OPTIMIZATION NOTES

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Optimization Notes, First Edition

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Chapter - 2

MODELLING WITH LINEAR PROGRAMMING.

§ Linear Programming problem

eg. 1) Problem to maximise profit per day :
A manufacturer produces A : 10 gms
B : 20 gms
Cost of plastic per gm = Rs. 350
He can spend max. of Rs. 74000 / day.
He saves a profit of Rs 40/ A type toy
Rs 80/ B type toy
Supplier cannot supply more than 30 kg plastic/day

Sol^n :- Let x = no. of toys of type A } How many x & y is
y = no. of toys of type B } reqd to max. profit/day

① Objective f^n :- $z = 40x + 80y$ (∵ we need to maximise) → ①

② Constrains/ Restrictions
Cannot spent more than 74000/day.
So, for x = 3500 Rs. [10×350].
y = 4000 Rs [20×350]. (30 kg)
$3500x + 4000y \leq 74000$. ⇒ $10x + 20y \leq 30,000$ → ②

③ Non negativity cond^n :-
$x, y \geq 0$.

As (1) & (2) are linear, it's called Linear Programming problem.

* FORMULATION OF LPP.

S1) Identify the decision variables

S2) To formulate the objective f^n which is to be optimized.

S3) Formulate the constraints / restrictions.

S4) Write down the non-ve $cond^{ns}$.

S1) $x_1, x_2, x_3 --- x_n$ are decision variables, say

S2) Optimize $Z = c_1x_1 + c_2x_2 + --- + c_nx_n$.

S3)
$$a_{11}x_1 + a_{12}x_2 + --- + a_{1n}x_n \ (\leq \text{ or } = \text{ or } \geq) \ b_1$$
$$a_{21}x_1 + a_{22}x_2 + --- + a_{2n}x_n \ (\leq \text{ or } = \text{ or } \geq) \ b_2$$
$$\vdots \qquad \vdots \qquad \cdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + --- + a_{mn}x_n \ (\leq \text{ or } = \text{ or } \geq) \ b_m$$

S4) $x_1, x_2, x_3, --- x_n \geq 0$ (non negativity)

* Note :- If no. of variables are 2, then, we can use GRAPHIC METHOD (discussed later).

ex Reddy-Mikks Company — A paint company
M_1 & M_2 are raw materials
Table is given between Tons of raw material vs tons of Exterior/interior paint

	Ext	Int	Max. daily availability
M_1	6	4	24
M_2	1	2	6
Profit (/1000)	5	4	–

Given : Diff. b/w ext & int is 1 ton.
Max. daily demand for int. paint is 2 ton

S1) Let x & y be amount of ext. or int (paint (in tons)) to be produced daily.

S2) Objective f^n :- Maximize profit

$$Z = 5000x_1 + 4000x_2$$

S3) Constraints :- $6x + 4y \leq 24$ (for M_1)
$1x + 2y \leq 6$ (for M_2)
$y - x \leq 1$ or $-x + y \leq 1$
$y \leq 2$

S4) Non negative cond^n :-
$x, y \geq 0$.

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Q. Diet Problem:
A diet should contain
Carbohydrates (C) : 4000
Fat (F) : 500
Protein (P) : 300.

Cost per unit of food A = \$ 2
food B = \$ 4

		C	F	P
per unit	food A :	10	20	15
	food B :	25	10	20

Make an LPP to get the req^d nutrition.

S1 : Identify Decision Variables

Sol^n :- let x be no. of units of food A
y " " food B

S2 : Write Objective f^n

Minimize $Z = 2x + 4y$

S3 : ~~Optimize~~ Write constraints.

(i) $10x + 25y \geq 4000$
(ii) $20x + 10y \geq 500$
(iii) $15x + 20y \geq 300$

S4: Non negative restriction.

$x, y \geq 0$

Ans

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Q Toy making problem

2 types of toys are being produced by 3 m/c

Types of toys → type 1, type 2
Machine → A, B, C

		Max time available	Time reqd to make 1 unit	
			Toy type 1	Toy type 2
M/C	A	380 hrs	6 hrs	8 hrs
	B	300 hrs	8 hrs	4 hrs
	C	404 hrs.	12 hrs.	4 hrs

Profit earned by selling every unit of
type 1 toy : \$5
type 2 toy : \$3.

