i\left(\frac{5\pi}{2}\right)$

$= \frac{i5\pi}{2},\ r > 0,\ \frac{2\pi}{3} < \theta < \frac{8\pi}{3}.$

Result ★ For any 2 non zero complex nos: z_1 & z_2, we have,

① $\log(z_1 z_2) = \log(z_1) + \log(z_2)$

② $\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$

V. Imp.

Note: The above result cannot be applied when principal branches are considered.

(Also, the results need not be true, when, any single valued f^{ns} are considered.)

i.e, $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$

$\operatorname{Log}\left(\frac{z_1}{z_2}\right) \neq \operatorname{Log}(z_1) - \operatorname{Log}(z_2)$

Q Show:

$\operatorname{Log}(z_1 z_2) \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$,

when $z_1 = z_2 = -1$.

$|z_1| = |z_2| = 1$

$\operatorname{Arg}(z_1) = \operatorname{Arg}(z_2) = \pi$

$\arg(z_1) = \theta_1 = \pi + 2n\pi = (2n+1)\pi,\ n \in \mathbb{Z}$

~~arg(~~

~~&~~ $z_1 . z_2 = 1$.

$|z_1 z_2| = 1$

$\operatorname{Arg}(z_1 z_2) = 0$.

So, $\operatorname{Log}(z_1 z_2) = \ln|z_1 z_2| + i\operatorname{Arg}(z_1 z_2)$

$= \ln 1 + i\,0$.

$\rightarrow \operatorname{Log}(z_1 z_2) = 0 \rightarrow ②$

$\text{Log}\, z_1 = \ln|z_1| + i\,\text{Arg}(z_1)$
$= \ln 1 + i(\pi)$
$= \pi i$

$\Rightarrow \text{Log}\, z_1 + \text{Log}\, z_2 = 2\pi i \longrightarrow (1)$

Clearly,
$(2) \neq (1)$, so,

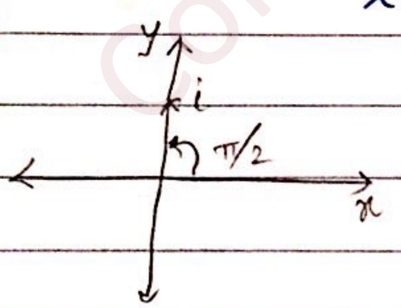
$\text{Log}(z_1 z_2) \neq \ln(|z_1 z_2|) + i\,\text{Arg}(z_1 z_2)$

Q. Show that

(1) $\log(i^2) = 2\log i$ when
$\log z = \ln r + i\theta$,
$r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}$.

(2) $\log(i^2) \neq 2\log i$ when
$\log z = \ln r + i\theta$
$r > 0,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4}$

Let $z = i$
$\Rightarrow r = |z| = 1$
$\text{Arg}(z) = \frac{\pi}{2}$.



$\theta = \arg(z) = \text{Arg}(z) + 2n\pi$
$= \frac{\pi}{2} + 2n\pi,\ n \in \mathbb{Z}$.

(1) $r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4} \equiv 45° < \theta < 405°$.
$\theta = \frac{\pi}{2} = 90°,\ n = 0$

$= \frac{\pi}{2} + 2\pi = \cancel{405°}\ 450°,\ n = 1$

$\therefore$ Only $\theta = \frac{\pi}{2}$ lies in given range.

$\Rightarrow \log(z) = \ln r + i\theta$
$= \ln 1 + i\frac{\pi}{2}$

$\Rightarrow \log(i) = \frac{i\pi}{2}$

$\Rightarrow 2\log(i) = \pi i \longrightarrow$ ①

Now, ~~Let~~ $z^2 = i^2 = -1$
$\Rightarrow |z^2| = 1$
$Arg(z^2) = Arg(-1) = \pi$
$\Rightarrow \theta = arg(z^2) = Arg(z^2) + 2n\pi$
$= \pi + 2n\pi, \; n \in \mathbb{Z}$.

(i) When $r>0$, $\frac{\pi}{4} < \theta < \frac{9\pi}{4}$ $(45° < \theta < 405°)$

So only, $\theta = \pi = 180°$ $(n=0)$ ~~for~~ lies in given range.

$\Rightarrow \log(z^2) = \ln|z^2| + i\theta$.
$= \ln 1 + i\pi$

$\Rightarrow \log(i^2) = \pi i \longrightarrow$ ②

From ① & ②,
$\log(i^2) = 2\log(i)$; $r>0$, $\frac{\pi}{4} < \theta < \frac{9\pi}{4}$

② Let $z = i$
$\Rightarrow |z| = 1$
$Arg(z) = \frac{\pi}{2}$.

$\theta = arg(z) = \frac{\pi}{2} + 2n\pi, \; n \in \mathbb{Z}$.

For $r>0$, $\frac{3\pi}{4} < \theta < \frac{11\pi}{4} = 135^\circ < \theta < 495$

$\theta = \frac{\pi}{2} = 90^\circ, \ n=0$

$= \frac{\pi}{2} + 2\pi = 450^\circ, \ n=1$

So, only $\theta = 450^\circ = \frac{5\pi}{2}$ lies in the given range

& $\log z = \ln|z| + i\theta$

$= \ln(1) + i\left(\frac{5\pi}{2}\right)$

$\Rightarrow \log i = 5\pi i/2$

$\Rightarrow 2\log i = 5\pi i \longrightarrow (1)$.

Now, $z^2 = i^2 = -1$

$|z^2| = 1$

$Arg(z^2) = \pi$.

So, $\theta = \arg(z^2) = \pi + 2n\pi, \ n \in \mathbb{Z}$.

$\theta = \pi = 180^\circ, \ n=0$

$= 3\pi = 540^\circ, \ n=1$

So, only $\theta = \pi$ satisfies given range

$\log(z^2) = \ln|z^2| + i\theta$

$= \ln 1 + i(\pi)$

$\Rightarrow \log(i^2) = \pi i. \longrightarrow (2)$

Clearly, $(1) \neq (2)$

So, $\log(i^2) \neq 2\log i$; $r>0$, $\frac{3\pi}{4} < \theta < \frac{11\pi}{4}$

① $\text{Log}(-ei) = 1 - \frac{\pi i}{2}$

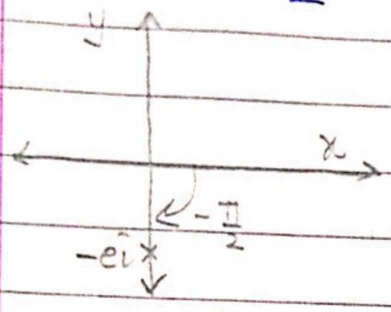
② $\text{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi i}{4}$. $(r>0,\ -\pi < \text{Arg}\,z < \pi)$

① Let $z = -ei$ $\therefore \text{Log}\,z = \ln|z| + i\,\text{Arg}(z)$

$|z| = r = e$. $= \ln e + i\left(-\frac{\pi}{2}\right)$

$\text{Arg} = -\frac{\pi}{2}$

$= 1 - i\frac{\pi}{2}$.

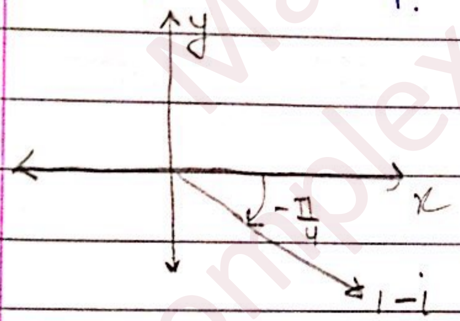


② $z = 1-i$

$\Rightarrow |z| = r = \sqrt{2}$ $\text{Log}\,z = \ln|z| + i\,\text{Arg}(z)$

$\text{Arg}(z) = -\frac{\pi}{4}$. $\Rightarrow \text{Log}(1-i) = \ln(\sqrt{2}) + i\left(-\frac{\pi}{4}\right)$

$\Rightarrow \text{Log}(1-i) = \frac{1}{2}\ln 2 - i\frac{\pi}{4}$.



Q. Verify for $n = 0, \pm 1, \pm 2, \ldots$

① $\log e = 1 + 2n\pi i$

② $\log i = \left(2n + \frac{1}{2}\right)\pi i$

③ $\log(-1+\sqrt{3}i) = \ln(2) + 2\left(n + \frac{1}{3}\right)\pi i$

① Let $z = e$.

$\Rightarrow |z| = e$

$\text{Arg}(z) = 0$. $\Rightarrow \theta = \arg(z) = 2n\pi,\ n \in \mathbb{Z}$

$\cancel{Log(z)}$ $\log(z) = \ln|z| + i\arg(z)$

$\log(e) = \ln e + i(2n\pi)$, $n \in \mathbb{Z}$

$\Rightarrow \log e = 1 + 2n\pi i$, $n \in \mathbb{Z}$.

③ Let $z = -1 + \sqrt{3}i$

$\Rightarrow |z| = 2$

$Arg(z) = \frac{2\pi}{3} \Rightarrow \theta = \arg(z) = \frac{2\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$

$\Rightarrow 2\pi\left(\frac{1}{3} + n\right)$, $n \in \mathbb{Z}$

So, $\log(z) = \ln|z| + i\arg(z)$

$= \ln(2) + i\left(\frac{2\pi}{3} + 2n\pi\right)$, $n \in \mathbb{Z}$.

$\Rightarrow \log(z) = \ln 2 + i\,2\pi\left(n + \frac{1}{3}\right)$, $n \in \mathbb{Z}$

Q. Show :-

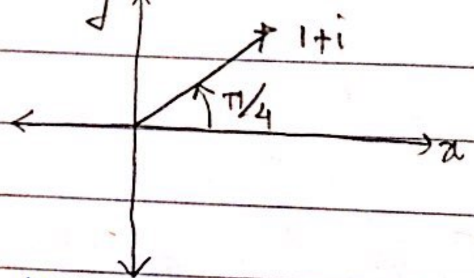
(i) $Log(1+i)^2 = 2\,Log(1+i)$

(ii) $Log(-1+i)^2 \neq 2\,Log(-1+i)$

(i) Let $z = 1+i$

$|z| = r = \sqrt{2}$

$Arg(z) = \frac{\pi}{4}$.



$Log(z) = \ln|z| + i\,Arg(z)$

$= \ln\sqrt{2} + i\left(\frac{\pi}{4}\right) = \frac{1}{2}\ln 2 + i\frac{\pi}{4}$.

$\Rightarrow 2\,Log(1+i) = \ln 2 + i\frac{\pi}{2}$. $\longrightarrow$ (1)

Let $z^2 = (1+i)^2 = 1-1+2i = 2i$

$\Rightarrow |z^2| = 2$

$Arg(z^2) = \frac{\pi}{2}$

$\therefore Log(z^2) = \ln|z^2| + i\,Arg(z^2)$

$\Rightarrow Log(1+i)^2 = \ln 2 + i\frac{\pi}{2} \rightarrow (2)$.

From (1) & (2),

$2\,Log(1+i) = Log(1+i)^2$

(ii) RHS

Let $z = -1+i$

$\Rightarrow |z| = \sqrt{2}$

$Arg(z) = \frac{3\pi}{4}$.

$\Rightarrow Log(z) = \ln|z| + i\,Arg(z)$

$= \ln\sqrt{2} + i\frac{3\pi}{4} = \frac{1}{2}\ln 2 + i\left(\frac{3\pi}{4}\right)$

$\Rightarrow 2\,Log(-1+i) = \ln 2 + i\left(\frac{3\pi}{2}\right) \rightarrow (1)$

LHS

Let $z^2 = (-1+i)^2 = 1-1-2i = -2i$

$\Rightarrow |z^2| = 2$

$Arg(z) = -\frac{\pi}{2}$.

$Log(z^2) = \ln(|z^2|) + i\,Arg(z^2)$

$= \ln 2 - i\frac{\pi}{2}$

$\Rightarrow Log(-1+i)^2 = \ln 2 - i\frac{\pi}{2} \rightarrow (2)$

Clearly ① ≠ ②

So, $Log(-1+i)^2 \neq 2\,Log(-1+i)$.

HP

Complex Exponents.

If z is a non zero complex no., then,

$$z^c = \exp\{c \log z\}$$

$$\Rightarrow z^c = e^{c \log z}$$

The principal value of z^c is denoted & defined as

$$p.v\{z^c\} = e^{c\,Log z}.$$

Note:- In the similar manner, we can express c^z.

$$c^z = \exp(z \log c)$$

Q. Show that:

① $i^{-2i} = \exp\{(4n+1)\pi\}$; $n \in \mathbb{Z}$

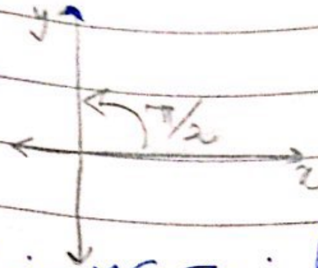
② $(1+i)^i = \exp\{-\frac{\pi}{4} + 2n\pi\}\exp\{\frac{i}{2}\ln 2\}$, $n \in \mathbb{Z}$

③ $(-1)^{1/\pi} = e^{(2n+1)i}$; $n \in \mathbb{Z}$.

(1) Let $z = i$

$|z| = 1$

$Arg(z) = \frac{\pi}{2}$.



$\Rightarrow \theta = arg(z) = \frac{\pi}{2} + 2n\pi$; $n \in \mathbb{Z}$; $\log(z) = \ln|z| + i\theta$

$= \ln 1 + i(\frac{\pi}{2} + 2n\pi)$

$= i(\frac{\pi}{2} + 2n\pi)$; $n \in \mathbb{Z}$

By defin^n,

$$z^c = e^{c \log z}$$

$$\Rightarrow i^{-2i} = e^{(-2i)\log i}$$

$$= e^{-2i(\frac{\pi}{2} + 2n\pi)i}$$

$$= e^{2(\pi/2 + 2n\pi)} \; ; n \in \mathbb{Z}$$

$$= e^{(\pi + 4n\pi)} \; ; n \in \mathbb{Z}$$

$$\Rightarrow i^{-2i} = e^{(4n+1)\pi} \; ; n \in \mathbb{Z}$$

(2) Let $z = 1+i$

$c = i$

$|z| = \sqrt{2}$

$Arg(z) = \frac{\pi}{4}$.

By defin^n,

$$z^c = e^{c \log z}$$

$$= e^{i \log(1+i)}$$

$$= e^{i(\frac{1}{2}\ln 2) + (-\frac{\pi}{4} + 2n\pi)}$$

$\theta = \frac{\pi}{4} + 2n\pi$. $\Rightarrow z^c = \exp\{\frac{i}{2}\ln 2\} \cdot \exp(-\frac{\pi}{4} + 2n\pi)$; $n \in \mathbb{Z}$.

$\log(1+i) = \ln|z| + i\,Arg(z)$

$= \frac{1}{2}\ln 2 + i(\frac{\pi}{4} + 2n\pi)$

(3) Let $z = -1$

$\Rightarrow |z| = 1$

$Arg(z) = \pi$.

$\theta = \pi + 2n\pi = (2n+1)\pi$; $n \in \mathbb{Z}$.

$$\log z = \ln|z| + i\arg(z)$$
$$= \ln 1 + i(2n+1)\pi$$

Now,

$$z^c = \exp(c \log z)$$
$$= e^{c\log z}$$
$$= e^{\frac{1}{\pi}\log i}$$
$$= e^{1/\pi\,(i(2n+1)\pi)}$$
$$\Rightarrow z^c = e^{(2n+1)i}$$

Q. Find the principal value of :-

(1) $(-i)^i$

(2) $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$

(3) $(1-i)^{4i}$

(1) Let $z = -i$

$\Rightarrow |z| = 1$

$\operatorname{Arg}(z) = -\frac{\pi}{2}$.

So, $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$

$$= \ln 1 + i\left(-\frac{\pi}{2}\right) = -i\frac{\pi}{2}.$$

Now

$$z^c = e^{c\log z}$$
$$= e^{i\left(-\frac{\pi i}{2}\right)}$$
$$\Rightarrow z^c = e^{\pi/2}.$$
$$\Rightarrow \text{p.v}\{(-i)^i\} = e^{\pi/2}.$$

(2) Let $z = \frac{e}{2}(-1-\sqrt{3}i)$

$\Rightarrow |z| = \frac{e}{2}(2) = e.$

$Arg(z) = -\frac{2\pi}{3}.$

$Log(z) = \ln|z| + i\,Arg(z)$

$= \ln e + i\left(-\frac{2\pi}{3}\right) = \left(1 - \frac{2\pi i}{3}\right).$

Now, $z^c = e^{3\pi i\, Log(z)}$

$= e^{3\pi i\left(1 - \frac{2\pi i}{3}\right)}$

$= e^{3\pi i} \cdot e^{2\pi^2}$

$\Rightarrow z^c = \exp\{3\pi i\}\exp\{2\pi^2\}$

$\Rightarrow p.v\left\{\frac{e}{2}(-1-\sqrt{3}i)\right\} = e^{3\pi i + 2\pi^2} = (-1)^3 \cdot e^{2\pi^2} = -e^{2\pi^2}$

(3) Let $z = 1-i$

$\Rightarrow |z| = \sqrt{2}$

$Arg(z) = -\frac{\pi}{4}$

$\Rightarrow Log(z) = \ln|z| + i\,Arg(z)$

$= \frac{1}{2}\ln 2 - i\frac{\pi}{4}.$

$z^c = e^{4i\, Log(z)}$

$= e^{4i\left(\frac{1}{2}\ln 2 - i\frac{\pi}{4}\right)}$

$= e^{2i\ln 2} \cdot e^{\pi}.$

$\Rightarrow z^c = \exp(2i\ln 2) \cdot \exp(\pi)$

$\Rightarrow p.v\left\{(1-i)^{4i}\right\} = e^{\pi + (2\ln 2)i} = -e^{(2\ln 2)i}$

$= -2^{2i}$

Q. Shows :- $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

Let $z = -1+\sqrt{3}i$

$\Rightarrow |z| = 2$

$\cancel{Ar}\cdot \operatorname{Arg}(z) = +\frac{2\pi}{3}$

$\theta = \arg(z) = +\frac{2\pi}{3} + 2n\pi$.

$$\log(z) = \ln|z| + i\theta$$

$$= \ln 2 + i\left(+\frac{2\pi}{3} + 2n\pi\right)$$

$$z^c = e^{c\log(z)} = e^{\frac{3}{2}\left(\ln 2 + \left(\frac{\pi}{3}(+2) + 2n\pi\right)i\right)}$$

$$= e^{3/2 \ln 2} \cdot e^{\frac{3}{2}i\left(+\frac{2\pi}{3} + 2n\pi\right)}$$

$$= e^{3/2 \ln 2} \cdot e^{i\pi} \cdot e^{i(3n\pi)}$$

$$= (2^{3/2}) \cdot (-1)\, e^{i(3n\pi)} \quad ; n \in \mathbb{Z}$$

$$= (-2)^{3/2} e^{i(3n\pi)} \qquad = (+2)^{3/2} \cdot e^{(3n+1)}$$

$$= \cancel{(-2)^{1/2} \cdot (-2) \cdot (-1)}$$

$$= 2^{3/2}\{\cos(3n+1)\pi + i\sin(3n+1)\pi\}$$

$$= 2\sqrt{2}\,(-1)^{3n+1}$$

$$= 2\sqrt{2}\,(-1)^{n+1}$$

$$= 2\sqrt{2}\,(\pm 1) \qquad \left[\because (-1)^{3n+1} = \pm 1\right.$$

$$\Rightarrow [(-1) + \sqrt{3}i]^{3/2} = \pm 2\sqrt{2}.$$

Ans

Q. Show that :

HW ① $\text{Log}(-ei) = 1 - \frac{\pi i}{2}$

Done before ② $\text{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4} i$.

* For finding +ve dirⁿ for a closed curve:- move along the boundary s.t., the region is towards your left

Date 14/5/12

Chapter – 4
COMPLEX INTEGRATION

Let $w(t) = u(t) + i\,v(t)$, $a \leq t \leq b$, be, a complex valued f^n. Then,

$$\int_{t=a}^{b} w(t)\,dt = \int_{t=a}^{b} u(t)\,dt + i\int_{t=a}^{b} v(t)\,dt$$

Note:- 1. $\mathrm{Re}\left[\int_{t=a}^{b} w(t)\,dt\right] = \int_{t=a}^{b} \mathrm{Re}(w(t))\,dt$

2. & $\mathrm{Im}\left[\int_{t=a}^{b} w(t)\,dt\right] = \int_{t=a}^{b} \mathrm{Im}(w(t))\,dt$.