Formulate it as LPP to maximize profit

Solⁿ S1 : Identify decision variables
let x be no. of toys of type 1 to be produced
y " " type 2 "

S2 : Write objective fⁿ
Maximize :- $Z = 5x + 3y$

S3 : Constraints
$6x + 8y \leq 380$
$8x + 4y \leq 300$
$12x + 4y \leq 404$

S4 : Non negative restriction
$x, y \geq 0$

Ans

Q. Candidate election problem

Candidate has to advertise with max of \$40,000
Ads can be given by 2 mediums - Radio & TV

	Radio	TV
Cost	\$200	\$500
No. of viewers which each Ad can reach	3000	7000

Each type of Ad should be atleast 10 } Cond^n
No. of radio ads > No. of TV ads }

Sol^n - S1 : Decision variables
Let x : no. of Radio ads
y : " TV "

S2 : Objective f^n
Maximize :- $Z = 3000x + 7000y$

S3 : Constraints
$$200x + 500y \leq 40{,}000$$
$$x \geq 10, \ y \geq 10$$
$$x > y$$

S4 : Non-negative restriction
$$x, y \geq 0$$

Ans

GRAPHICAL METHOD FOR SOLVING LPP

eg: Maximize :- $Z = 2x + 3y$

Constraint :- $3x + y \leq 2$

$x + 2y \leq 4$

Non-ve :- $x, y \geq 0$

* Feasible solⁿ : The values of x & y which satisfies all the constraints & non-ve restriction.
* Infeasible solⁿ :- Any value of x & y which doesn't satisfy any of constraint or non-ve restriction.
* Optimal solⁿ :- The value of x & y out of ∞ feasible solⁿs, that maximizes/optimises the Objective fⁿ
* Basic solⁿ :- Let the eqⁿs be m
  variables be n.
  & if, $n > m$
  then, substitute $n-m$ no. of variable equal to zero.
  Now, solve the eqⁿs. & , if you get a solⁿ (if eqⁿs are solvable), then, solⁿ is Basic solⁿ.

eg :- $3x + y + 2z = 3$ $\quad n = 3, m = 2. \Rightarrow$ Put any Var = 0.

$x + 2y + z = 5.$

Let $x = 0 \Rightarrow y = \frac{7}{3}$ & $z = \frac{1}{3}$. So $\left(0, \frac{7}{3}, \frac{1}{3}\right)$: Basic solⁿ

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Quiz Question * Max. possible basic $sol^n \le {}^nC_m$

$\hookrightarrow$ n : no. of variables
m : no. of eq^{ns}.

* If a Basic sol^n is non -ve, it is called Basic Feasible Solution (BSF).

* Graphical Method :-
With the eq^{ns}, plot on graph & shade the region enclosed.
The region : Feasible solutions
Imp The corner pts : Optimum sol^n (one out of all corner pts)

$\hookrightarrow$ mainly used when $\exists$ 2 decision variables in a problem

* STEPS for Solving LPP by Graphical Method.

S1) Find the region defined by constraints and non-negativity restriction.
This region is called ~~as~~ FEASIBLE REGION.
S2) Find the values of the objective f^n, 'Z' at all the corner pts. (vertices) of the feasible region.
S3) That corner pt. will give us the optimum sol^n (Maximum / Minimum).

Note:- To avoid confusion: If no. of decision variables are >3, use variables as $x_1, x_2, \dots$

∵ z is always objective f^n.