3. $\left|\int_{t=a}^{b} w(t)\,dt\right| \leq \int_{t=a}^{b} |w(t)|\,dt$

& 4. $\left|\int_{t=a}^{\infty} w(t)\,dt\right| \leq \int_{t=a}^{\infty} |w(t)|\,dt$.

☆ An arc or a curve in the complex plane is given by:-

$$z(t) = x(t) + i(y(t)) \;;\; a \leq t \leq b$$

$$(x(t),\, y(t))$$

Imp.
s.t. x & y are its f^{ns} of t :

* A curve (/an arc) is said to be a ~~two~~ simple curve (or, simple arc) if it doesn't cross itself.
  A simple closed curve (or arc) is said to be ~~one~~ a Jordon curve (or Jordon Arc)

* A contour in the complex plane is made up of simple arcs joined from end to end. It is said to be simple if it doesn't cross itself.

* The length of a simple curve
$$Z(t) = x(t) + i\,y(t), \quad a \leq t \leq b$$
is given by
$$\boxed{L = \int_{t=a}^{b} |Z'(t)|\,dt}.$$

* Let $f(z)$ be a complex valued f^n defined at all pts. on a smooth curve -
$$Z = Z(t), \quad a \leq t \leq b,$$
represented by curve c, then,
$$\int_c f(z)\,dz = \int_{t=a}^{b} f(Z(t))\,.\,\underbrace{z'(t)\,dt}_{L}$$
/* expressing integral in terms of parameter t */

* RESULTS

1. $$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz.$$

2. $$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz \; ; \; z_0 : \text{constt}.$$

3. $$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

Imp

4. ML INEQUALITY

If $f(z)$ is a bounded f^n at every pt. on a simple curve C, i.e,

$$|f(z)| \leq M \text{ on } C,$$

→ bound of $f(z)$

then,

$$\left|\int_C f(z)\,dz\right| \leq ML.$$

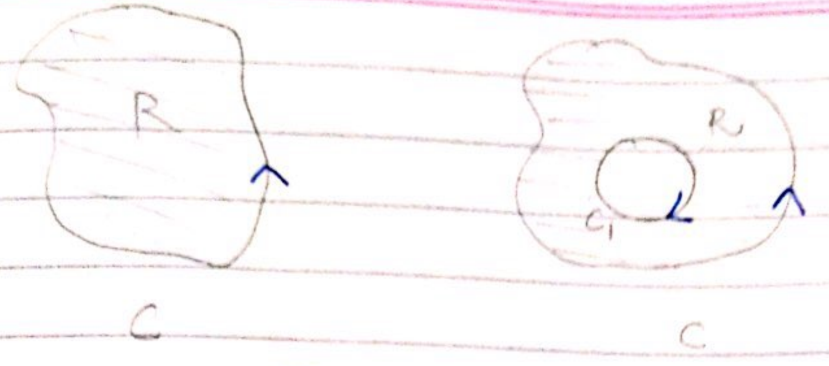
, where L is the length of the curve C.

5. $$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots$$

; where, $C = C_1 + C_2 + \cdots$.

* If C is a smooth curve, then, the increasing dir^n is the +ve dir^n.

If C is a SCC (simple closed curve), enclosing a region R, then, the +ve dir^n is that dir^n through which one walks, finds the region R to his LEFT.



Q. Evaluate

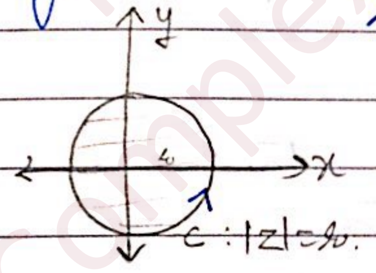
$$I = \int_C \bar{z}\, dz \; ; \; C \text{ is the right hand half of the curve } z = 2e^{i\theta}, \; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

use above solved result

Hence, evaluate $\int_C \frac{dz}{z}$ on the same curve.

The eq^n of a circle with its centre at origin & radius r units is given by $|z| = r$. & any pt. on this circle is given by

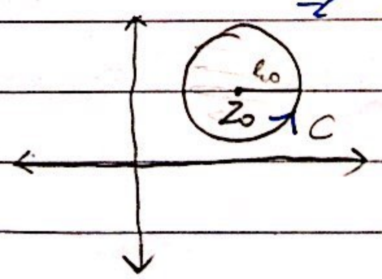
$$z = re^{i\theta}$$



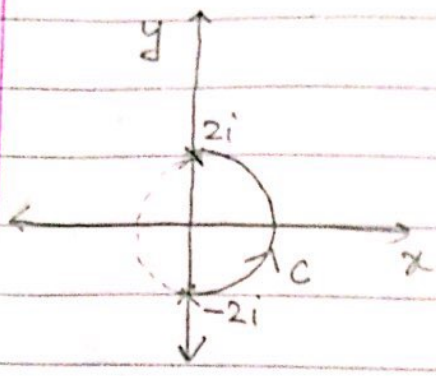
The eq^n of a circle with centre at z_0 & radius r_0 units is given by $|z - z_0| = r_0$ & any pt. on this circle is given by

$$z - z_0 = r_0 e^{i\theta}$$

$$\Rightarrow z = z_0 + r_0 e^{i\theta}.$$



Here, c is given by $c : |z| = 2$; the RHS half of the circle joining pts. $-2i$ to $+2i$ as shown in the diagram.



Here, $z = 2e^{i\theta}$.

$$\Rightarrow \overline{z} = 2\overline{e^{i\theta}}$$

$$\Rightarrow \overline{z} = 2e^{-i\theta}$$

Also, $dz = 2e^{i\theta} \cdot i\,d\theta$.

$$\therefore \int_c \overline{z}\,dz = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} 2e^{-i\theta} \cdot 2e^{i\theta} \cdot i\,d\theta$$

$$= 4i \int_{\theta=-\pi/2}^{\pi/2} d\theta.$$

$$\Rightarrow \int_c \overline{z}\,dz = 4\pi i \longrightarrow (1).$$

Now $$\int_c \frac{dz}{z} = \int_c \frac{\overline{z}}{z \cdot \overline{z}}\,dz$$

$$= \int_c \frac{\overline{z}}{|z|^2}\,dz$$

$$= \int_c \frac{\overline{z}\,dz}{2^2}$$

$$= \frac{1}{4} \int_c \overline{z}\,dz$$

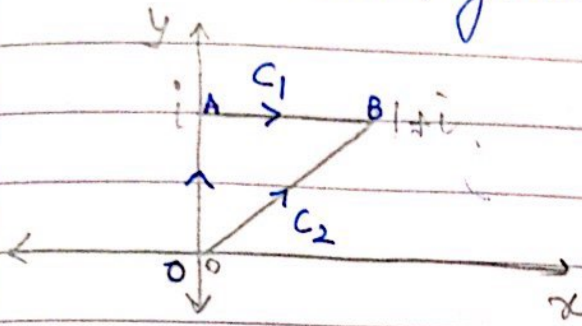
$$= \frac{1}{4}(4\pi i) \quad \text{(from (1))}$$

$$\Rightarrow \int_c \frac{dz}{z} = \pi i.$$

Q. Evaluate :

$$\int_C f(z)\,dz, \text{ where } f(z) = y - x - i(3x^2) \ \&$$

C is the curve, given by :



Here, $$\int_C f(z)\,dz = \int_{C_1 - C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz. \longrightarrow (1).$$

From the diagram,

$$\int_{C_1} f(z)\,dz = \left(\int_{OA} + \int_{AB}\right) f(z)\,dz$$

Along OA : $x = 0$, $y = 0$ to 1.

$\Rightarrow dx = 0$ & dy

$z = x + iy$

$\Rightarrow dz = dx + i\,dy$

$\Rightarrow dz = i\,dy.$

$$f(z) = y - x - i(3x^2) = y \Big|_{x=0}.$$

So, $f(z)\,dz = y(i\,dy)$

$$\therefore \int_{OA} f(z)\,dz = \int_{y=0}^{1} y\,i\,dy = i\left[\frac{y^2}{2}\right]_0^1 = \frac{i}{2} \rightarrow (2)$$

Along $\underline{AB}$. $y = 1,\ x = 0$ to 1

$dy = 0$.

$z = x + iy$

$\Rightarrow dz = dx + i\,dy$

$\Rightarrow dz = dx$.

$f(z) = y - x - i(3x^2)$

$\Rightarrow f(z) = 1 - 1(x + i3x^2)$

$\Rightarrow f(z)\,dz = (1 - x - 3ix^2)(dx)$

$$\Rightarrow \int_{AB} f(z)\,dz = \int_{x=0}^{1} (1 - x - 3ix^2)\,dx$$

$$= \left[x - \frac{x^2}{2} - ix^3\right]_0^1$$

$$= \left(1 - \frac{1}{2} - i\right)$$

$$\Rightarrow \int_{AB} f(z)\,dz = \frac{1}{2} - i \rightarrow (3)$$

$$\therefore \int_{C_1} f(z)\,dz = \frac{i}{2} + \frac{1}{2} - i = \frac{1-i}{2} \quad ((3) + (2)) \rightarrow (4)$$

Along $\underline{C_2}$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = t$$

$\Rightarrow x = y = t \quad ; \quad t = 0$ to 1.

$dz = dx + i\,dy$

$= dt + i\,dt = (1+i)\,dt$.

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$$f(z) = y - x - 3x^2 i$$
$$= t - t - 3t^2 i$$
$$= -3t^2 i$$
$$\Rightarrow f(z)dz = (-3t^2 i)(1+i)dt$$
$$= (-3t^2 i + 3t^2)\,dt$$
$$= (3t^2 - 3t^2 i)\,dt$$

$$\Rightarrow \int_{C_2} f(z)dz = \int_{t=0}^{1} (3t^2 - 3t^2 i)\,dt$$
$$= \left[t^3 - t^3 i\right]_0^1$$
$$\Rightarrow \int_{C_2} f(z)dz = 1 - i \longrightarrow (5)$$

Using (4) & (5) in (1), we get

$$\int_C f(z)dz = \frac{1-i}{2} - 1 - i = \cancel{\frac{3}{2}(1-i) = \frac{3}{2} - \frac{3}{2}i}$$
$$= -\frac{1}{2}(1-i) = \frac{-1}{2} + \frac{i}{2} = \frac{i-1}{2} \quad \text{Ans}$$

Q. Use parametric represent^n of C for the f^n f & the curve C to evaluate
$$\int_C f(z)dz, \text{ if}$$

(1) $f(z) = \dfrac{z+2}{z}$ & C is the semicircle $z = 2e^{i\theta}$, $0 \le \theta \le \pi$

(2) $f(z) = z - 1$ & C is the arc, from $z=0$ to $z=2$.
consisting of semicircle
$z = 1 + e^{i\theta}; \pi \le \theta \le 2\pi$

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③ $f(z) = \pi \exp(\pi \bar{z})$ where C is the boundary of the sq. with vertices 0, 1, 1+i, i & the orientⁿ is in the +ve dirⁿ.

④ $f(z) = \begin{cases} 1, & y < 0 \\ 4y, & y > 0 \end{cases}$

& C is from $z = -1-i$ to $z = 1+i$ along the curve $y = x^3$.

① Given $z = 2e^{i\theta} \Rightarrow C : |z| = 2$.

$$dz = 2(i d\theta) e^{i\theta} ; \quad 0 \leq \theta \leq \pi$$

$$f(z) = \frac{z+2}{z} = \frac{2e^{i\theta}+2}{2e^{i\theta}} = \frac{e^{i\theta}+1}{e^{i\theta}}.$$

$$\Rightarrow \int_C f(z)dz = \int_{\theta=0}^{\pi} \left(\frac{e^{i\theta}+1}{\cancel{e^{i\theta}}}\right)(2\cancel{e^{i\theta}})\, i d\theta .$$

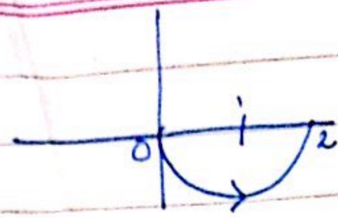
$$= 2i \int_{\theta=0}^{\pi} (e^{i\theta}+1)\, d\theta.$$

$$= 2i \left[\frac{e^{i\theta}}{i} + \theta\right]_0^{\pi}$$

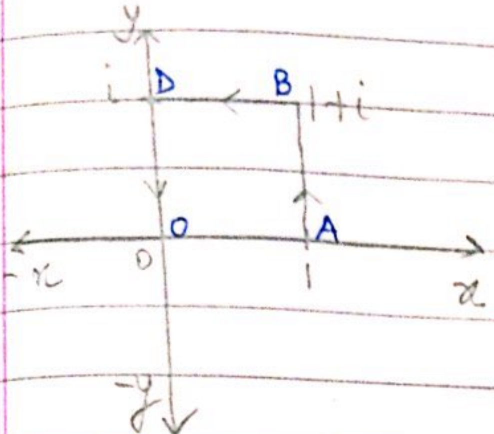
$$= \left(\frac{2\cancel{i}(-1)}{\cancel{i}} + 2\pi i\right) - \left[\frac{2\cancel{i}}{\cancel{i}}(1)\right]$$

$$= -2 + \pi - 2 = 2\pi i - 4 .$$

(2) $z = 1+e^{i\theta}$; $C: |z-1| = 1$.
HW Ans = $\frac{3}{2\pi}$



(3)



Here,

$$\int_C f(z)\,dz = \left(\int_{OA} + \int_{AB} + \int_{BD} + \int_{DO}\right) f(z)\,dz \longrightarrow (1)$$

Along $\underline{OA}$: $y=0$, $x=0$ to 1.
$\Rightarrow dy = 0$
$\Rightarrow dz = dx + i\,dy = dx$

$\cancel{\frac{x-0}{1-0} = \frac{y-0}{0-0}}$

$f(z) = \pi e^{\pi \bar{z}} = \pi e^{\pi(x-iy)} = \pi e^{\pi x}$.

$$\Rightarrow \int_{OA} f(z)\,dz = \int_{x=0}^{1} \pi e^{\pi x}\,dx = \cancel{\pi}\left(\frac{e^{\pi x}}{\cancel{\pi}}\right)_0^1 = e^{\pi} - 1.$$

Along AB : $x=1$, ~~x~~ $y=0$ to 1
$dx = 0$
$dz = dx + i\,dy$
$\Rightarrow dz = i\,dy$

$f(z) = \pi e^{\pi \bar{z}} = \pi e^{\pi(x-iy)} = \pi e^{\pi(1-iy)}$
$= \pi e^{\pi} \cdot e^{-\pi i y}$.

$$\Rightarrow \int_{AB} f(z)\,dz = \int_{y=0}^{1} (\pi e^{\pi} \cdot e^{-\pi i y})\, i\,dy.$$

$$= \pi i e^{\pi} \int_{y=0}^{1} e^{-\pi i y}\,dy.$$

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$$= \pi i e^{\pi} \left(\frac{e^{-\pi i y}}{-\pi i} \right)_0^1$$

$$\cancel{=} \cancel{\pi e}$$

$$= -e^{\pi} \left(e^{-\pi i} - 1 \right)$$

$$= -e^{\pi} \left(-1 - 1 \right)$$

$$\Rightarrow \int_{AB} f(z)\,dz = 2e^{\pi}$$

Now Along BD: $y = 1,\ x = 1$ to 0.

$$\Rightarrow dy = 0$$

$$dz = dx.$$

$$\int_{BD} f(z)\,\cancel{dz} = \pi e^{\pi(x - iy)} = \pi e^{\pi(x - i)}$$

$$= \pi e^{\pi x} \cdot \cancel{e}\, e^{-\pi i}$$

$$= \pi e^{\pi x} (-1)$$

$$= -\pi e^{\pi x}$$

$$\Rightarrow \int_{BD} f(z)\,dz = \int_{x=1}^{0} \left(-\pi e^{\pi x} \right) dx$$

$$= -\cancel{\pi} \left(\frac{e^{\pi x}}{\cancel{\pi}} \right)_1^0$$

$$= -(1 - e^{\pi})$$

$$= e^{\pi} - 1$$

Along DO: $x = 0,\ y = 1$ to 0.

$$dz = i\,dy.$$

$$f(z) = \pi e^{\pi(x - iy)} = \pi e^{\pi(-iy)}$$

$$\int_{DO} f(z)\,dz = \int_{y=1}^{0} \pi e^{-\pi i y} \cdot (i\,dy)$$

$$= i\pi \int_{y=1}^{0} e^{-i\pi y}\,dy = \cancel{i\pi} \left(\frac{e^{-i\pi y}}{-\cancel{i\pi}} \right)_1^0$$

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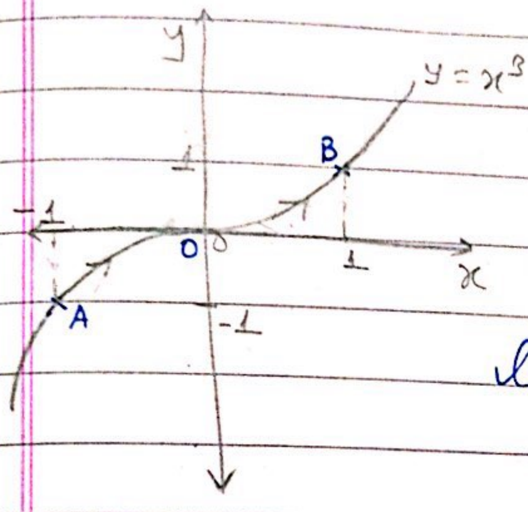
$$= -1 - e^{-i\pi}$$
$$= -1 - (+1)$$
$$= -2.$$

So, $\int_C f(z)\,dz = (e^{\pi}-1) + 2e^{\pi} + (e^{\pi}-1) \cancel{+2} - 2$

$= 4e^{\pi} - 4 = 4(e^{\pi}-1)$

Ans

(4)



Here,

$$\int_C f(z)\,dz = \left(\int_{AO} + \int_{OB}\right) f(z)\,dz \rightarrow (1)$$

Along AO:

$y = x^3$, $x = -1$ to 0

$dy = 3x^2\,dx$

$dz = dx + i\,dy = dx + i(3x^2)dx$

$= (1 + 3ix^2)\,dx$

$f(z) = 1$

$\Rightarrow \int_{AO} f(z)\,dz = \int_{x=-1}^{0} 1.(1+3ix^2)\,dx$

$= \left(x + ix^3\right)_{-1}^{0}$

$= -(-1 - i) = 1 + i.$

Along OB: $y = x^3$, $x = 0$ to 1.

$dy = 3x^2\,dx$

$dz = dx + i\,dy = dx + i(3x^2)dx$

$= (1 + 3x^2 i)\,dx.$

$$f(z) = 4y = 4x^3.$$

$$\int_{OB} f(z)\,dz = \int_{x=0}^{1} (4x^3)(1+3x^2 i)\,dx$$

$$= \left[x^4 + \frac{\overset{2}{\cancel{12}}}{\cancel{5}} x^6 i\right]_0^1$$

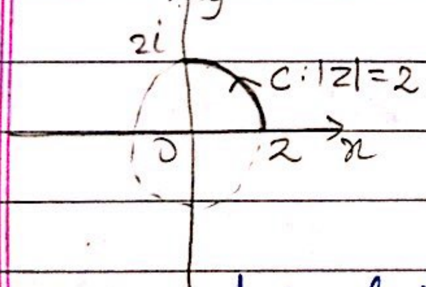
$$= 1 + 2i$$

So, $$\int_C f(z)\,dz = (1+i) + (1+2i)$$
$$= 2 + 3i$$ Ans

Q. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$, lying in the first quadrant. Then, show that

$$\left| \int_C \frac{dz}{z^2-1} \right| \leq \frac{\pi}{3}.$$

(By Δ inequality, we have $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$)



By Δ inequality,

$$\left| \int_C f(z)\,dz \right| \leq ML \longrightarrow (1).$$

Let $f(z) = \dfrac{1}{z^2-1}$

Consider

$$|z^2-1| \geq \left||z^2|-|1|\right|$$

$$\Rightarrow |z^2-1| \geq |2^2-1|$$

$$= |z^2-1| \geq 3.$$

$$\Rightarrow \frac{1}{|z^2-1|} \leq \frac{1}{3}.$$

$$\Rightarrow \left|\frac{1}{z^2-1}\right| \leq \frac{1}{3}$$

$$\Rightarrow |f(z)| \leq M \quad ; M = \frac{1}{3}$$

L = Length of ~~at~~ C

$$= \frac{1}{4}(\text{Circumference of } |z|=2)$$

$$= \frac{1}{4}(2\pi \times 2) = \pi.$$

$\therefore$ from (1),

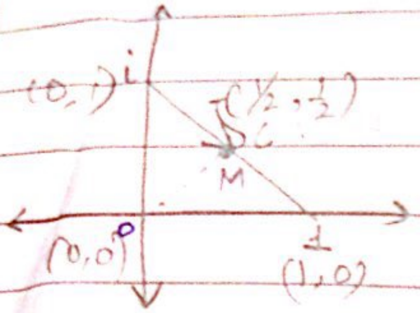
$$\left|\int_C f(z)\,dz\right| \leq \pi \cdot \frac{1}{3}.$$

$$\Rightarrow \left|\int_C f(z)\,dz\right| \leq \frac{\pi}{3}$$

H.P

Q. Let C denote the line segment from $z=i$ to $z=1$. Show that

$$\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}.$$



By ML inequality, we have

$$\left| \int_C f(z)\, dz \right| \le ML \quad \text{--- (1)}$$

~~Let $f(z) = \frac{1}{z^4}$~~

Of all the pts. on the line C, the mid pt. M is closest to the origin.