Q. Solve the following LPP by graphical method :-

① Maximize $z = 5x + 4y$

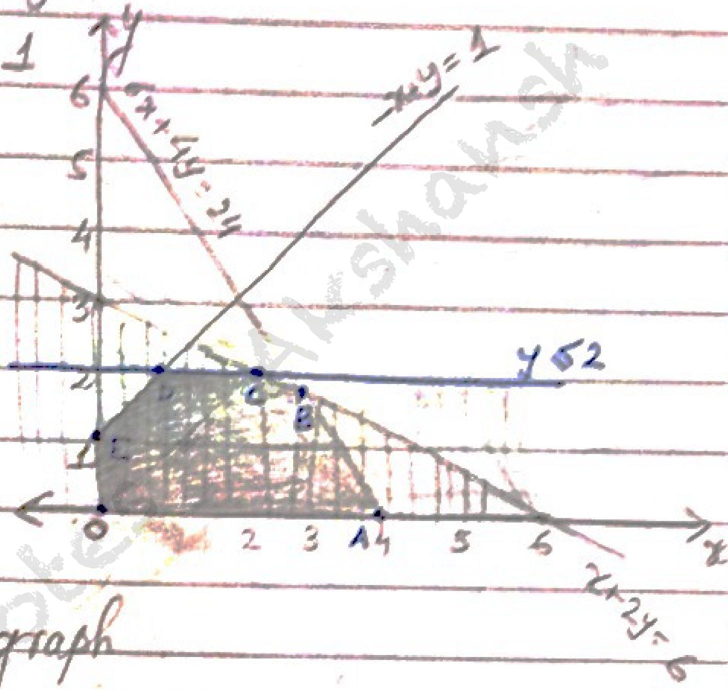
s.t ① $6x + 4y \leq 24$

② $x + 2y \leq 6$

③ $-x + y \leq 1$

④ $y \leq 2$

non-ve : $x, y \geq 0$



Solving & shading region, we get the graph

Now, finding corner pts :-

	Value of $Z = 5x + 4y$
S2) O(0,0)	0
A(4,0)	20
B(3,1.5)	21
C(2,2)	18
D(1,2)	13
E(0,1)	4

S3) We have to maximize in our problem So, B(3,1.5) is the pt.

So, optimum sol^n of given LPP is

$x = 3, \; y = 1.5, \; z = 21.$

Aliter :- To find optimum solⁿ :-

Idea :- We want optimum value of z in our eqⁿ. So, assume any value of z.

eg :- for $z = 5x + 4y$

let eqⁿ become $5x + 4y = 40$, say ($z = 40$)

Draw that line. It will be crossing our region

↳ This line is called

ISO-PROFIT LINE : If we're maximizing

ISO-LOSS LINE : If we're minimizing

Now, gradually move the line away from the origin (for maximize) or towards origin (for minimize).

The LAST CORNER pt. where this line touches, that pt is corner pt. & its coordinates give the optimum solⁿ

Note :- If the line is becoming $\parallel$ to boundary line, then $\exists$ ∞ solⁿˢ.

Q Solve graphically :-

Maximize : $z = 3x + 2y$

s.t : $-2x + 3y \leq 9$

$3x - 2y \leq -20$

non-ve : $x, y \geq 0$

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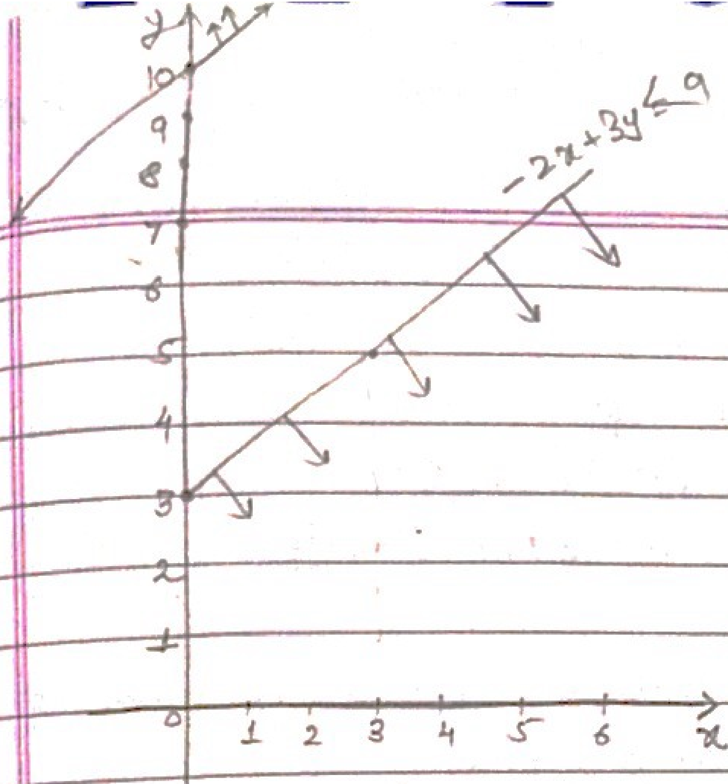


$-2x+3y=9$

$2x-3y=-9$

$3x-2y=-20$

$\exists$ no feasible region for this LPP. So, $\exists$ no optimum solⁿ.