$\therefore |z| = OM = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Let $f(z) = \frac{1}{z^4}$

$\forall\ z$ on C, $|z| \geq \frac{1}{\sqrt{2}}$.

$\Rightarrow |z^4| \geq \left(\frac{1}{\sqrt{2}}\right)^4$

$\Rightarrow |z^4| \geq \frac{1}{4}$

$\Rightarrow \left|\frac{1}{z^4}\right| \leq 4. (= M)$.

So, $|f(z)| \leq 4 (= M)$.

L = Length of C
$= \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$.

$\therefore$ from (1),

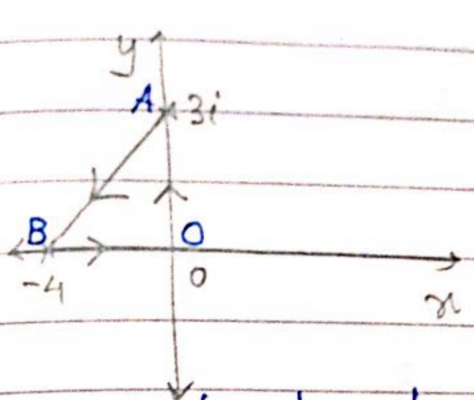
$$\left| \int_C f(z)\, dz \right| \leq 4 \cdot \sqrt{2}.$$

$$\Rightarrow \left| \int_C \frac{dz}{z^4} \right| \leq 4\sqrt{2}. \quad \text{Ans}$$

Q. If C is the boundary of the Δ, with vertices $0, 3i, -4$, oriented in the $\circlearrowleft$ dir^n, then, Show :-

$$\left|\int_C (e^z - \bar{z})\, dz\right| \le 60.$$

$$\left|\int_C f(z)\,dz\right| \le ML \to (1)$$



Let $f(z) = e^z - \bar{z}$

$|f(z) = |e^z - \bar{z}|$

$\le |e^z| + |\bar{z}|$.

Consider $|e^z| = |e^{x+iy}|$

$= |e^x||e^{iy}|$

$= e^x \sqrt{\cos^2 y + \sin^2 y}$

$= e^x$

$\Rightarrow |e^z| \le 1$ as $e^0 = 1$ is max value when $x = -4$ to 0.

$|\bar{z}| = |z| \le 4$ on C ($z = -4$ is farthest from origin).

$\therefore |f(z)| \le 1+4 \Rightarrow |f(z)| \le 5 (= M)$.

L = Length of c

= Length $(OA + AB + BO)$

$OA = 3$, $AB = \sqrt{3^2+4^2} = 5$; $BO = 4$

$\therefore L = 3+4+5 = 12$.

So, from ①

$$\left|\int_C f(z)\,dz\right| \leq 5.12$$

$$\Rightarrow \left|\int_C [e^z - \bar{z}]\,dz\right| \leq 60 \qquad \underline{\text{H.P}}$$

☆ ANTI DERIVATIVE

Let f be a complex valued f^n, cts at all pts in some domain s.t.

$$F'(z) = f(z).$$

Then, F is called an antiderivative of f & hence, we have

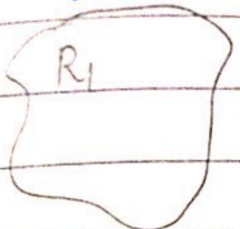
$$\int_C f(z)\,dz = F(z)\Big]_{z=z_1}^{z_2}$$

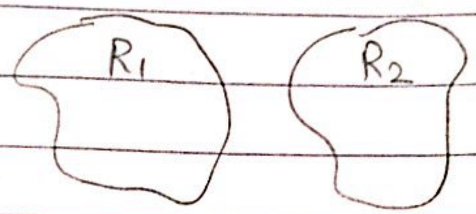
$$= F(z_2) - F(z_1);$$

where C is the curve joining z_1 & z_2.

✡ Connected domain:

A domain D in the z plane is said to be connected, if, some curve joining any 2 pts. in the region, completely lies within the region.

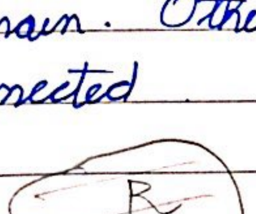


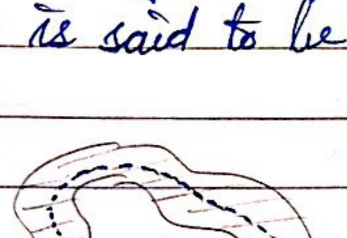


connected

$R_1 \cup R_2$: not connected

A connected domain is said to be simply connected, if, any simple closed curve lying within the region includes <u>only</u> the pts. of the domain. Otherwise, it is said to be multiply connected.





simply connected

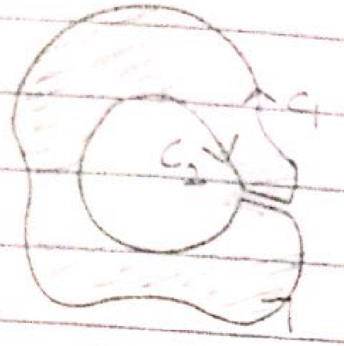
multiply connected

In other words, any domain without a hole is simply connected.

A domain with holes is multiply connected.

Note:

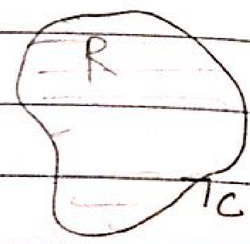
A multiply connected domain can be made into a simply connected domain by introducing STRIP CUTS, as shown in the diagram:



Simply connected region

✶ CAUCHY-GOURSAT THEOREM (or) CAUCHY INTEGRAL THEOREM

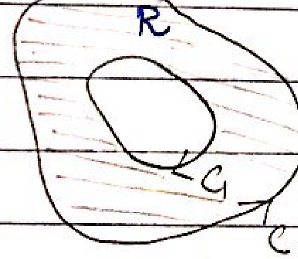
If $f(z)$ is analytic at all pts. within & on a simple closed curve (SCC), C, then



$$\int_C f(z)dz = 0$$

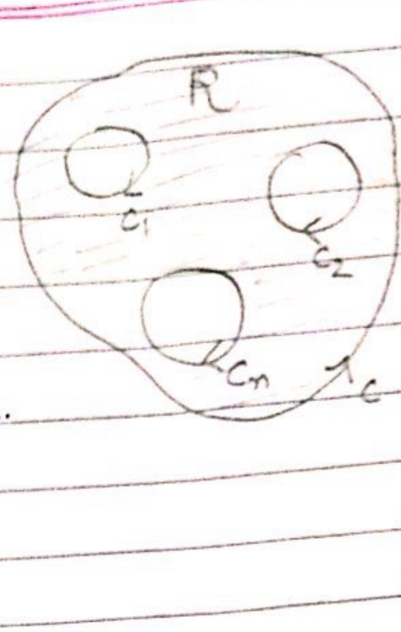
We can extend the Cauchy's integral theorem to multiply connected domains, as follows:

Result 1) If $f(z)$ is analytic at all pts. in a multiply connected domain R, bounded by 2 SCC, C & C_1, as shown in diagram, then



$$\int_C f(z)dz = \int_{C_1} f(z)\,dz.$$

Result 2) If f is analytic at all pts. in a multiply connected domain R, bounded by the SCCs $C, C_1, C_2, C_3, \ldots, C_n$, then,



$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz.$$

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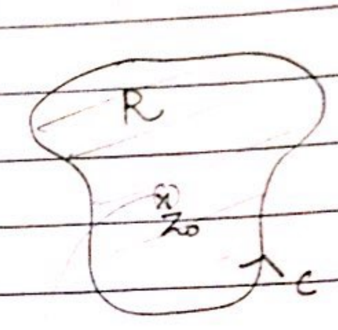
* CAUCHY'S INTEGRAL FORMULA :-

If f is analytic at all pts. within & on a SCC, C, & z_0 is any interior pt., then,



$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\, dz} \rightarrow (1)$$

some interior pt.

↳ Cauchy's Integral formula

In addition, if f has derivatives upto order n, then,

n^{th} derivative

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz} \; ; \; n = 1, 2, \ldots$$

Note: While doing problems, we proceed as follows:

(1) $\int_C f(z)\,dz = 0$, if f is analytic at all pts. within & on C.

Valid only when only one singular pt inside C (Otherwise, partial fractions are used)

(2) $\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$, by Cauchy's integral formula.

(3) $\displaystyle\int_C \frac{f(z)}{(z-z_0)^n}\,dz = 2\pi i \frac{f^{(n-1)}(z_0)}{(n-1)!}$; $n = 1, 2, \ldots$

, using Cauchy's integral formula for derivatives

—x—.

Q. Apply Cauchy's theorem to show that

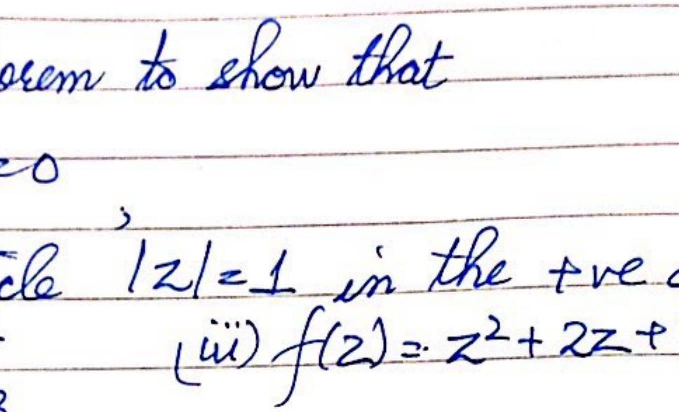
$$\int_C f(z)\,dz = 0,$$

where C is the circle $|z| = 1$ in the +ve dir^n,

(i) when $f(z) = \dfrac{z^2}{z-3}$. (iii) $f(z) = z^2 + 2z + 3$.

(ii) $f(z) = \dfrac{1}{z^2 + 2z + 2}$

(i) here, $f(z) = \dfrac{z^2}{z-3}$



The singular pts. are given by

$z - 3 = 0 \Rightarrow \boxed{z = 3}$

$\hookrightarrow$ Order $= 1$.

The singularity $z = 3$ lies outside c.

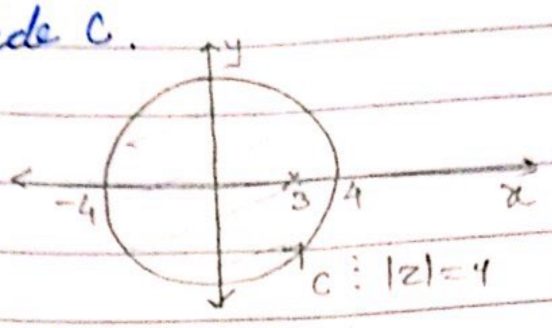
Hence, f is analytic at all pts. within & on C.

$\therefore$ By Cauchy's integral theorem

$\displaystyle\int_C f(z)\,dz = 0$ or $\displaystyle\int_C \frac{z^2}{z-3}\,dz = 0$.

* Let C be the circle $|z|=4$, then the singularity $z=3$ lies inside C.

$\therefore$ Let $\frac{g(z)}{z-3} = f(z) = \frac{z^2}{z-3}$.



$\Rightarrow g(z) = z^2$

$g(z_0) = g(3) = 3^2 = 9$.

$\therefore$ By Cauchy's Integral formula,

$$\int_C f(z)\,dz = \int_C \frac{z^2}{z-3}\,dz = \int_C \frac{g(z)}{z-3}\,dz$$

$$= 2\pi i\,(g(z_0))$$

$$= 2\pi i\,(g(3))$$

$$= 2\pi i\,(9)$$

$$= 18\pi i$$

(ii) $f(z) = \frac{1}{z^2+2z+2}$

Singular pts are given by

$$z^2+2z+2=0$$

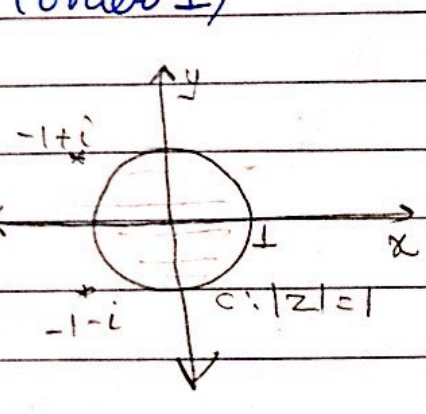
$$\Rightarrow (z+1)^2+1=0$$

$$\Rightarrow z+1 = \pm i$$

$$\Rightarrow z = -1 \pm i \quad \text{(order 1)}$$

Both the singularities lie outside the unit circle $|z|=1$.

$\therefore$ f is analytic at all pts. inside & on C.



$\therefore$, By Cauchy's Integral theorem, we have

$$\int_C f(z)dz = 0.$$

$$\Rightarrow \int_C \frac{dz}{z^2+2z+2} = 0.$$

(iii) $f(z) = z^2 + 2z + 3$

Here f is analytic at all pts. in z-plane. So, its analytic within & on the curve $C : |z|=1$).

$\therefore$ By Cauchy's Integral thm,

$$\int_C f(z)dz = 0.$$

$$\Rightarrow \int_C (z^2+2z+3)dz = 0$$

Impt

Note: By Cauchy's integral formula, we can evaluate an integral only when the integrand has one singularity inside C.

If $\exists$ more than 1 singularity inside C, we use the method of partial fractions & then proceed as above.

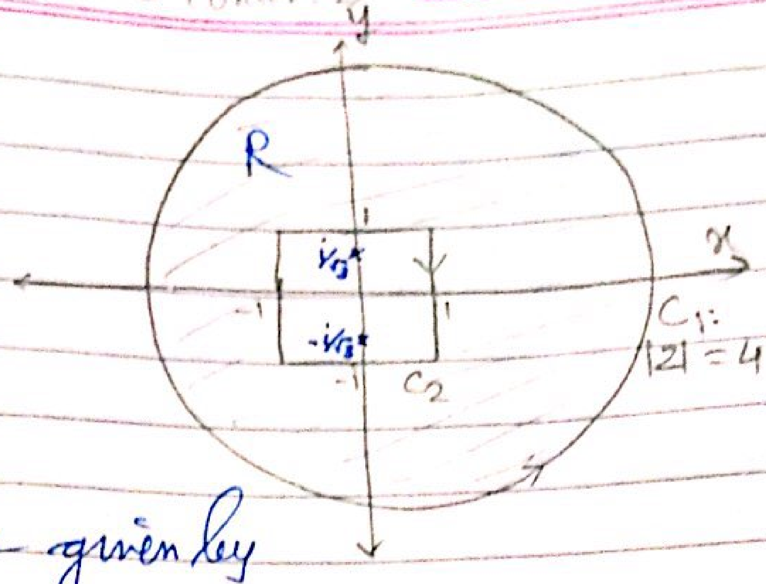
Q. Let C_1 be a positively oriented circle $|z|=4$. & C_2 be a +vely oriented boundary of the square, whose sides lie along the lines $x = \pm 1$, $y = \pm 1$. Show that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz. \text{ when}$$

* Order: Multiplicity of root; i.e, how many times @ root has come.
eg: $\sin(z) = 0$ (order = 1) $\sin^2 z = 0$ (order: 2)

① $f(z) = \dfrac{1}{3z^2+1}$

② $f(z) = \dfrac{z+2}{\sin(z/2)}$



① $f(z) = \dfrac{1}{3z^2+1}$

The singularities are given by

$$3z^2+1 = 0$$

$$\Rightarrow z^2 = \frac{-1}{3} \Rightarrow z = \pm\frac{i}{\sqrt{3}} \quad \text{(order = 1)}$$

Both the singularities lie outside the region.

$\therefore$ f is ~~anal~~ analytic at all pts in the multiply connected region R, bounded by C_1 & C_2.

$\therefore$ By the extension of Cauchy's integral thm, we have

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

② $f(z) = \dfrac{z+2}{\sin(z/2)}$

The singularities are given by

$$\sin\left(\frac{z}{2}\right) = 0 \Rightarrow \frac{z}{2} = n\pi \; ; \; n \in \mathbb{Z}$$

$$\Rightarrow z = 2n\pi \quad \text{(order 1)}$$

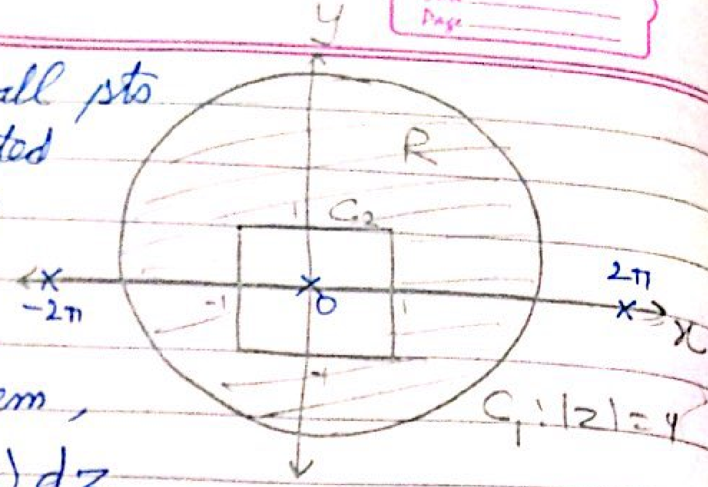
$$= 0, \pm 2\pi, \pm 4\pi, \ldots$$

Here, all the singularities lie outside the given region.

$\therefore f$ is analytic at all pts in the multiply connected region R, bounded by C_1 & C_2.

$\therefore$ By extension of Cauchy's integral theorem,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$



Q. Evaluate

① $\displaystyle\int_C \frac{z\,dz}{(9-z^2)(z+i)}$; (i) $|z| = 2$ (iv) $|z-3| = 1$
(ii) $|z+i| = \frac{1}{2}$.
(iii) $|z| = \frac{1}{4}$.

Let $f(z) = \dfrac{z}{(9-z^2)(z+i)}$

The singularities are given by

$(9-z^2)(z+i) = 0$.

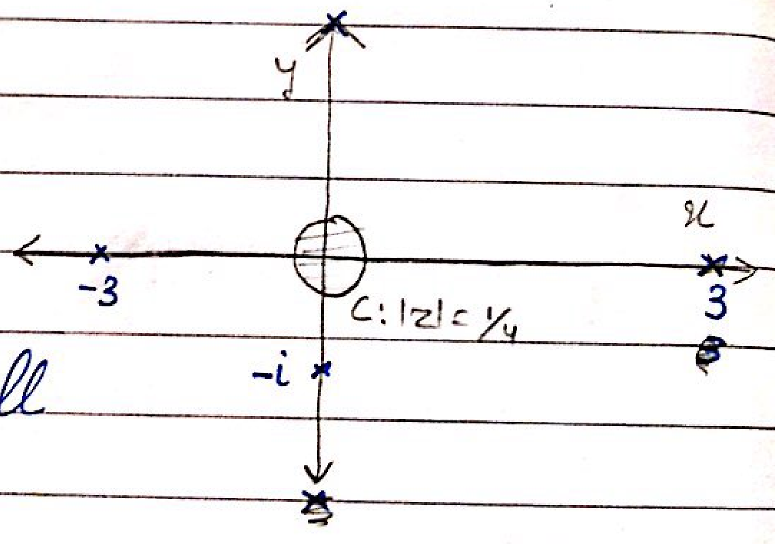
$\Rightarrow 9 - z^2 = 0$ or $z + i = 0$

$\Rightarrow z = \pm 3., -i$ (order = one)

(iii) $C : |z| = \frac{1}{4}$

All these singularities lie outside C.