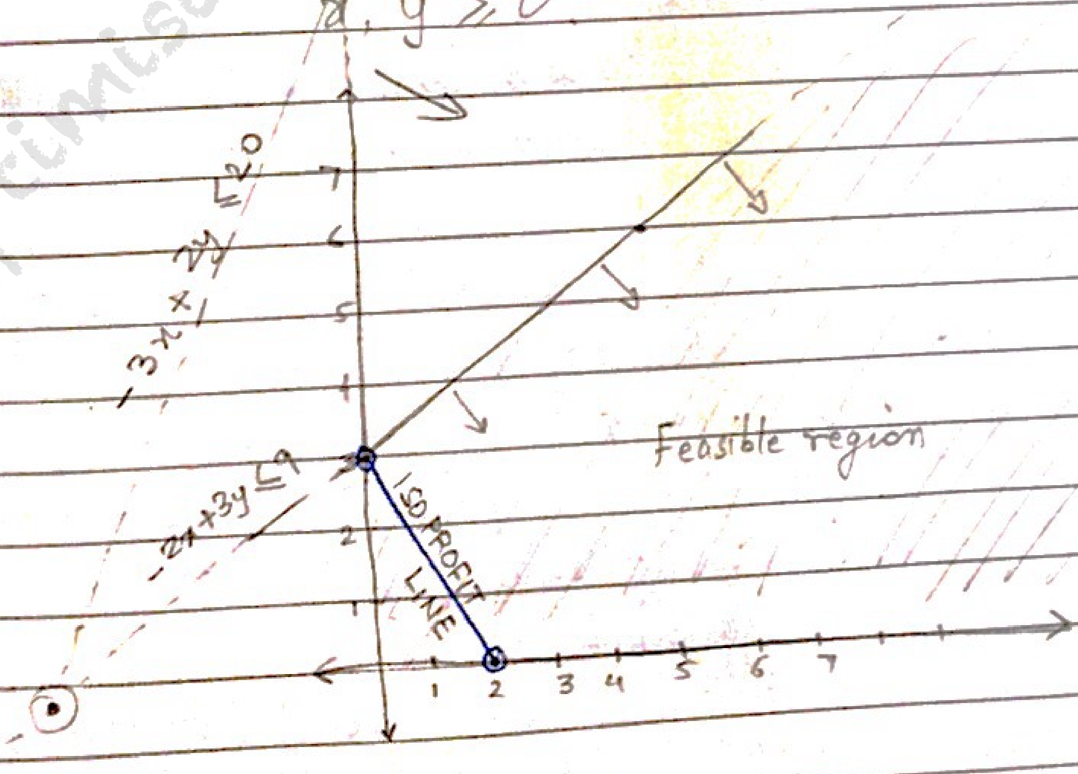
Q Solve the following LPP graphically.

Maximize $z = 3x+2y$

s.t. $-2x+3y \le 9 \longrightarrow (1)$

$-3x+2y \le 20 \longrightarrow (2)$

$x, y \ge 0$.



Making iso-profit line for maximizing problem
Taking any value of $z = 6$, say.
$\Rightarrow \; 3x + 2y = 6$
$\rightarrow (0,3), (2,0)$
Now extending it outwards, Z bound on upper side.

Hence,
the given feasible region is unbounded on upper side.
Thus, given LPP has ∞ sol^n

Part B :- Minimize $z = 3x + 2y$
Clearly, from graph, moving the line (iso-cost line) towards the origin.
Z min value $= 0$