$\therefore f$ is analytic at all pts. inside & on C.

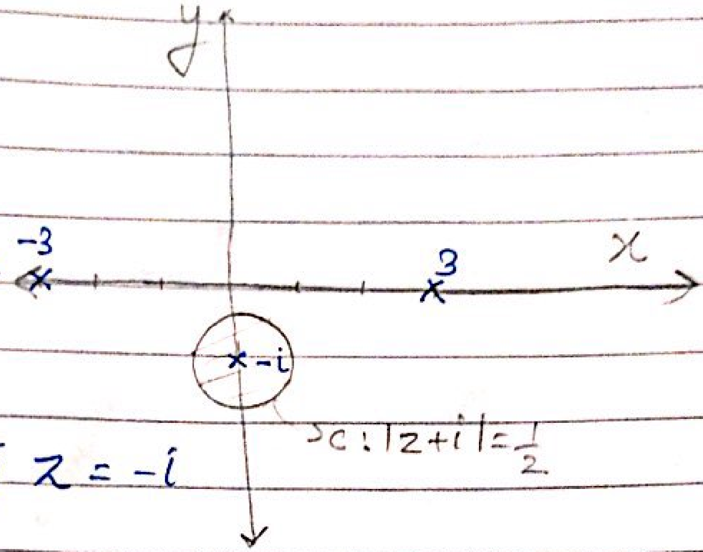


$\therefore$ By Cauchy's integral theorem,

$$\int_c f(z)\,dz = 0$$

$$\Rightarrow \int_c \frac{z\,dz}{(9-z^2)(z+i)} = 0.$$

(ii) $|z+i| = \frac{1}{2}$



Here, C is the circle $|z+i| = \frac{1}{2}$ with centre at $(0, -1)$ and radius $\frac{1}{2}$ units.

Only the singularity $z = -i$ lies inside C.

Let $f(z) = \frac{\phi(z)}{z - z_0}$

$$\frac{z}{(9-z^2)(z+i)} = \frac{\phi(z)}{(z-(-i))}$$

$$\Rightarrow \frac{z}{(9-z^2)\cancel{(z+i)}} = \frac{\phi(z)}{\cancel{(z+i)}}$$

$$\Rightarrow \phi(z) = \frac{z}{9-z^2}.$$

$$\phi(z_0) = \phi(-i) = \frac{-i}{9+1} = \frac{-i}{10}$$

$\therefore$ By Cauchy's integral formula,

$$\int_c f(z)\,dz = \int_c \frac{\phi(z)}{z - z_0}\,dz$$

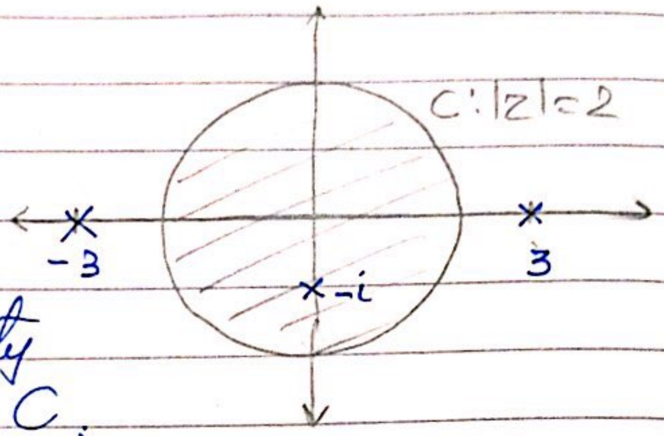
$$= \int_C \frac{\phi(z)}{z+i}\, dz$$

$$= 2\pi i\ \phi(-i)$$

$$= 2\pi i \left(\frac{-i}{10}\right)$$

$$\Rightarrow \int_C f(z)\,dz = \frac{\pi}{5}.$$

(ii) Here, $C : |z| = 2$
Center : origin
Radius : 2 units.



The only singularity
$z = -i$ lies inside C.
~~So~~ Let $f(z) = \frac{\phi(z)}{z - z_0}$

$$\Rightarrow \phi(z) = \frac{z}{9-z^2} \quad \text{(done in previous part)}$$

$$\text{So, } \phi(z_0) = \frac{-i}{10}.$$

By Cauchy's integral formula,

$$\int_C f(z)\,dz = \int_C \frac{\phi(z)}{z+i}\, dz$$

$$= 2\pi i\, \phi(-i)$$

$$= 2\pi i \left(\frac{-i}{10}\right)$$

$$\Rightarrow \int_C f(z)\,dz = \frac{\pi}{5} \quad \text{Ans}$$

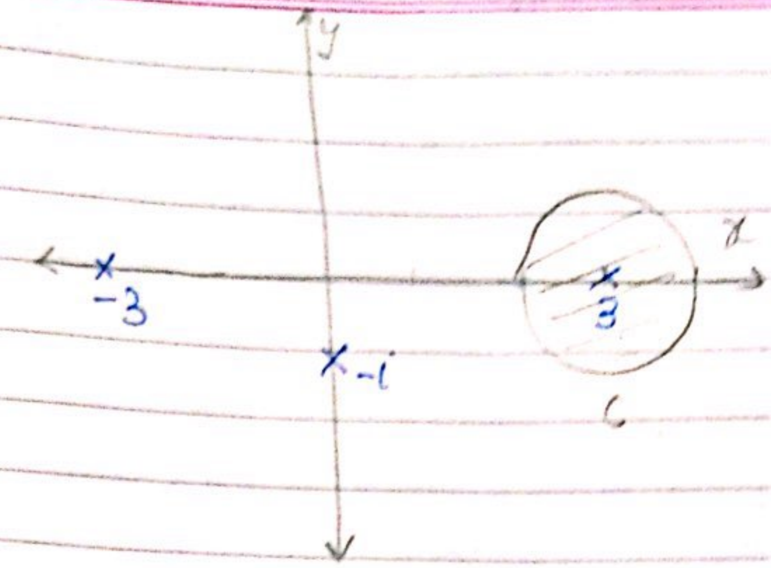
(iv) $C : |z-3| = 1$

Here C is a circle

Centre: $z = 3$

Radius: 1 unit

Only the singularity $z = 3$ lies inside C.



Let $f(z) = \dfrac{\phi(z)}{z-3}$

$$\Rightarrow \frac{z}{(9-z^2)(z+i)} = \frac{\phi(z)}{z-3}$$

$$\Rightarrow \phi(z) = \frac{-z}{(3+z)(z+i)}$$

$$\phi(z_0) = \phi(3) = \frac{-3}{(3+3)(3+i)}$$

$$= \frac{-1}{2(3+i)}$$

$$= \frac{-(3-i)}{10(2)} = \frac{i-3}{20}.$$

$\therefore$ By Cauchy's integral form,

$$\int_C f(z)\,dz = \int_C \frac{\phi(z)}{z-z_0}\,dz$$

$$= 2\pi i\,\phi(3)$$

$$= 2\pi i\left(\frac{i-3}{20}\right)$$

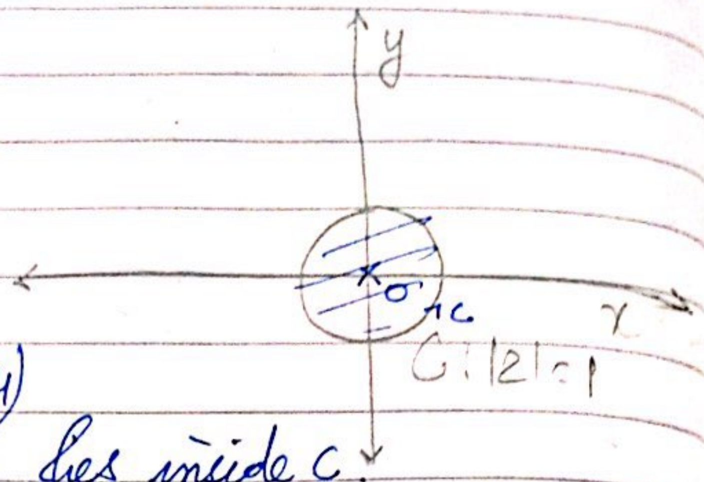
$$\Rightarrow \int_C f(z)\,dz = \frac{\pi i}{10}(i-3)$$

Ans.

Q. Find the value of the integral

① $\int_C \frac{\exp(2z)}{z^4}\,dz$; $C : |z| = 1$.

Let $f(z) = \frac{e^{2z}}{z^4}$



The singularities are given by

$z^4 = 0$

$\Rightarrow z = 0$ (order = 4)

The singularity, $z=0$ lies inside C.

Let $f(z) = \frac{\phi(z)}{(z-0)^4}$.

$\Rightarrow \frac{e^{2z}}{z^4} = \frac{\phi(z)}{z^4}$

$\Rightarrow \phi(z) = e^{2z}$.

$\Rightarrow \phi'(z) = 2e^{2z}$

$\phi''(z) = 4e^{2z}$

$\phi'''(z) = 8e^{2z}$ (Derivative to be found upto order − 1)

$\phi'''(z_0) = \phi''(0)$

$= 8e^0 = 8$

By Cauchy's integral formula,

$\int_C f(z)\,dz = \int_C \frac{\phi(z)}{z^4}\,dz$

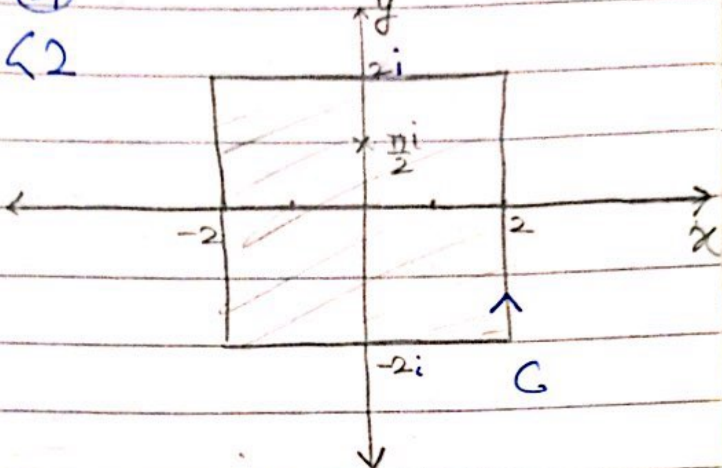
$= 2\pi i \frac{\phi'''(z_0)}{3!}$

$= 2\pi i \frac{(8)}{6} = \frac{8\pi i}{3}$ Ans

Q. Let C denote the +vely oriented boundary of the square whose sides lie along the lines $x = \pm 2$, $y = \pm 2$. Find

(1) $\int_c \frac{e^{-z}}{\left(z - \frac{\pi i}{2}\right)} dz$. (2) $\int_c \frac{\cos z}{z(z^2+8)} dz$.

(3) $\int_c \frac{\tan(z/2)}{(z-x_0)^2} dz$, $-2 < x_0 < 2$ (4)



(1) $f(z) = \frac{e^{-z}}{z - \frac{\pi i}{2}}$

The singularity is given by
$z - \frac{\pi i}{2} = 0 \Rightarrow z = \frac{\pi i}{2}$ (order = 1)

The singularity $z = \frac{\pi i}{2}$ lies inside c

Let $f(z) = \frac{\phi(z)}{z - z_0}$

$\Rightarrow \frac{e^{-z}}{z - \frac{\pi i}{2}} = \frac{\phi(z)}{z - \frac{\pi i}{2}}$ $\therefore \phi(z) = e^{-z}$.

$\therefore \phi(z_0) = \phi\left(\frac{\pi i}{2}\right) = e^{-i\pi/2} = \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)$
$= -i$

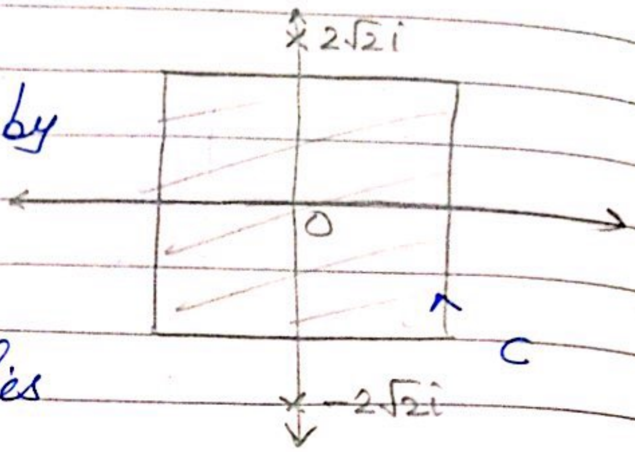
$\therefore$ By ~~Gaut~~ Cauchy's integral formula,

$\int_c f(z)\,dz = \int_c \frac{\phi(z)}{z - \frac{\pi i}{2}} dz$.

$$= 2\pi i\, \phi\left(\frac{\pi i}{2}\right)$$

$$= 2\pi i(-i) = 2\pi$$

(2) $f(z) = \dfrac{\cos z}{z(z^2+8)}$.



The singularity is given by

$z(z^2+8) = 0$

$\Rightarrow z = 0, \pm 2\sqrt{2}i$

(order 1)

The singularity $z = 0$ lies inside C

Let $f(z) = \dfrac{\phi(z)}{z - z_0}$

$\Rightarrow \dfrac{\cos z}{\cancel{z}(z^2+8)} = \dfrac{\phi(z)}{\cancel{z}}$

$\Rightarrow \phi(z) = \dfrac{\cos z}{z^2+8}$

$\Rightarrow \phi(z_0) = \phi(0) = \dfrac{1}{0+8} = \dfrac{1}{8}$.

By cauchy's integral formula

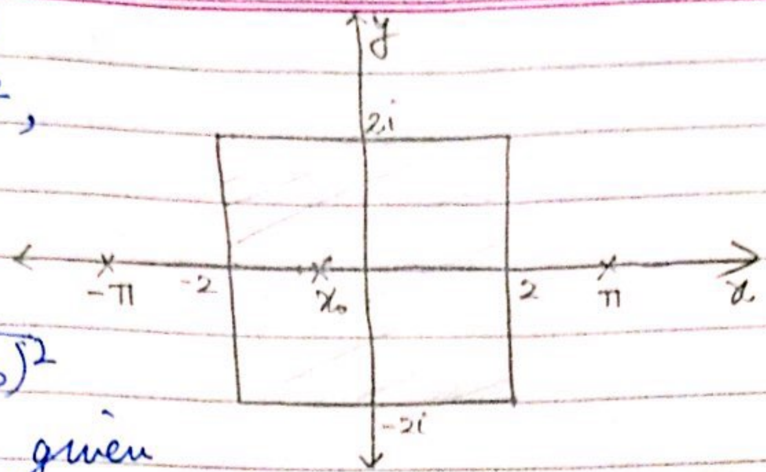
$$\int_C f(z)\,dz = \int_C \frac{\phi(z)}{z}\,dz$$

$$= 2\pi i\, \phi(z_0)$$

$$\Rightarrow \int_C f(z)\,dz = \frac{2\pi i}{8} = \frac{\pi i}{4}.$$

(3) $f(z) = \dfrac{\tan(z/2)}{(z-\cancel{z}x_0)^2}$,

$-2 < x_0 < 2$



$* = \dfrac{\sin(z/2)}{\cos(z/2)(z-z_0)^2}$

The singularities are given by

$\cos(z/2)(z-x_0)^2 = 0$

$\Rightarrow \cos\left(\dfrac{z}{2}\right) = 0$ or $(z-x_0)^2 = 0$

$\Rightarrow \dfrac{z}{2} = (2n+1)\dfrac{\pi}{2} \Rightarrow z = (2n+1)\pi$ or $z = x_0$

(order = 1) (order = 2)

The singularity $z = x_0$ lies inside C.

Let $f(z) = \dfrac{\phi(z)}{z-z_0}$

$\Rightarrow \dfrac{\sin(z/2)}{\cos(z/2)\cancel{(z-x_0)^2}} = \dfrac{\phi(z)}{\cancel{(z-x_0)^2}}$

$\Rightarrow \phi(z) = \dfrac{\sin(z/2)}{\cos(z/2)} = \tan(z/2)$

$\Rightarrow \phi'(\cancel{z_0}) = \sec^2\left(\dfrac{z}{2}\right)\left(\dfrac{1}{2}\right)$

$\Rightarrow \phi'(z_0) = \phi'(x_0) = \dfrac{1}{2}\sec^2\left(\dfrac{x_0}{2}\right)$

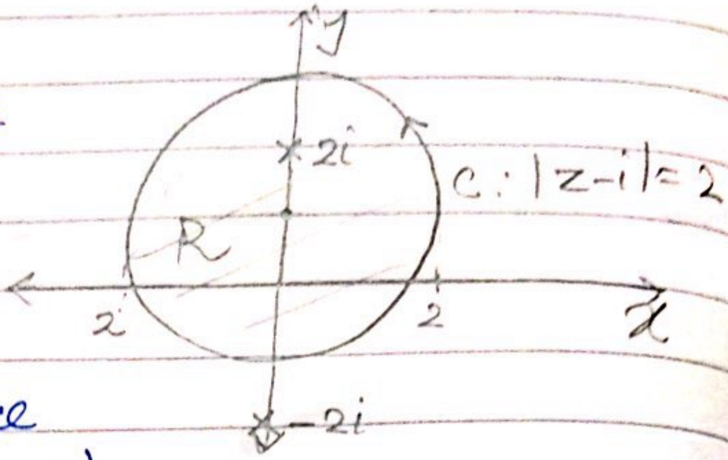
$\therefore \int_C f(z)\,dx = \int_C \dfrac{\phi(z)}{(z-x_0)^2}\,dz$

$= (2\pi i)\dfrac{\phi'(z_0)}{1!} = \cancel{2}\pi i\left(\dfrac{1}{\cancel{2}}\sec^2\left(\dfrac{x_0}{2}\right)\right)$

$= \pi i \sec^2(x_0/2),\ x_0 \in (-2,2)$

Q. Find the value of the integral of $g(z)$ around the circle $|z-i|=2$ in the +ve sense, when
(i) $g(z) = \frac{1}{z^2+4}$ (ii) $g(z) = \frac{1}{(z^2+4)^2}$

(ii) Here, $g(z) = \frac{1}{(z^2+4)^2}$



The singularities are given by
$(z^2+4)^2 = 0$
$\Rightarrow z^2 = -4$ twice
$\Rightarrow z = \pm 2i$ (order 2)
Here, c ; $|z-i| = 2$
Centre : i
radius = 2 units.
Only the singularity $z = 2i$ lies inside c.

$g(z) = \frac{\phi(z)}{(z-2i)^2}$

$\Rightarrow \frac{1}{(z^2+4)^2} = \frac{\phi(z)}{(z-2i)^2}$

$= \frac{1}{(z+2i)^2 \cancel{(z-2i)^2}} = \frac{\phi(z)}{\cancel{(z-2i)^2}}$

$\Rightarrow \phi(z) = \frac{1}{(z+2i)^2}$.

$\cancel{\phi(z_0) = \phi(2i) = \frac{1}{(4i)^2}} = \frac{-1}{16}$

$\phi'(z) = \frac{-2}{(z+2i)^3}$

$\phi'(z_0) = \phi'(2i) = \frac{-2}{(4i)^3} = \frac{-1}{32(-i)} = \frac{1}{32i}$

Therefore, By Cauchy Integral formula,

$$\int_C f(z)\,dz = \int_C \frac{\phi(z)}{(z-2i)^2}.$$

$$= 2\pi i \left(\frac{\phi'(z_0)}{1!} \right)$$

$$= \frac{2\pi i\, \phi'(2i)}{1!}$$

$$= 2\pi i \left(\frac{1}{32i} \right)$$

$$\Rightarrow \int_C f(z)\,dz = \frac{\pi}{16} \quad \text{Ans}$$

(ii) $g(z) = \frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$

The singularities are

$$z^2 + 4 = 0$$

$$\cancel{\Rightarrow z = \pm 2i}$$

$$z = \pm 2i \text{ (order 1)}$$

Only singularity $z = 2i$ lies inside C

$$g(z) = \frac{\phi(z)}{(z-2i)}$$

$$\Rightarrow \frac{1}{(z+2i)\cancel{(z-2i)}} = \frac{\phi(z)}{\cancel{(z-2i)}}$$

$$\Rightarrow \phi(z) = \frac{1}{z+2i}$$

$$\Rightarrow \phi(z_0) = \phi(2i) = \frac{1}{4i}$$

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$\therefore$ By Cauchy's integral formula

$$\int_c f(z)dz = \int_c \frac{\phi(z)}{z-2i}\,dz = 2\pi i(\phi(z_0))$$

$$= 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$$

Q Let C be the circle $|z|=3$; Described in the +ve sense. If

$$g(w) = \int_c \frac{(2z^2-z-2)}{z-w}\,dz, \quad |w| \neq 3.$$

Then, show that

(i) $g(2) = 8\pi i$

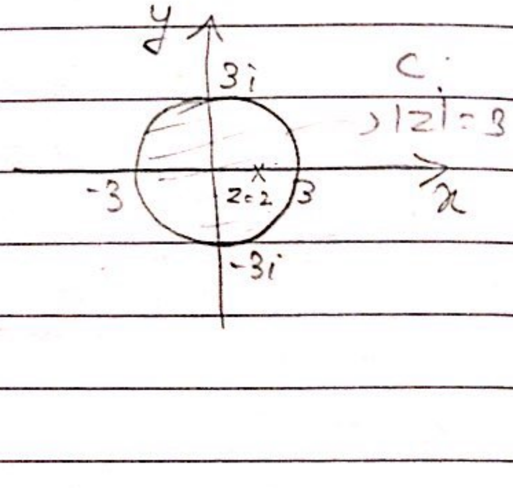
(ii) What is the value of $g(w)$, when $|w|>3$.

(i) Singularities are

$z = w$.

When $w=2$, we have

$$g(2) = \int_c \frac{2z^2-z-2}{z-2}\,dz$$



The singularity $z=2$ lies inside C. $= \int_c \frac{\phi(z)}{z-2}\,dz$

where, $\phi(z) = 2z^2-z-2$.

$\Rightarrow \phi(z_0) = \phi(2) = 2\cdot 2^2-2-2$

$= 4.$

$\therefore$ By Cauchy's integral formula,

$$g(2) = \int_c \frac{\phi(z)}{z-2}\,dz = 2\pi i\,\phi(2) = 8\pi i.$$

(ii) If $|w| > 3$, then, w lies outside C.
$\therefore$ By Cauchy's integral thm,
$g(w) = 0$.

Q. Let C be any simple closed contour described in the +ve sense & write

$$g(w) = \int_C \frac{z^3+2z}{(z-w)^3}\,dz.$$

Show that : $g(w) = \begin{cases} 6\pi i w, & \text{if } w \text{ lies inside } C. \\ 0, & \text{if } w \text{ lies outside } C. \end{cases}$

Here, the singularity is $z = w$ (order 3).

Case 1: Let w lie inside C.

$$g(w) = \int_C \frac{\phi(z)}{(z-w)^3}\,dz \text{ ; where } \phi(z) = z^3+2z.$$

$\Rightarrow \phi'(z) = 3z^2+2$

$\phi''(z) = 6z$

$\phi''(w) = 6w$.

$$= 2\pi i \frac{(\phi''(w))}{2!}$$

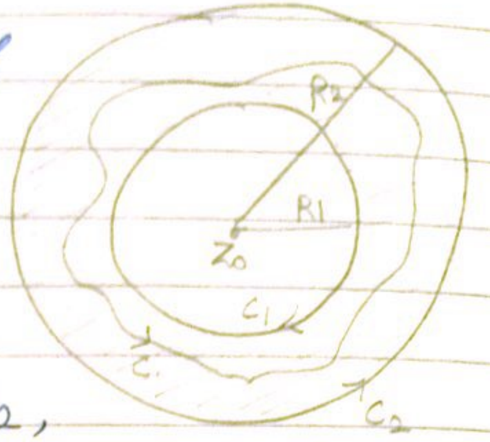
$= 6\pi i w$, by Cauchy's integral formula for derivatives.

TAYLOR'S SERIES

Let $f(z)$ be analytic at all pts. in an open disk, centered at z_0, with finite radius, R. (i.e, $|z-z_0| < R$), then, f can be expanded as an infinite series -

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ ; where } a_n = \frac{f^{(n)}(z_0)}{n!}$$

LAURENT'S SERIES



Let f be analytic at all pts. in an ANNULAR region,

$$R_1 < |z-z_0| < R_2,$$

bounded by 2 concentric circles with radii R_1 & R_2, centered at z_0.

Let C be any simple closed curve in the annular region which includes the pt. z_0. Then, $f(z)$ can be expanded as an infinite series given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \longrightarrow \text{(I)}$$

$$= \left\{ a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \right\} + \left\{ \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots \right\} \longrightarrow \text{(I)}.$$

where,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz \; ; \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} \, dz \; ; \quad n = 1, 2, 3 \dots$$

Note: We use the following formulae to do problems in Laurent's series,

① $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots$

② $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$

③ $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$

④ $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$

⑤ $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$

$$\left.\begin{aligned}
&⑥\ (1+z)^{-1} = 1 - z + z^2 - z^3 + \dots \\
&⑦\ (1-z)^{-1} = 1 + z + z^2 + z^3 + \dots \\
&⑧\ (1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \dots \\
&⑨\ (1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \dots
\end{aligned}\right\}\ \text{Imp: Valid only if } |z|<1$$

Q. Expand f ~~cos~~ as a Laurent's series, valid in the given domains:

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

(i) $|z|<1$

(ii) $1<|z|<2$

(iii) $|z|>2$

(iv) $|z-1|<1$.

Let $f(z) = \frac{A}{z-1} + \frac{B}{z-2} = \frac{-1}{(z-1)(z-2)}$

By partial fraction method,

$\Rightarrow A = 1, \; B = -1$.

$\therefore f(z) = \frac{1}{z-1} - \frac{1}{z-2} \longrightarrow (1)$

$= \cancel{\frac{-1}{1-z} + \frac{1}{}}$

$z-1 = -1(1-z)$ ✓
$= z\left(1-\frac{1}{z}\right)$

(i) Here,

$|z|<1$

$\Rightarrow \left|\frac{z}{2}\right| < \frac{1}{2} < 1$

$z-2 = -2\left(1-\frac{z}{2}\right)$ ✓

$= z\left(1-\frac{2}{z}\right)$

$\Rightarrow \left|\frac{z}{2}\right| < 1$

$\therefore$ (1) becomes

$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$.

$= -1\left(\frac{1}{1-z}\right) + \frac{1}{2}\left(\frac{1}{1-z/2}\right)$

$= (-1)(1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$

$\Rightarrow f(z) = (-1)\left(1+z+z^2+\ldots\right) + \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\right)$

This is the required Laurent's series.

(ii) $1<|z|<2$

$z-1 = -1(1-z)$
$= z\left(1-\frac{1}{z}\right)$ ✓

$\Rightarrow 1<|z| \;\&\; |z|<2$

$\Rightarrow \frac{1}{|z|} < 1 \;\&\; \left|\frac{z}{2}\right| < 1$.

$z-2 = -2\left(1-\frac{z}{2}\right)$ ✓

$= z\left(1-\frac{2}{z}\right)$

$\therefore$ (1) becomes

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z(1-1/z)} - \frac{1}{(-2)(1-\frac{z}{2})}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$\Rightarrow f(z) = \frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\ldots\right) + \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\right)$$

This is the required Laurent's expansion.

(iii) Here, $|z| > 2$ i.e $2 < |z|$

$\Rightarrow \frac{2}{|z|} < 1$

$z-1 = -1(1-z)$

$= z\left(1-\frac{1}{z}\right)$ ✓

$\Rightarrow \left|\frac{1}{z}\right| < \frac{1}{2} < 1$

$z-2 = -2\left(1-\frac{z}{2}\right)$

$= z\left(1-\frac{2}{z}\right)$ ✓

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z(1-\frac{1}{z})} - \frac{1}{z(1-\frac{2}{z})}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$f(z) = \frac{1}{z}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\ldots\right) - \frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\ldots\right)$$

This is the required Laurent's expansion.

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(iv) Here, $|z-1|<1$

Let $u = z-1 \Rightarrow |u|<1$

$\Rightarrow z-2 = u+1-2 = u-1$

$\therefore f(z) = \frac{1}{u} - \frac{1}{u-1}$ $\quad u-1 = u(1-\frac{1}{u})$ ✗

$= \frac{1}{u} - \frac{1}{(-1)(1-u)}$ $\quad = -1(1-u)$ ✓

$= \frac{1}{u} + (1-u)^{-1}$

$= \frac{1}{u} + (1+u+u^2+\dots)$

$\Rightarrow f(z) = \frac{1}{z-1} + (1+(z-1)+(z-1)^2+\dots)$

This is the required Laurent's series.

Q. Find the Laurent's expansion of $f(z) = \frac{e^z}{(z+1)^2}$ in $0<|z+1|<\infty$

Let $u = z+1 \Rightarrow z = u-1$

$\therefore f(z) = \frac{e^{u-1}}{u^2}$

$= \frac{1}{u^2} \cdot e^{-1} \cdot e^{u}$

$= \frac{1}{eu^2} \cdot e^{u} = \frac{1}{eu^2}\left(1+\frac{u}{1!}+\frac{u^2}{2!}+\dots\right)$

$= \frac{1}{e(z+1)^2}\left(1+\frac{z+1}{1!}+\frac{(z+1)^2}{2!}+\dots\right)$

This is the required Laurent's expansion.

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Q. Find 2 Laurent's expansion & state the regions in which those expansions are valid.

$$f(z) = \frac{1}{z^2(1-z)}$$

We shall expand $f(z)$ as Laurent's series, valid in the domains

(i) $|z|<1$ & (ii) $|z|>1$.

(i) Let $|z|<1$

$$f(z) = \frac{1}{z^2(1-z)}$$

$1-z = 1(1-z)$ ✓

$= -z\left(1-\frac{1}{z}\right)$

$$= \frac{1}{z^2}(1-z)^{-1}$$

$$= \frac{1}{z^2}\left(1+z+z^2+z^3+\dots\right)$$

valid in $|z|<1$

(ii) $|z|>1$

$\Rightarrow 1 < |z|$

$1-z = 1-z$

$\Rightarrow \frac{1}{|z|} < 1$

$= -z\left(1-\frac{1}{z}\right)$ ✓

$$\Rightarrow f(z) = \frac{1}{z^2(-z)\left(1-\frac{1}{z}\right)}$$

$$= \frac{-1}{z^3}\left(1-\frac{1}{z}\right)^{-1}$$

$$\Rightarrow f(z) = \frac{-1}{z^3}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\dots\right)$$

valid for $|z|>1$.

These are the required 2 Laurent's series.

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Q. Show that when $0<|z-1|<2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}$$

(i) H.W: Expand the above f^n as Laurent's series valid in the regions
(i) $|z|<1$
(ii) $1<|z|<3$
(iii) $|z|>3$.

Let $f(z) = \frac{z}{(z-1)(z-3)}$

Let $f(z) = \frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$.

$\Rightarrow z = A(z-3) + B(z-1)$

At $z=3 \Rightarrow B = \frac{3}{2}$.

$z=1$, $A = \frac{-1}{2}$.

$\therefore f(z) = \frac{-1}{2(z-1)} + \frac{3}{2(z-3)}$

Let $u = z-1$, $\Rightarrow z = u+1$
$\Rightarrow z-3 = u-2$

Given $0<|z-1|<2$
$\Rightarrow 0<|u|<2$
$\Rightarrow \left|\frac{u}{2}\right|<1$

$\therefore f(z) = \frac{-1}{2u} + \frac{3}{2(u-2)} = \frac{-1}{2u} + \frac{3}{2(-2)(1-\frac{u}{2})}$

$\Rightarrow f(z) = \dfrac{-1}{2u} - \dfrac{3}{4\left(1-\frac{u}{2}\right)}$

$= \dfrac{-1}{2u} - \dfrac{3}{4}\left(1-\dfrac{u}{2}\right)^{-1}$

$\Rightarrow f(z) = \dfrac{-1}{2u} - \dfrac{3}{4}\left(1+\dfrac{u}{2}+\left(\dfrac{u}{2}\right)^2+\cdots\right)$

So, This is the required Laurent's expansion:

$f(z) = \dfrac{-1}{2(z-1)} - \dfrac{3}{4}\left(1+\left(\dfrac{z-1}{2}\right)+\left(\dfrac{z-1}{2}\right)^2+\cdots\right)$

$= \dfrac{-1}{2(z-1)} - 3\left(\dfrac{1}{2^2}+\dfrac{z-1}{2^3}+\cdots\right)$

$\Rightarrow f(z) = -3\left(\displaystyle\sum_{n=0}^{\infty} \dfrac{(z-1)^n}{2^{n+2}}\right) - \dfrac{1}{2(z-1)}$

Ans

Q. Write 2 Laurent's expansion for the f^n:-
$f(z) = \dfrac{1}{z(1+z^2)}$ & state the regions of validity.

For $|z|<1$ $\qquad$ $z = \cancel{\pm i}, 0$

Let $z^2 = u$ $\qquad$ $|z|<1$ & $|z|>1$

$\Rightarrow f(z) = \dfrac{1}{\sqrt{u}(1+u)}$

$= \dfrac{1}{\sqrt{u}}\left(1-u+u^2-u^3+\cdots\right)$

$\Rightarrow f(z) = \dfrac{1}{z}\left(1-z^2+z^4-z^6+\cdots\right)$

This is valid for $|z|<1$
& is the required Laurent's expansion

For $|z|>1$ $\qquad 1+z^2 = 1+z^2$ X

$$f(z) = \frac{1}{z(1+z^2)} \qquad = z^2\left(1+\frac{1}{z^2}\right)$$

$$= \frac{1}{z(z^2)(1+1/z^2)}$$

$$= \frac{1}{z^3\left(1+\frac{1}{z^2}\right)}$$

$$= \frac{1}{z^3}\left(1+\frac{1}{z^2}\right)^{-1}$$

$$\Rightarrow f(z) = \frac{1}{z^3}\left(1-\frac{1}{z^2}+\left(\frac{1}{z^2}\right)^2-\left(\frac{1}{z^2}\right)^3+\dots\right)$$

This is the required Laurent's expansion, valid for $|z|>1$

§ ★ POLES, RESIDUES

★ *Isolated Singularity*:
A singular pt. z_0 is said to be an isolated singularity, if, the f^n f is analytic at all pts. in the deleted neighbourhood of z_0,
i.e, $0<|z-z_0|<\delta$.

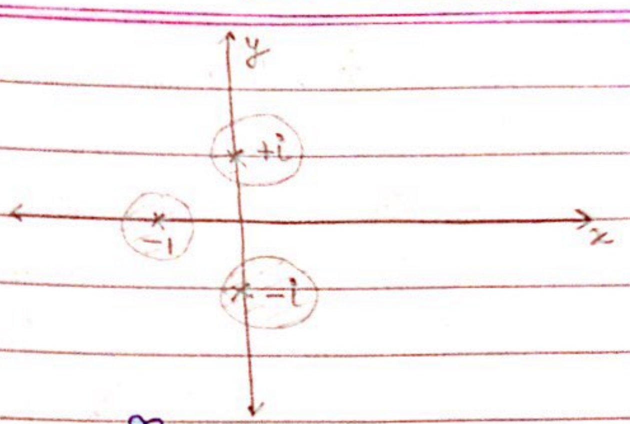
eg① Let $f(z) = \dfrac{z^2}{(z+1)^3(z^2+1)}$

The singular pts. are:-

$$(z+1)^3(z^2+1)=0$$

$$\Rightarrow z = \underset{(\text{order } 3)}{-1\ (\text{thrice})},\ \underset{(\text{order } 1)}{\pm i}$$

All these singularities are isolated singularities.



* POLES:

Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \longrightarrow ①$

be the Laurent's expansion of $f(z)$ about an isolated singular pt. z_0.

In the expansion ①, the first part

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is called the analytic part.

, the second part

$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ is called the principal part.

(i) Removable singularity:

If, in the Laurent's expansion ①, $\exists$ no principal part (all $b_n = 0$), then z_0 is said to be a removable singularity.

~~i.e.~~ ~~eg:~~

eg. ② $z=0$ is a removable singularity of the f^n

$$f(z) = \frac{\sin z}{z}$$

$$= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

(no -ve powers of z)

(ii) Pole of order m :
If the principal part contains only the first m terms, then z_0 is said to be a pole of order m.
In this case, the Laurent's expansion — ① would be

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}$$

In the previous eg. ①, $z=-1$ is a triple pole (pole of order 3) & $z = \pm i$ are simple poles (pole of order 1).

(iii) Essential singularity :
If, in the Laurent's expansion :- ①, the principal part doesn't terminate, then z_0 is called an essential singularity.

eg ③ : $z=4$ is an essential singularity of the f^n : $f(z) = e^{\frac{1}{z-4}}$

$$= 1 + \frac{(1/z-4)}{1!} + \frac{(1/z-4)^2}{2!} + \dots$$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z-4}\right) + \frac{1}{2!}\left(\frac{1}{z-4}\right)^2 + \dots$$

(-ve powers of $z-4$ don't end)

Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m} + \dots$

be the Laurent's expansion of $f(z)$ about an isolated singular pt. z_0. Then,

b_1 : coeff. of $\frac{1}{z-z_0}$ in the Laurent's expansion.

b_1 : called as the residue of $f(z)$ & we write

$$\boxed{b_1 = \text{Res}\{f(z)\}_{z=z_0}}$$

By defin^n, $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$

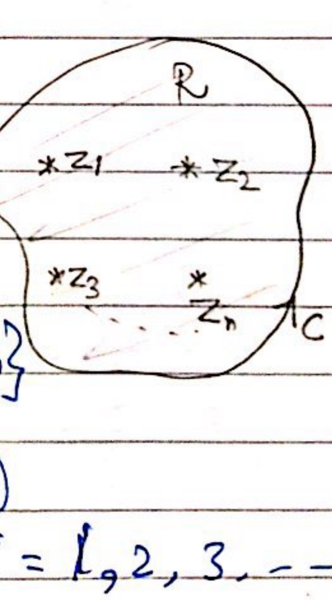
$$\Rightarrow b_1 = \frac{1}{2\pi i}\int_C \frac{f(z)}{\cancel{z}}\,dz$$

$$\Rightarrow \int_C f(z)\,dz = 2\pi i\, b_1$$

$$= 2\pi i\, \text{Res}\{f(z)\}_{z=z_0}$$

CAUCHY's RESIDUE THEOREM

Let f be analytic at all pts. inside an on a SCC : C, except at the isolated singularities $z_1, z_2, \dots z_n$ which lie within C.



Then,

$$\int_C f(z)\,dz = 2\pi i\{R_1 + R_2 + \dots + R_n\}$$

$$= 2\pi i\left(\sum R\right) \longrightarrow (1)$$

, where $R_i = \text{Res}\{f(z)\}_{z=z_i}$; $i = 1, 2, 3, \dots, n$

& $\sum R$ = sum of residues.

eqn ① is the Cauchy's Residue thm.

* Formulae to calculate the residues
(can fail sometimes).

(I) Let $z = z_0$ be a simple pole (pole of order 1).

(i) $\text{Res}\{f(z)\}_{z=z_0} = \phi(z_0)$

, where $\frac{\phi(z)}{z-z_0} = f(z)$.

(ii) Let $f(z) = \frac{P(z)}{Q(z)}$

Then, $\text{Res}\{f(z)\}_{z=z_0} = \frac{P(z_0)}{Q'(z_0)}$

(II) Let z_0 be a pole of order m.

Then, $f(z) = \frac{\phi(z)}{(z-z_0)^m}$,

where $\phi(z) \neq 0$ & ϕ is analytic at z_0.

$$\text{Res}\{f(z)\}_{z=z_0} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad ; m = 1, 2, \ldots$$

Q Find the residue at $z=0$ for the fns

① $\frac{1}{z+z^2}$

② $z\cos\left(\frac{1}{z}\right)$

① let $f(z) = \dfrac{1}{z+z^2} = \dfrac{1}{z(z+1)}$

The poles are given by $z(z+1) = 0$

$\Rightarrow z = 0, -1$ (order 1).

M1 $f(z) = \dfrac{1}{z+z^2}$

$= \dfrac{1}{z(z+1)}$

$= \dfrac{1}{z}(1+z)^{-1}$

$= \dfrac{1}{z}(1 - z + z^2 - \cdots)$

$= \dfrac{1}{z} - 1 + z - z^2 + \cdots$

$\therefore$ $\text{Res}\{f(z)\}_{z=0}$ = coeff. of $\left(\dfrac{1}{z-0}\right)$ = ~~cof~~ coeff of $\dfrac{1}{z}$

$= 1$.

M2 Let $f(z) = \dfrac{\phi(z)}{z-0}$.

$\Rightarrow \dfrac{1}{z(1+z)} = \dfrac{\phi(z)}{z}$

$\Rightarrow \phi(z) = \dfrac{1}{1+z}$.

$\text{Res}\{f(z)\}_{z=0} = \phi(0) = \left(\dfrac{1}{z+1}\right)_{z=0} = 1$.

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(2) Let $f(z) = z\cos\left(\frac{1}{z}\right)$

$$= z\left[1 - \frac{(1/z)^2}{2!} + \frac{(1/z)^4}{4!} - \cdots\cdots\right]$$

$$= z\left[1 - \frac{1}{z^2 . 2!} + \frac{1}{z^4 . 4!} - \cdots\cdots\right]$$

$$= \left[z - \frac{1}{z . 2!} + \frac{1}{z^3 . 4!} - \cdots\cdots\right]$$

$$\underset{z=0}{\text{Res}}\{f(z)\} = \text{coeff. of } \frac{1}{z-0}\left(= \frac{1}{z}\right)$$

$$= \frac{-1}{2!} = \frac{-1}{2}.$$

Q. Evaluate (1) $\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz$.

(i) Taken counterclockwise around the circle (a) $|z-2| = 2$

(b) $|z| = 4$

(c) $|z| = \frac{1}{2}$.

Let $f(z) = \frac{3z^3+2}{(z-1)(z^2+9)}$.

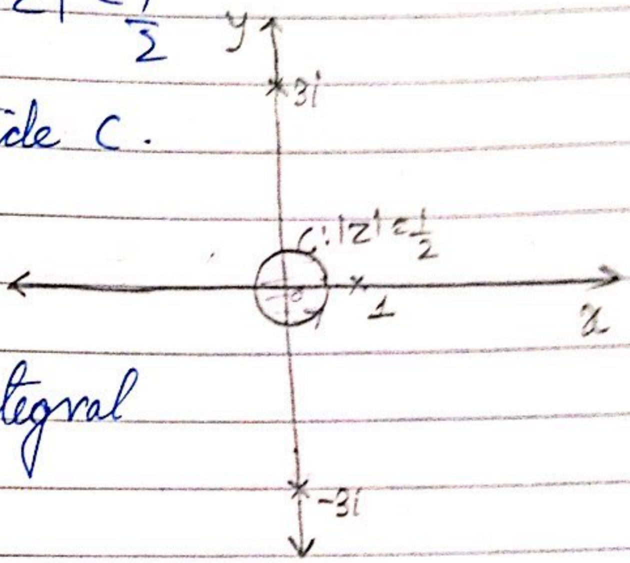
The poles are given by

$(z-1)(z^2+9) = 0$

$\Rightarrow z = 1, \pm 3i$ (order 1)

(simple poles)

(iii) Here, C is the circle $|z| = \frac{1}{2}$



All these poles lie outside C.

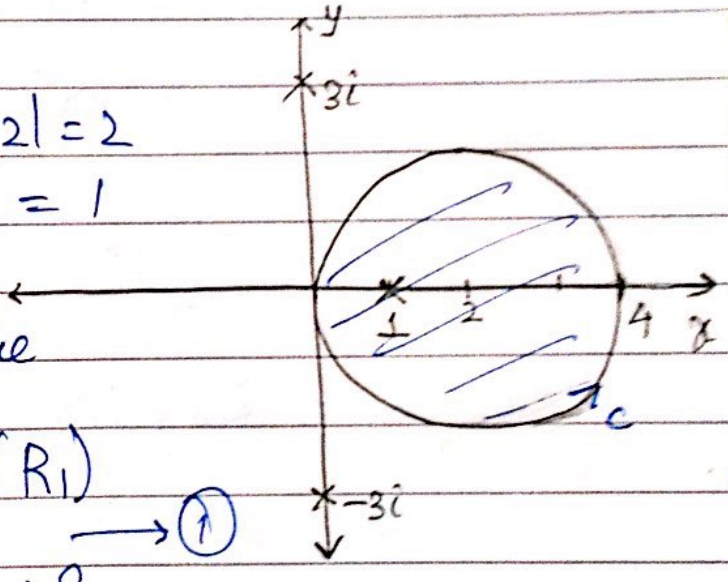
~~So, By Cauchy~~

$\therefore f$ is analytic at all pts within & on C.

$\therefore$ By the Cauchy's integral thm,

$$\int_C f(z)\,dz = 0$$

or $$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 0.$$



(ii) Here, C is circle $|z-2| = 2$

of the 3 poles, only $z = 1$ lies inside C.

So By Cauchy's residue thm,

$$\int_C f(z)\,dz = 2\pi i\,(R_1) \longrightarrow (1)$$

, where $R_1 = \text{Res}\{f(z)\}_{z=1}$

$$= \phi(1), \text{ where } \frac{\phi(z)}{z-1} = f(z)$$

$$= \frac{3(1)^3+2}{1^2+9}$$

$$= \frac{5}{10}$$

$$= \frac{1}{2}.$$

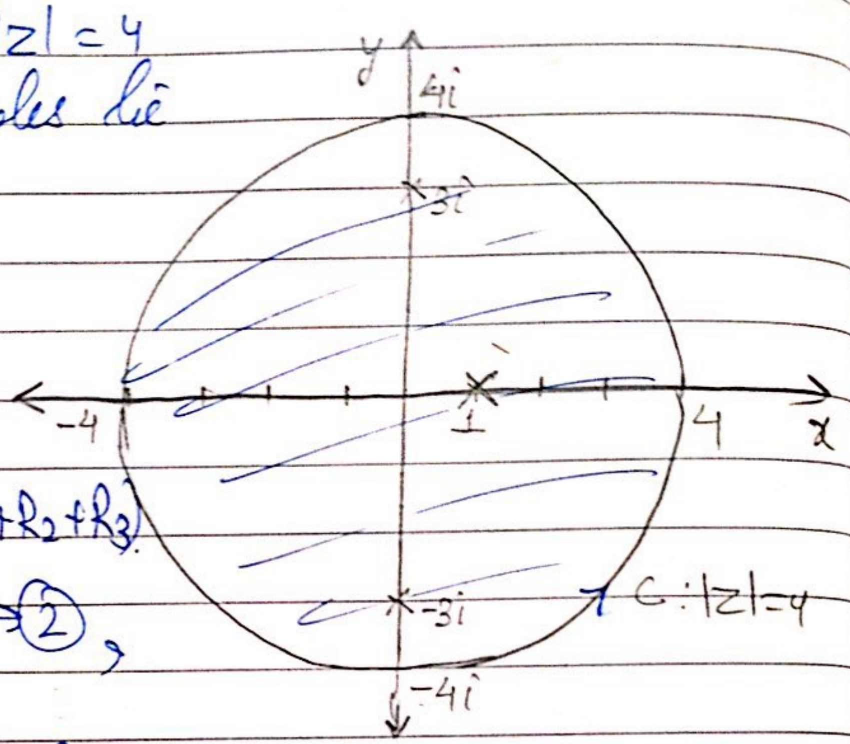
$$\frac{\phi(z)}{z-1} = \frac{3z^3+2}{(z-1)(z^2+9)}$$

$$\Rightarrow \phi(z) = \frac{3z^3+2}{z^2+9}.$$

$\therefore$, from (1),

$$\int_C f(z)\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i \quad \text{Ans.}$$

(ii) The curve C is $|z| = 4$
Here, all the poles lie inside C.
So, By Cauchy's Residue thm,



$$\int_C f(z)\,dz = 2\pi i(R_1 + R_2 + R_3) \longrightarrow (2),$$

where

$$R_1 = \text{Res}\{f(z)\}_{z=1} = \frac{1}{2}.$$

$$R_2 = \text{Res}\{f(z)\}_{z=3i}$$

$$R_3 = \text{Res}\{f(z)\}_{z=-3i}$$

$R_2 = \text{Res}\{f(z)\}_{z=3i} = \phi(3i)$, where $\dfrac{\phi(z)}{z-3i} = f(z)$

$= \phi(3i)$

$= \dfrac{3(3i)^3+2}{(3i-1)(3i+3i)}$

$= \dfrac{-81i+2}{(3i-1)(6i)}$

$\Rightarrow \dfrac{\phi(z)}{\cancel{z-3i}} = \dfrac{3z^3+2}{(z-1)(z+3i)\cancel{(z-3i)}}$

$\Rightarrow \phi(z) = \dfrac{3z^3+2}{(z-1)(z+3i)}$

$$\Rightarrow R_2 = \frac{-81i+2}{-18-6i} = \frac{2-81i}{6(i+3)}\frac{(3-i)}{(3-i)} = \frac{-245+75i}{60i}$$

$$R_3 = \text{Res}\{f(z)\}_{z=-3i} = \phi(-3i)$$

~~where~~ where $\frac{\phi(z)}{z-(-3i)} = f(z)$

$$\Rightarrow \frac{\phi(z)}{\cancel{z+3i}} = \frac{3z^3+2}{(z-1)(z-3i)\cancel{(z+3i)}}$$

$$\Rightarrow \phi(z) = \frac{3z^3+2}{(z-1)(z-3i)}$$

$$= \phi(-3i)$$

$$= \frac{3(-3i)^3+2}{(-3i-1)(-3i-3i)}$$

$$= \frac{-81(-i)+2}{(-3i-1)(-6i)}$$

$$= \frac{81i+2}{(3i+1)(6i)} = \frac{81i+2}{-18+6i} = \frac{81i+2}{6(-3+i)}\frac{(-3-i)}{(-3-i)}$$

$$= \frac{-243i+81-6-2i}{60}$$

$$= \frac{245+75i}{60i}$$

$$= \frac{-245i+75}{60}$$

From (2),

$$\Rightarrow \int_C f(z)\,dz = 2\pi i\left\{\frac{1}{2}+\left(\frac{-245+75i}{60i}\right)+\left(\frac{245+75i}{60i}\right)\right\}$$

$$= 2\pi i\left\{\frac{1}{2}+\frac{150i}{60i}\right\}$$

$$= 2\pi i\left(\frac{1}{2}+\frac{5}{2}\right)$$

$$= 6\pi i. \quad \text{Ans}$$

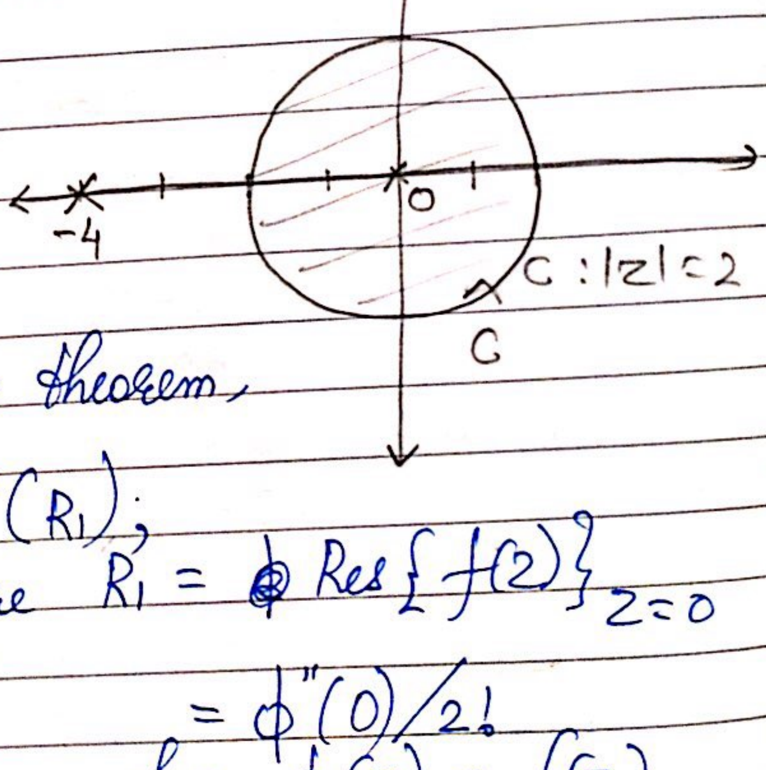
Q Evaluate $\int_C \frac{dz}{z^3(z^{\cancel{2}}+4)}$

when C is the circle: (i) $|z| = 2$
(ii) $|z+2| = 3$
(iii) $|z| = 4$.

Let $f(z) = \frac{1}{z^3(z^{\cancel{2}}+4)}$.

The poles are given by
$z^3(z+4) = 0$
$\Rightarrow z = 0$ (~~th~~ triple pole)
$z = -4$ (Simple pole)

(i) $C : |z| = 2$



The only pole $z=0$ lies inside C.

By Cauchy's Residue theorem,

$\int_C f(z)\,dz = 2\pi i (R_1)$;
where $R_1 = \text{Res}\{f(z)\}_{z=0}$

$= \phi''(0)/2!$

where $\frac{\phi(z)}{(z-0)^3} = f(z)$

$\Rightarrow \frac{\phi(z)}{z^3} = \frac{1}{\cancel{z^3}(z+4)}$

$\Rightarrow \phi(z) = \frac{1}{\cancel{z^3}(z+4)}$

$$\cancel{\Rightarrow \phi'(z) = \frac{-2}{z^3}}$$

$$\cancel{\Rightarrow \phi''(z) = \frac{-6}{z^4}}$$

$$\phi'(z) = \frac{-1}{(z+4)^2}$$

$$\phi''(z) = \frac{+2}{(z+4)^3}.$$

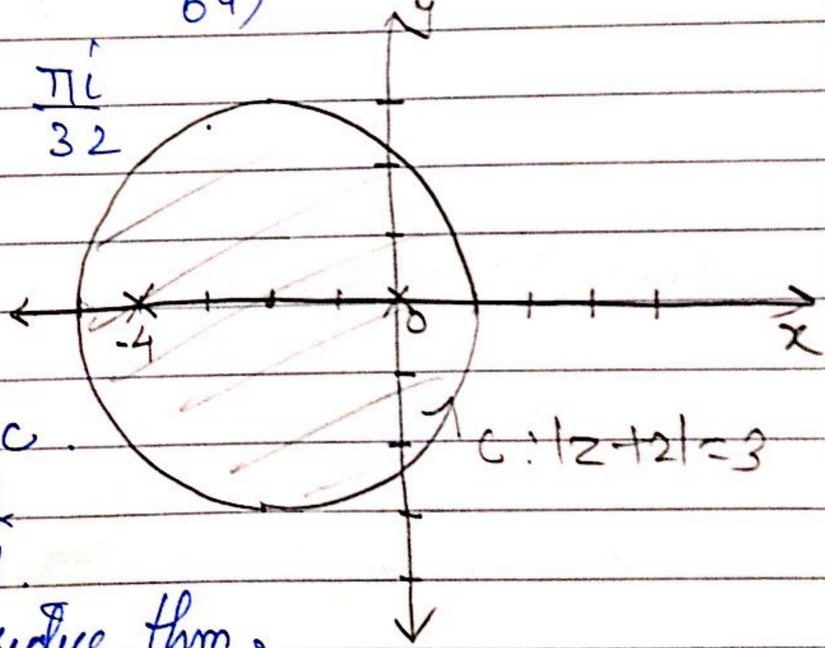
$$\Rightarrow \frac{\phi''(0)}{2!} = \frac{1}{2}\left(\frac{2}{(0+4)^3}\right)$$

$$= \frac{1}{64}.$$

So, $$\int_C f(z)\,dz = 2\pi i\left(\frac{\phi''(0)}{2!}\right)$$

$$= 2\pi i\left(\frac{1}{64}\right)$$

$$= \frac{\pi i}{32}$$



(ii) $C : |z+2| = 3$

2 poles lie inside c.
These are $z = 0$ &
$z = -4$.

So By Cauchy's Residue thm,

$$\int_C f(z)\,dz = 2\pi i(R_1 + R_2).$$

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$R_1 = \text{Res}\{f(z)\}_{z=0} = \frac{1}{64}$

$R_2 = \text{Res}\{f(z)\}_{z=-4}$

$= \phi(-4)$, where $\frac{\phi(z)}{z+4} = f(z)$

$\Rightarrow \frac{\phi(z)}{\cancel{z+4}} = \frac{1}{z^3\cancel{(z+4)}}$

$\Rightarrow \phi(z) = \frac{1}{z^3}$.

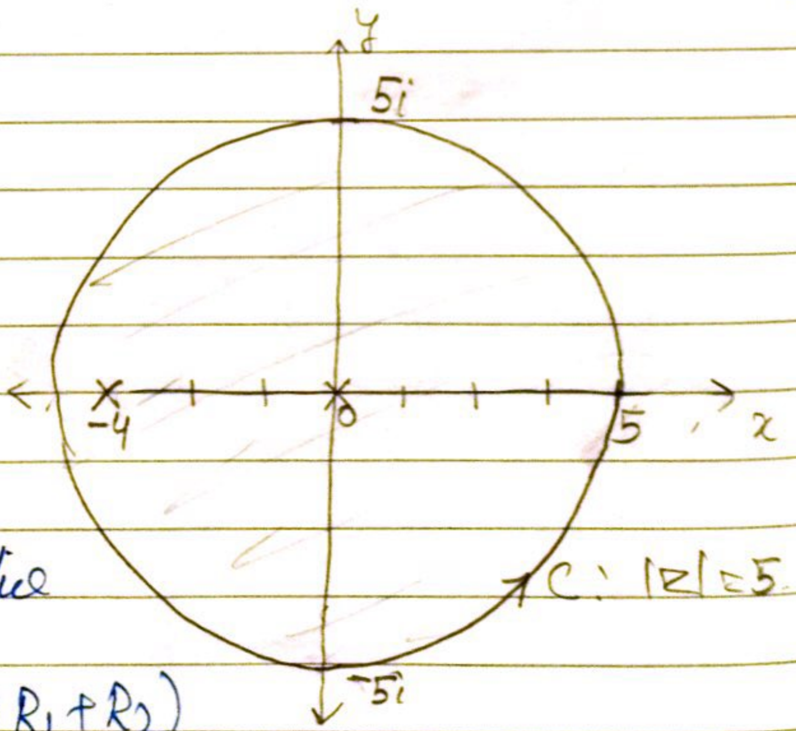
$\Rightarrow \phi(-4) = \left(\frac{1}{-4}\right)^3$

$= \frac{-1}{64}$

So, $\int_C f(z)\,dz = 2\pi i(R_1 + R_2)$

$= 2\pi i\left(\frac{1}{64} - \frac{1}{64}\right) = 0.$ Ans

(iii) $C : |z| = 4$



Both the poles $z=0$ & $z=-4$ lie inside C.

So,

By Cauchy's Residue thm,

$\int_C f(z)\,dz = 2\pi i(R_1 + R_2)$

$= 0.$

Ans

Q. Let C be the circle $|z| = 2$. Then, evaluate

① $\int_C \frac{e^{-z}}{(z-1)^3} dz$.

② $\int_C \tan z \, dz$

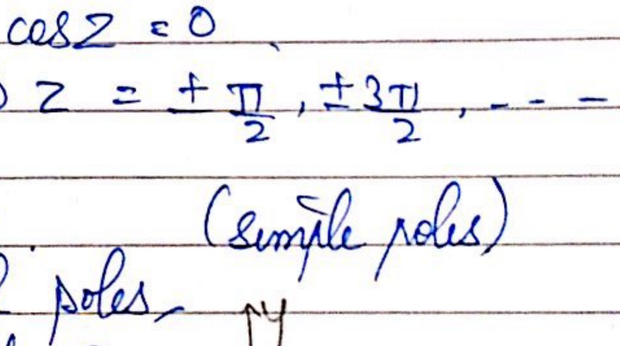
③ $\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-3)} dz$

④ $\int_C \frac{\cosh(\pi z)}{z(z^2+1)}$

② Let $f(z) = \tan z = \frac{\sin z}{\cos z} = \frac{P(z)}{Q(z)}$

The poles are given by $\cos z = 0$
$\Rightarrow z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$

Here, C is $|z| = 2$.
centre : origin, radius : 2. (simple poles)
Out of the infinite no. of poles, only $z = \pm \frac{\pi}{2}$ lie inside C.



So By Cauchy's Residue thm,

$\int_C f(z) dz = 2\pi i (R_1 + R_2)$

$R_1 = \operatorname{Res}\{f(z)\}_{z=\frac{\pi}{2}}$

$= \cancel{\phi\left(\frac{\pi}{2}\right)}, \cancel{Q(z)}$

$= \frac{P(z)}{Q'(z)} \Big]_{z=\pi/2} = \frac{\sin z}{-\sin z} \Big|_{z=\pi/2} = -1.$

$$R_2 = \text{Res}\{f(z)\}_{z=-\frac{\pi}{2}}$$

$$= \left.\frac{P(z)}{Q'(z)}\right|_{z=-\pi/2}$$

$$= \left.\frac{\sin(z)}{-\sin z}\right|_{z=-\frac{\pi}{2}}$$

$$= (-1).$$

So, $\int_C f(z)\,dz = 2\pi i(-1-1)$

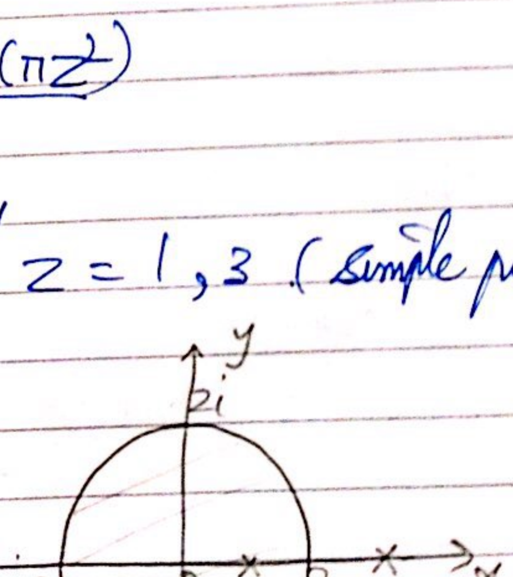
$$= \underline{\underline{-4\pi i}} \quad \text{Ans}$$

(3) $\int_C \dfrac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-3)}\,dz.$

Let $f(z) = \dfrac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-3)}$

The poles are given by

$(z-1)(z-3)=0 \Rightarrow z = 1, 3$ (simple poles)



Only the pole $z=1$ lies inside C.

So By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(R_1)$$

$$R_1 = \text{Res}\{f(z)\}_{z=1}$$

$$= \phi(1) \quad \text{where } \frac{\phi(z)}{z-1} = f(z)$$

$$\Rightarrow \frac{\phi(z)}{\cancel{z-1}} = \frac{\sin(\pi z^2) + \cos(\pi z^2)}{\cancel{(z-1)}(z-3)}$$

$$\Rightarrow \phi(z) = \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-3)}$$

So, $\phi(1) = \frac{\sin(\pi) + \cos(\pi)}{-2} = \frac{-1}{-2} = \frac{1}{2}$.

So, $\int_C f(z)\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i$ Ans.

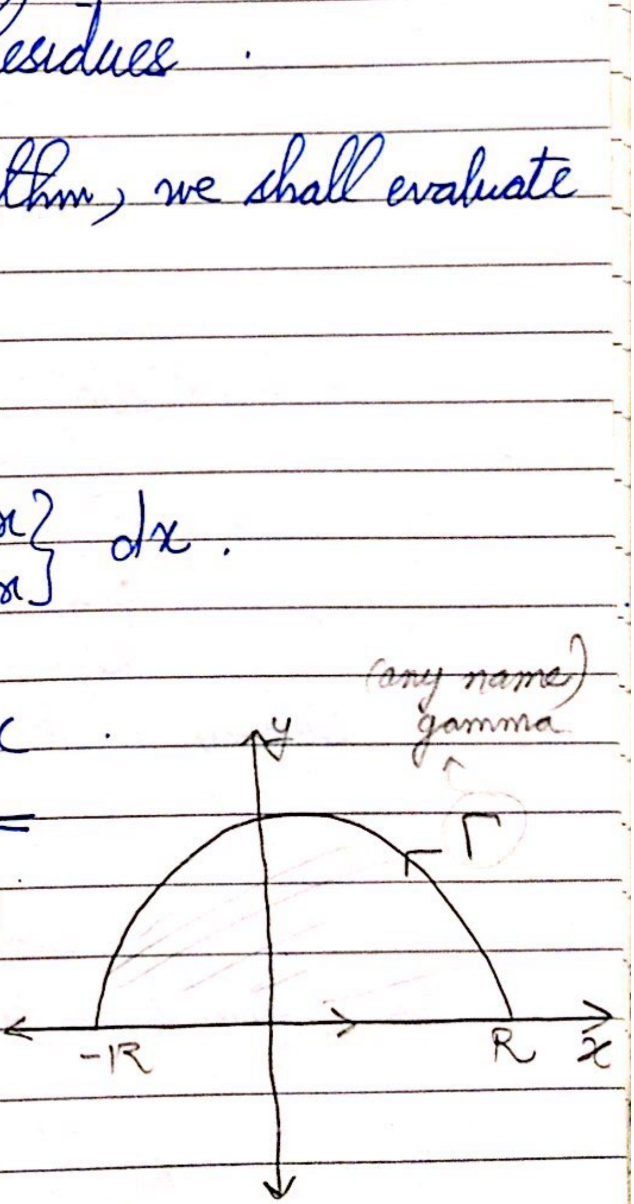
§ ★ Application of Residues.

Using the Cauchy's residue thm, we shall evaluate the integrals:

(1) TYPE I: $\int_{-\infty}^{\infty} f(x)\,dx$

(2) Type II: $\int_{-\infty}^{\infty} f(x) \left.\begin{matrix}\sin mx \\ \cos mx\end{matrix}\right\} dx$.

(1) TYPE - (I) $\int_{-\infty}^{\infty} f(x)\,dx$.



{ The principal value (p.v) of the above integral is denoted and defined by :-

PTO

$$p.v\left\{\int_{-\infty}^{\infty} f(x)\,dx\right\} = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

}.

Let C be a SCC, consisting of
(i) The line segment from $-R$ to R.
(ii) The semicircle $\Gamma : |z| = R$, which lies in the upper half plane, *s.t it includes all the poles of $f(z)$ which lie in the upper half plane (not lower half).

By Cauchy's residue thm,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz.$$

(on the real line, $z = x + 0i = x$.
$\Rightarrow f(z) = f(x)$.)

not the same

$$\Rightarrow 2\pi i\left(\sum R\right) = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz.$$

, where $\sum R$ is the sum of the residues of $f(z)$ at the poles which lie in the upper half plane.

Taking limit $R \longrightarrow \infty$.

$$\Rightarrow 2\pi i\left(\sum R\right) = \int_{-\infty}^{\infty} f(x)\,dx + \lim_{R\to\infty}\int_{\Gamma} f(z)\,dz.$$

* Show that $\lim_{R\to\infty}\int_{\Gamma} f(z)\,dz = 0$ using the ML inequality for Type (1) integrals.

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Note ① • To evaluate type II integrals, we consider
$\int_{-\infty}^{\infty} f(z)\, e^{imx}\, dx$ & proceed as above.

& • We shall use the Jordon's Lemma to show that
$\lim_{R\to\infty} \int_{\Gamma} f(z)\, e^{imz}\, dz = 0.$

Q Evaluate the integrals:

① $\int_{0}^{\infty} \frac{dx}{(x^2+1)^2}$

② $\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2+4)(x^2+9)}$

③ $\int_{0}^{\infty} \frac{dx}{x^4+1}$

(If $f(x)$ is an even f^n, then,
$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
&
$\int_{-a}^{a} f(x)\,dx = 0$, if $f(x)$ is odd)

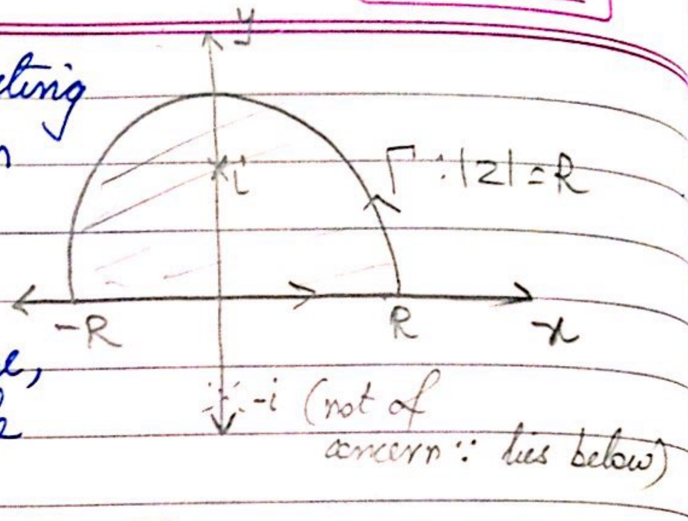
① Consider $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$

Let $f(x) = \frac{1}{(x^2+1)^2}$

$\Rightarrow f(z) = \frac{1}{(z^2+1)^2}$

The poles are given by
$z = \pm i$ (order = 2) (double poles)

Let C be a SCC consisting of the Line segment from $-R$ to R & the semicircle $\Gamma : |z| = R$, in the upper half plane, which includes the pole $z = i$



Then, by Cauchy's Residue thm,

$$\int_C f(z)\,dz = 2\pi i\,(R_1) \longrightarrow (1)$$

where $R_1 = \text{Res}\{f(z)\}_{z=i}$

$$= \frac{\phi'(i)}{1!} \quad ; \quad \frac{\phi(z)}{(z-i)^2} = f(z)$$

$$\Rightarrow \frac{\phi(z)}{\cancel{(z-i)^2}} = \frac{1}{\cancel{(z-i)^2}\,(z+i)^2}$$

$$\Rightarrow \phi(z) = \frac{1}{(z+i)^2}$$

$$\Rightarrow \phi'(z) = \frac{-2}{(z+i)^3}$$

$$\phi'(i) = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$\Rightarrow R_1 = \left(\frac{1}{4i}\right)\Big/ 1! = \frac{1}{4i}$$

So, $\int_C f(z)\,dz = 2\pi \cancel{i}\left(\frac{1}{4\cancel{i}}\right) = \frac{\pi}{2}$.

Now, $\int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \frac{\pi}{2}$.

Taking the limit $R \longrightarrow \infty$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx + \lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = \frac{\pi}{2} \longrightarrow (2)$$

We shall show that

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0 \quad \text{(By ML inequality)}.$$

Here, $f(z) = \frac{1}{(z^2+1)^2}$.

$$\big|\, |z|^2 - 1 \big| \leq |z^2+1|$$

$$\Rightarrow |R^2-1| \leq |z^2+1|.$$

$$\Rightarrow R^2 - 1 \leq |z^2+1|.$$

$$\Rightarrow (R^2-1)^2 \leq |(z^2+1)^2|.$$

$$\Rightarrow \frac{1}{|(z^2+1)^2|} \leq \frac{1}{(R^2-1)^2}$$

$$\Rightarrow \left|\frac{1}{(z^2+1)^2}\right| \leq \frac{1}{(R^2-1)^2}.$$

$$\Rightarrow |f(z)| \leq \frac{1}{(R^2-1)^2} \quad (= M, \text{ say}).$$

L: the circumference of semicircle Γ.
$= \pi R$.

So, By ML inequality,

$$\left|\int_{\Gamma} f(z)\,dz\right| \leq ML.$$

$$\Rightarrow \left|\int_\Gamma f(z)\,dz\right| \leq \frac{1}{(R^2-1)^2}\left((\pi R)\right) \to 0 \text{ as } R\to\infty.$$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0.$$

$\therefore$ (2) becomes.

$$\int_{-\infty}^{\infty} f(x)\,dx + 0 = \frac{\pi}{2}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}.$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} \quad (\because f(x) \text{ is an even fn})$$

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

Ans.

(2) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+4)(x^2+9)}$

Let $f(x) = \dfrac{x^2}{(x^2+4)(x^2+9)}$

$\Rightarrow f(z) = \dfrac{z^2}{(z^2+4)(z^2+9)}$

The poles are given by

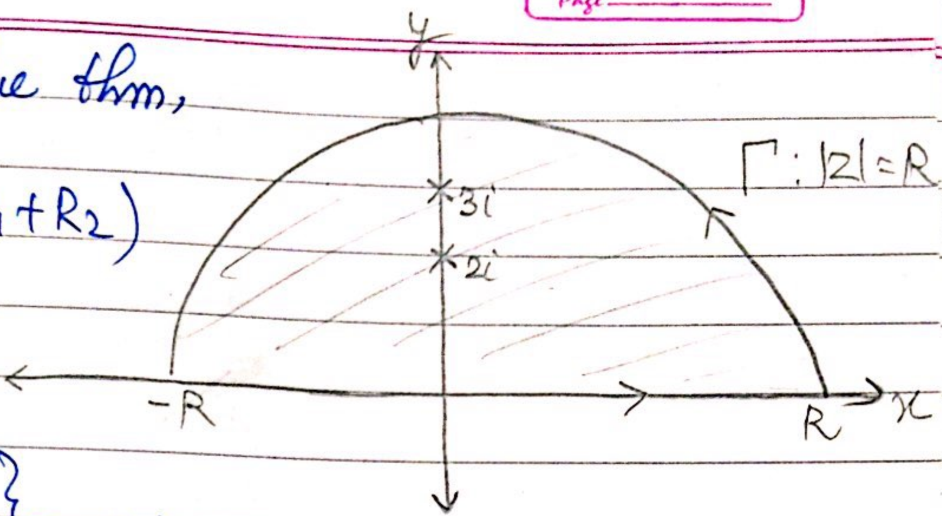
$(z^2+4)(z^2+9) = 0$

$\Rightarrow z = \pm 2i, \pm 3i$ (simple poles)

Let C be a SCC, consisting of the line segment from $-R$ to R, & the semicircle $\Gamma: |z|=R$, which includes both the poles which lie in the upper half plane ($z = 2i, 3i$).

By Cauchy's Residue thm,

$$\int_C f(z)\,dz = 2\pi i\,(R_1 + R_2) \longrightarrow (A)$$



$$R_1 = \text{Res}\{f(z)\}_{z=2i}$$

$$= \phi(2i) \quad \text{where } f(z) = \frac{\phi(z)}{z-2i}$$

$$\Rightarrow \frac{z^2}{(z+2i)\cancel{(z-2i)}(z^2+9)} = \frac{\phi(z)}{\cancel{(z-2i)}}$$

$$\Rightarrow \phi(z) = \frac{z^2}{(z+2i)(z^2+9)}$$

$$\Rightarrow \phi(2i) = \frac{-4}{(4i)(5)}$$

$$= -\frac{1}{5i}.$$

$$\Rightarrow R_1 = -\frac{1}{5i}.$$

$$R_2 = \text{Res}\{f(z)\}_{z=3i}$$

$$= \phi(3i) \; ; \text{ where } \frac{\phi(z)}{z-3i} = f(z)$$

$$\Rightarrow \frac{\phi(z)}{\cancel{z-3i}} = \frac{z^2}{\cancel{(z-3i)}(z+3i)(z^2+4)}$$

$$\Rightarrow \phi(z) = \frac{z^2}{(z+3i)(z^2+4)}$$

$$\therefore R_2 = \phi(3i) = \frac{(3i)^2}{(3i+3i)((3i)^2+4)} = \frac{-9}{6i(-5)} = \frac{3}{10i}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\{R_1 + R_2\}$$
$$= 2\pi i\left(-\frac{1}{5i} + \frac{3}{10i}\right)$$
$$= 2\pi i\left(\frac{1}{10i}\right)$$
$$= \frac{\pi}{5}$$

$$\Rightarrow \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \frac{\pi}{5}$$

Taking limit $R \to \infty$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx + \lim_{R\to\infty}\int_{\Gamma} f(z)\,dz = \frac{\pi}{5} \longrightarrow ①$$

Using Δ inequality,

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2| \leq |z_1| + |z_2|$$

$$|z^2+4| \geq \big||z|^2 - |4|\big|$$
$$\Rightarrow |z^2+4| \geq |R^2 - 4|$$
$$\Rightarrow |z^2+4| \geq [R^2 - 4]$$
$$\Rightarrow \frac{1}{|z^2+4|} \leq \frac{1}{(R^2-4)}$$

Ily, $|z^2+9| \geq (R^2-9)$

&

$$\frac{1}{|z^2+9|} \leq \frac{1}{R^2-9}$$

Also, $|z^2| = |z|^2 = R^2$.

$$\therefore |f(z)| = \left|\frac{z^2}{(z^2+4)(z^2+9)}\right| = \frac{|z|^2}{|z^2+4|\,|z^2+9|}$$
$$\leq \frac{R^2}{(R^2-4)(R^2-9)} \;(=M)$$

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Also, $L = \pi R$ = circumference of Γ

$\therefore$ By ML inequality,

$$\left| \int_\Gamma f(z)dz \right| \leq ML$$

$$\Rightarrow \left| \int_\Gamma f(z)dz \right| \leq \frac{\pi R^3}{(R^2-4)(R^2-9)} \to 0 \text{ as } R \to \infty.$$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)dz = 0 \longrightarrow (2)$$

$\therefore$, from (1) & (2)

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{5}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)} dx = \frac{\pi}{5}.$$

---

(3) $\int_0^\infty \frac{dx}{x^4+1}$; Consider $\int_{-\infty}^\infty \frac{dx}{x^4+1}$

Let $f(x) = \frac{1}{x^4+1}$

$$\Rightarrow f(z) = \frac{1}{z^4+1} = \frac{P(z)}{Q(z)}$$

Poles are given by $z^4+1=0 \Rightarrow z^4 = -1.$

$$\cancel{(z^2)^2 - (i)^2 = 0}$$

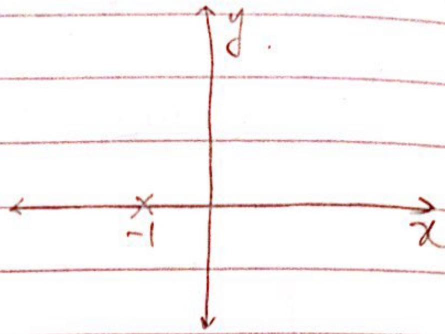
$$\cancel{\Rightarrow (z^2-i)}(z^2+i) = 0$$

$\Rightarrow$

$$\Rightarrow \cancel{(z^2 - i)}(z^2 + i) = 0$$

$$\Rightarrow z^4 = -1$$

$$r_0 = |z| = \sqrt{(-1)^2} = 1$$



$$\theta_0 = Arg(z) + 2n\pi.$$

$$\therefore \theta_0 = \pi + 2n\pi, \; n \in \mathbb{Z}$$

$$\Rightarrow z^4 = r_0 e^{i\theta_0}$$

$$= 1 . e^{i(\pi + 2n\pi)} \; ; n \in \mathbb{Z}$$

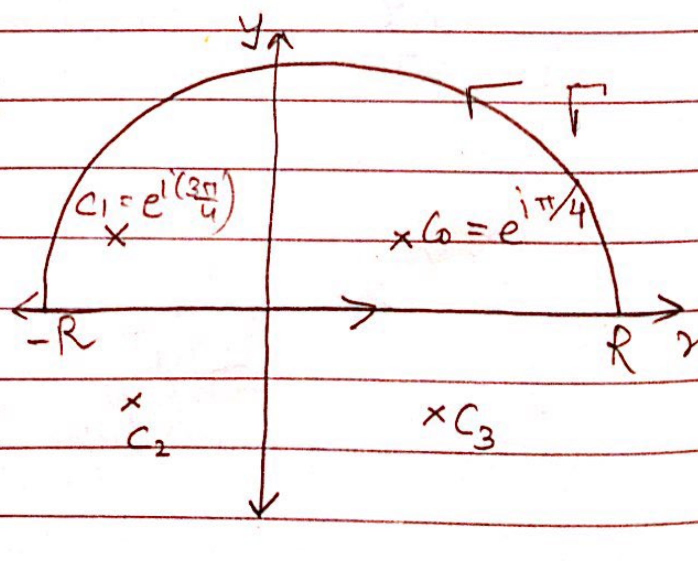
$\therefore$ The 4th roots of -1 are given by.

$$C_k = \left[e^{i\pi(2n+1)}\right]^{1/4}, \; n = 0, 1, 2, 3.$$

$$\Rightarrow C_k = e^{\frac{i\pi(2n+1)}{4}} \; ; n = 0, 1, 2, 3.$$

$$\Rightarrow \left.\begin{aligned} C_0 &= e^{i\pi/4} \\ C_1 &= e^{i(3\pi/4)} \\ C_2 &= e^{i5\pi/4} \\ C_3 &= e^{i7\pi/4} \end{aligned}\right\} \text{(simple poles)}$$

Of these poles, only $z = e^{i\pi/4}$ & $e^{i\frac{3\pi}{4}}$ lie in the upper half plane.



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Let C be a SCC consisting of the line segment from $-R$ to R & the semicircle $\Gamma : |z| = R$, which includes both the poles in the upper half plane ($z = e^{i\pi/4}, e^{i3\pi/4}$).

So, By Cauchy's Residue thm,

$$\int_C f(z)\,dz = 2\pi i (R_1 + R_2) \longrightarrow (1)$$

$$R_1 = \operatorname{Res}\{f(z)\}_{z = e^{i\pi/4}}$$

$= \phi(e^{i\pi/4})$; where $f(z) = \cancel{\dfrac{\phi(z)}{z - e^{i\pi/4}}}$ This way won't work (very lengthy)

$$= \left.\frac{P(z)}{Q'(z)}\right|_{z = e^{i\pi/4}}$$

$$= \left.\frac{1}{4z^3}\right)_{z = e^{i\pi/4}}.$$

$$= \left.\frac{z}{4z^4}\right|_{z = e^{i\pi/4}}$$

$$= \frac{e^{i\pi/4}}{4(-1)} \qquad (\because z^4 = -1)$$

$$= \left(\frac{-1}{4}\right)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \left(\frac{-1}{4}\right)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$R_2 = \text{Res}\left\{f(z)\right\}_{z=e^{i\frac{3\pi}{4}}}$$

$$= \left.\frac{P(z)}{Q'(z)}\right|_{z=e^{i\frac{3\pi}{4}}}$$

$$= \left.\frac{1}{4z^3}\right|_{z=e^{i3\pi/4}}$$

$$= \left.\frac{z}{4z^4}\right|_{z=e^{i3\pi/4}}$$

$$= \frac{e^{i3\pi/4}}{4(-1)}$$

$$= \left(\frac{-1}{4}\right)\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\frac{3\pi}{4}\right]$$

$$R_2 = \left(\frac{-1}{4}\right)\left[-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right]$$

So, $\int_C f(z)dz = 2\pi i\,(R_1 + R_2)$ (From ①)

$$= 2\pi i\left(\frac{-1}{4}\right)\left[\cancel{\frac{1}{\sqrt{2}}} + \frac{i}{\sqrt{2}} - \cancel{\frac{1}{\sqrt{2}}} + \frac{i}{\sqrt{2}}\right]$$

$$= \left(-\frac{\pi i}{2}\right)\left[\sqrt{2}\,i\right]$$

$$\Rightarrow \int_C f(z)dz = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz = \frac{\pi}{\sqrt{2}}$$

Taking limit $R \to \infty$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx + \lim_{R\to\infty}\int_\Gamma f(z)dz = \frac{\pi}{\sqrt{2}} \longrightarrow ②$$

By Δ inequality.

$$|z^4+1| \geq \left||z|^4 - 1\right|$$

$$\Rightarrow \frac{1}{|z^4+1|} \leq \frac{1}{\left||z|^4-1\right|}$$

$$\Rightarrow \frac{1}{|z^4+1|} \leq \frac{1}{|R^4-1|}$$

$$\Rightarrow \frac{1}{|z^4+1|} \leq \frac{1}{(R^4-1)}$$

So, $$|f(z)| = \left|\frac{1}{z^4+1}\right| \leq \frac{1}{(R^4-1)} (= M).$$

Also, $L = \pi R$, circumference of Γ.

$\therefore$ By ML inequality,

$$\left|\int_\Gamma f(z)\,dz\right| \leq ML \leq \frac{\pi R}{R^4-1} \to 0 \text{ as } R \to \infty.$$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0 \longrightarrow 2.$$

$\therefore$ from 2 & 3

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sqrt{2}}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

Because $f(x)$ is even f^n, so,

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

Ans.

HW Q : $\int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$; $a, b > 0$.

✲ JORDON's LEMMA.

$\forall$ pts. on the circle $C_R : |z| = R$, if $\exists$ a +ve constant M_R s.t.

$$|f(z)| \leq M_R$$

where $M_R \to 0$ as $R \to \infty$.

Then, $\lim_{R\to\infty} \int_{C_R} f(z)e^{imz}\,dz = 0.$

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Q. Evaluate

① $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+2x+2} dx$

② $\int_{0}^{\infty} \frac{\cos(ax)\, dx}{(x^2+b^2)^2}$; $a>0,\ b>0$

③ $\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)}$ $(a, b>0)$

④ $\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2+3} dx$

① ~~Let~~ We shall consider $\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+2x+2}$

Let $f(x) = \frac{x}{x^2+2x+2}$

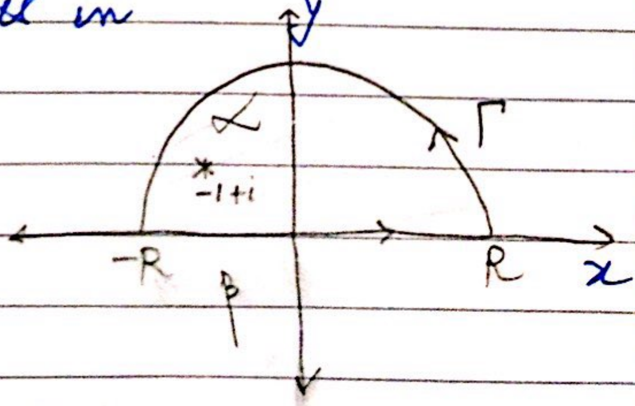
$\Rightarrow f(z) = \frac{z}{z^2+2z+2}$

The poles are given by

$z^2+2z+2=0$

$\Rightarrow z = -1 \pm i$ (Simple poles)

Only the pole $z=-1+i$ lies in the upper half pole.



Let C be a SCC consisting of the line segment $-R$ to R & the semicircle, $\Gamma : |z| = R$ in the upper half plane, which includes the pole $-1+i$.

$\therefore$ By Cauchy's residue thm,

$$\int_C f(z)\, e^{iz}\, dz = 2\pi i\, (R_1) \longrightarrow (1)$$

$$R_1 = \operatorname{Res}\left\{f(z)\, e^{iz}\right\}_{z=-1+i}$$

Let $f(z)\, e^{iz} = \left[\dfrac{\phi(z)}{z-(-1+i)} \cdot e^{iz}\right]$

$$\Rightarrow \cancel{f(z)}\ \frac{z}{[z-(-1-i)][z-\cancel{(-1+i)}]} = \frac{\phi(z)}{\cancel{z-(-1+i)}}$$

$$\Rightarrow \phi(z) = \frac{z}{z+1+i}$$

$$\therefore \operatorname{Res}\left\{f(z)e^{iz}\right\} = \left[\phi(-1+i) \cdot e^{i(-1+i)}\right]$$

$$= \frac{-1+i}{-1+i+1+i} \cdot e^{-1-i}$$

$$\Rightarrow R_1 = \left(\frac{-1+i}{2i}\right) e^{-1-i}$$

$$= \left(\frac{-1+i}{2i}\right) \cdot e^{-1}\left(\cos 1 - i \sin 1\right)$$

$$\Rightarrow \int_C f(z)\, e^{iz}\, dz = 2\pi i \left[\frac{-1+i}{2ie}\right]\left[\cos 1 - i \sin 1\right]$$

$$= \frac{\pi}{e}\left(\cancel{-1+i}\,(-\cos 1 + \sin 1) + i(\cos 1 + \sin 1)\right)$$

$$\Rightarrow \int_{-R}^{R} f(x)\, e^{ix}\, dx + \int_\Gamma f(z)\, e^{iz}\, dz = \frac{\pi}{e}\left(-\cos 1 + \sin 1 + i(\cos 1 + \sin 1)\right)$$

Taking limit $R \to \infty$

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx + \lim_{R\to\infty}\int_{\Gamma} f(z)e^{iz}dz = \frac{\pi}{e}\left[(-\cos 1 + \sin 1) + i(\cos 1 + \sin 1)\right]$$

$\longrightarrow (2)$

Consider $f(z) = \dfrac{z}{z^2+2z+2}$

$$= \frac{z}{(z-\alpha)(z-\beta)} \quad ; \alpha = -1+i, \quad \beta = -1-i$$

By Δ inequality.

$$|z-\alpha| \geq \left| |z| - |\alpha| \right|$$

$$\Rightarrow |z-\alpha| \geq |R - |\alpha||$$

$$\Rightarrow |z-\alpha| \geq (R - |\alpha|)$$

$$\Rightarrow |z-\alpha| \geq (R-\sqrt{2})$$

||ly, $|z-\beta| \geq (R-\sqrt{2})$

$$\therefore \quad |f(z)| = \left|\frac{z}{(z-\alpha)(z-\beta)}\right|$$

$$= \frac{|z|}{|(z-\alpha)||z-\beta|} \leq \frac{R}{(R-\sqrt{2})(R-\sqrt{2})}$$

$$\leq \frac{R}{(R-\sqrt{2})^2}$$

$$\Rightarrow |f(z)| \leq \frac{R}{(R-\sqrt{2})^2} \;(= M_R)$$

$\longrightarrow 0$ as $R \longrightarrow \infty$.

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$\therefore$, By Jordon's Lemma,

$$\lim_{R\to\infty} \int_{\Gamma} f(z) e^{iz}\, dz = 0 \quad (\text{Surely write it}).$$

$\therefore$, eqⁿ (2) becomes,

$$\int_{-\infty}^{\infty} f(x) e^{ix} dx = \frac{\pi}{e}\left(-\cos 1 + \sin 1 + i(\cos 1 + \sin 1)\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)(\cos x + i \sin x)\, dx = \quad ''$$

Equating the imaginary parts on both sides, we get

$$\int_{-\infty}^{\infty} f(x) \sin x\, dx = \frac{\pi}{e}(\cos 1 + \sin 1)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+2x+2}\, dx = \frac{\pi}{e}(\cos 1 + \sin 1)$$

—x—

(2) $\displaystyle\int_0^{\infty} \frac{\cos ax\, dx}{(x^2+b^2)^2}, \quad a, b > 0$

Consider $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}\, dx}{(x^2+b^2)^2}, \ (a, b > 0)$

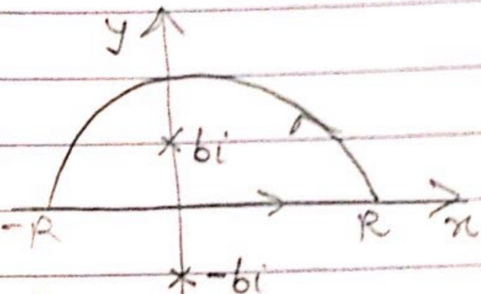
Let $f(x) = \dfrac{1}{(x^2+b^2)^2}$

$$\Rightarrow f(z) = \frac{1}{(z^2+b^2)^2}.$$

The poles are given by.

$$(z^2+b^2)^2 = 0$$
$$\Rightarrow z^2+b^2 = 0 \text{ (twice)}$$
$$\Rightarrow z = \pm ib \text{ (double poles)}$$

Here, $z = bi$ lies in the upper half plane ($\because b>0$).
Let C denote the line segment ~~|z|=~~ from $-R$ to R & semicircle, $\Gamma: |z|=R$ in the upper half plane, which has the pole bi.



$$\Rightarrow \int_C f(z)e^{iaz}\,dz = 2\pi i (R_1) \longrightarrow (1)$$

$$R_1 = \text{Res}\left\{f(z)e^{iaz}\right\}_{z=bi}$$

$$= \left[\frac{\phi'(z)}{1!}\right]_{z=bi}$$

$$\text{, where } \frac{\phi(z)}{(z-bi)^2} = f(z)e^{iaz}$$

$$= \frac{1}{(z-bi)^2(z+bi)^2}\cdot e^{iaz}$$

$$\Rightarrow \phi(z) = \frac{e^{iaz}}{(z+bi)^2}$$

$$\Rightarrow \phi'(z) = \frac{(z+bi)^2(a)e^{iaz} - e^{iaz}2(z+bi)}{(z+bi)^4}$$

$$\Rightarrow \phi'(z)\Big|_{z=bi} = -4b^2(a)e^{-ab} - e^{-ab}\cdot 2(2bi)$$

$$\text{or } \phi'(z) = \frac{(z+bi)(ai)(e^{iaz}) - e^{iaz}(2)}{(z+bi)^3}$$

$$= \frac{e^{iaz}((z+bi)ai - 2)}{(z+bi)^3}$$

$$\Rightarrow R_1 = \frac{\phi'(bi)}{1!}\Big|_{z=bi} = \frac{2bi(ai)-2}{(2bi)^3} e^{-ab}$$

$$= -\frac{2ab-2}{-8b^3 i} \cdot e^{-ab}$$

$$= \frac{ab+1}{4b^3 i} \cdot e^{-ab}$$

$\therefore$ From (1),

$$\int_C f(z)\, e^{iaz}.dz = 2\pi i\left(\frac{1+ab}{4b^3 i} \cdot e^{-ab}\right)$$

$$\Rightarrow \int_{-R}^{R} f(x)e^{iax}dx + \int_\Gamma f(z)e^{iaz}dz = \frac{\pi}{2b^3}(1+ab)(e^{-ab})$$

Taking $\lim_{R\to\infty}$, we have

$$\int_{-\infty}^{\infty} f(x)\, e^{iax}dx + \lim_{R\to\infty}\int_\Gamma f(z)e^{iaz}dz$$

$$= \frac{\pi}{2b^3}(1+ab)(e^{-ab}) \longrightarrow (2)$$

Now, $f(z) = \dfrac{1}{(z^2+b^2)^2}$

By Δ inequality,

$$|z^2+b^2| \geq ||z^2| - |b^2||$$

$$\Rightarrow |z^2+b^2| \geq |R^2 - b^2|$$

$$\Rightarrow |(z^2+b^2)| \geq (R^2-b^2)$$

$$\Rightarrow \frac{1}{|z^2+b^2|} \leq \frac{1}{(R^2-b^2)}$$

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$\Rightarrow \dfrac{1}{|z^2+b^2|^2} \leq \dfrac{1}{(R^2-b^2)^2}$.

$\Rightarrow |f(z)| = \left|\dfrac{1}{(z^2+b^2)^2}\right| = \dfrac{1}{|z^2+b^2|^2} \leq \dfrac{1}{(R^2-b^2)^2} \ (= M_r)$

$\longrightarrow 0$ as $R \longrightarrow \infty$.

$\therefore$ By Jordan's Lemma method.

$$\lim_{R\to\infty} \int_{\Gamma} f(z) e^{iaz} dz = 0$$

From (2)

$$\int_{-\infty}^{\infty} f(x) e^{iax} dx = \frac{\pi}{2b^3}(1+ab)e^{-ab}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)(\cos ax + i\sin ax)\, dx = \frac{\pi}{2b^3}(1+ab)e^{-ab}$$

Comparing the real parts, both sides –

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\cos ax\, dx = \frac{\pi}{2b^3}(1+ab)e^{-ab}.$$

$$\Rightarrow 2\int_{0}^{\infty} f(x)\cos ax\, dx = \frac{1}{2b^3}(1+ab)e^{-ab}$$

(Inversion of confi)

$$\Rightarrow \int_{0}^{\infty} f(x)\cos(ax)\, dx = \frac{1}{4b^3}(1+ab)e^{-ab}.$$

$$\Rightarrow \int_{0}^{\infty} \frac{\cos ax\, dx}{(x^2+b^2)^2} = \frac{1}{4b^3}(1+ab)\, e^{-ab}.$$

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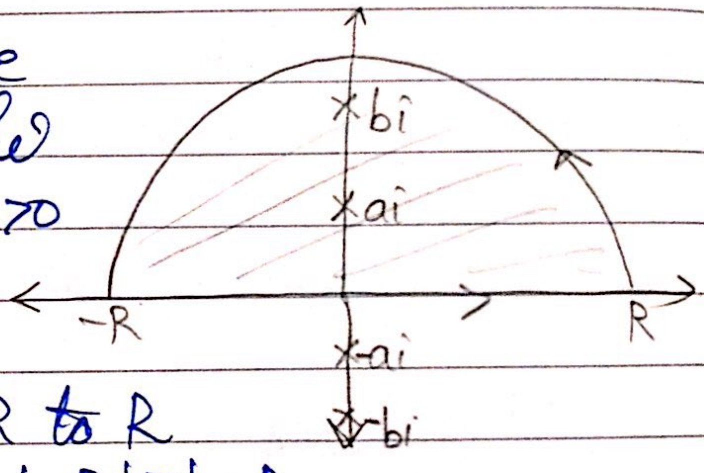
(3) $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)}$

Consider $\int_{-\infty}^{\infty} \frac{e^{ix} \, dx}{(x^2+a^2)(x^2+b^2)}$

Let $f(x) = \frac{1}{(x^2+a^2)(x^2+b^2)}$

$\Rightarrow f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$

The poles are given by
$z = \pm ai, \pm bi$ (Singular pts.)



Here, only 2 poles i.e $z = ai$ & $z = bi$ lie inside. ($\because a>0$ & $b>0$ given).

Let C denote the line segment from $-R$ to R & the semicircle $\Gamma : |z| = R$ in the upper half plane.

$\Rightarrow \int_C f(z) e^{iz} dz = 2\pi i (R_1 + R_2)$ ——→ (1)

$R_1 = \text{Res}\{f(z) e^{iz}\}_{z=ai}$

$R_2 = \text{Res}\{f(z) e^{iz}\}_{z=bi}$

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$\cancel{R_1 = \phi}$

Let $f(z) \cdot e^{iz} = \dfrac{\phi(z)}{z - ai}$

$\Rightarrow \dfrac{e^{iz}}{\cancel{(z-ai)}(z+ai)(z^2+b^2)} = \dfrac{\phi(z)}{\cancel{(z-ai)}}$

$\Rightarrow \phi(z) = \dfrac{e^{iz}}{(z+ai)(z^2+b^2)}$

So, $\text{Res}\left(f(z)e^{iz}\right)\Big|_{z=ai} = \phi(ai)$

$= \dfrac{e^{i(ai)}}{(2ai)(-a^2+b^2)}$

$\Rightarrow R_1 = \dfrac{e^{-a}}{2ai(b^2-a^2)}$

||ly, $R_2 = \dfrac{e^{-b}}{2bi(a^2-b^2)}$

$\Rightarrow \int_C f(z)e^{iz}dz = 2\pi i\left[\dfrac{e^{-a}}{2ai(b^2-a^2)} + \dfrac{e^{-b}}{2bi(a^2-b^2)}\right]$

$= \pi\left[\dfrac{-e^{-a}/a}{a^2-b^2} + \dfrac{e^{-b}}{b(a^2-b^2)}\right]$

$\Rightarrow \int_C f(z)e^{iz}dz = \dfrac{\pi}{a^2-b^2}\left[\dfrac{e^{-b}}{b} - \dfrac{e^{-a}}{a}\right]$

$\Rightarrow \int_{-R}^{R} f(x)e^{ix}dx + \int_\Gamma f(z)e^{iz}dz = \dfrac{\pi}{a^2-b^2}\left(\dfrac{e^{-b}}{b} - \dfrac{e^{-a}}{a}\right)$

Taking limit $R \to \infty$.

$\Rightarrow \int_{-\infty}^{\infty} f(x)e^{ix}dx + \lim_{R\to\infty}\int_\Gamma f(z)e^{iz}dz = \dfrac{\pi}{a^2-b^2}\left(\dfrac{e^{-b}}{b} - \dfrac{e^{-a}}{a}\right)$

$\longrightarrow (2)$

Now, consider $f(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)}$

By Δ inequality,

$$|z^2+a^2| \geq \left| |z|^2 - |a|^2 \right|$$

$$\Rightarrow |z^2+a^2| \geq (R^2 - a^2)$$

$$\Rightarrow \frac{1}{|z^2+a^2|} \leq \frac{1}{R^2-a^2}$$

IIly, $\dfrac{1}{|z^2+b^2|} \leq \dfrac{1}{R^2-b^2}$

$$\therefore |f(z)| = \left| \frac{1}{|z^2+a^2|\,|z^2+b^2|} \right|$$

$$\leq \left| \frac{1}{(R^2-a^2)(R^2-b^2)} \right|$$

$$\leq \frac{1}{(R^2-a^2)(R^2-b^2)} \; (= M_R)$$

$\longrightarrow 0$ as $R \to \infty$

$\therefore$ By Jordon's lemma,

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\, e^{iz}\, dz = 0$$

$\therefore$ eqn (2) becomes

$$\int_{-\infty}^{\infty} f(x) e^{ix}\, dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$$

Ans