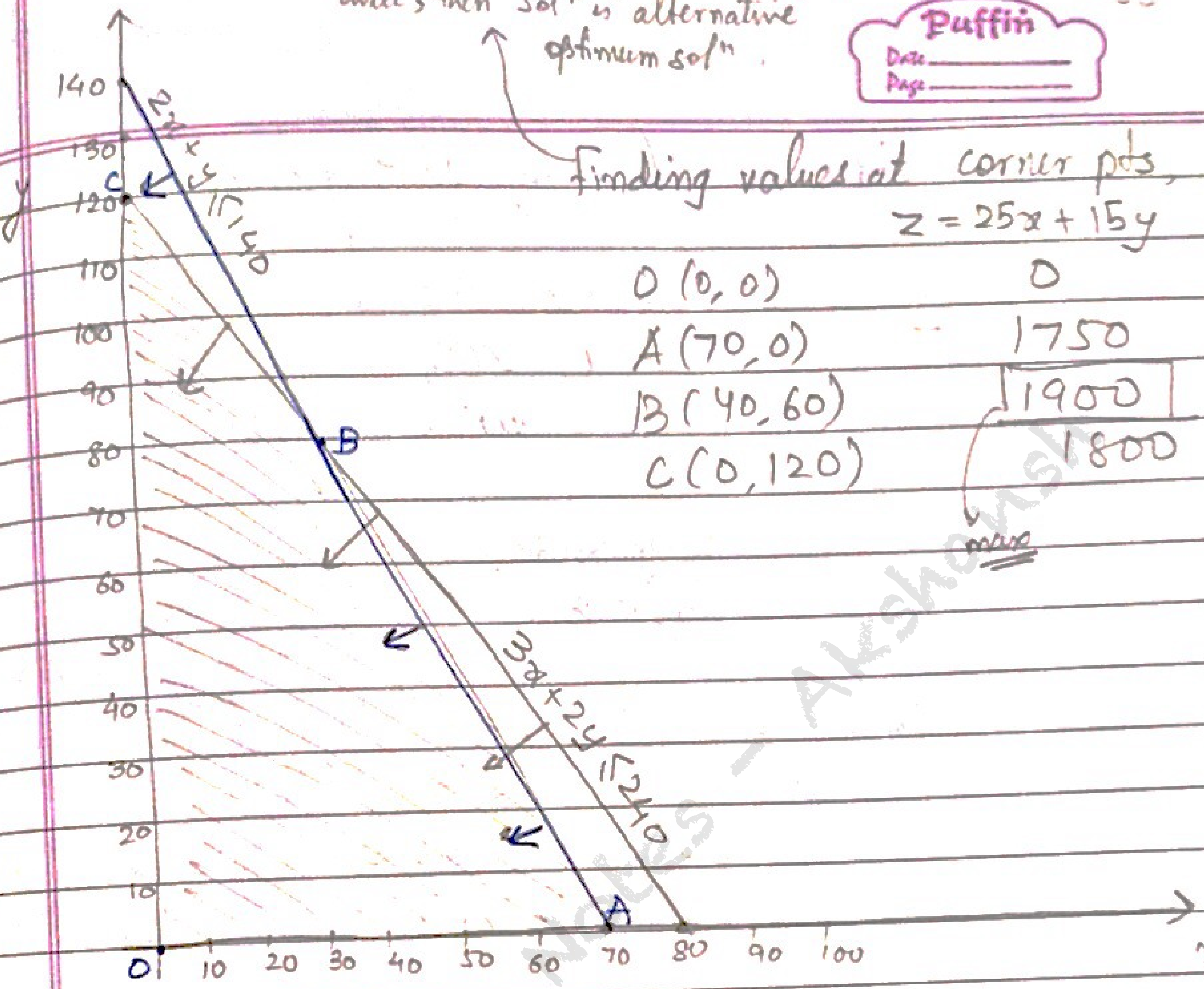
Q Solve graphically :-
Maximize $z = 25x + 15y$
s.t.
$3x + 2y \leq 240 \longrightarrow ①$
$2x + y \leq 140 \longrightarrow ②$
$x, y \geq 0$

0	80
120	0

70	0
0	140

Finding pt. of intersection of ① & ② :-
$3x + 280 - 4x = 240$
$x = 40$
$y = 60$
$z = 25(40) + 15(60)$
$= 1000 + 900$
$\boxed{z = 1900}$

Idea :- while making this table, if max value occurs twice, then solⁿ is alternative optimum solⁿ.



Finding values at corner pts.

	$Z = 25x + 15y$
O (0, 0)	0
A (70, 0)	1750
B (40, 60)	$\boxed{1900}$
C (0, 120)	1800

max

The optimum solⁿ of given LPP is :-
$x = 40,\ y = 60,\ Z_{max} = 1900.$

* Note :- Alternative optimum solⁿ (or alternate)
If in 2 consecutive corner pts. we get the same MAX value, then given LPP has Alternate solⁿ & any pt. on line segment joining these corner pts. will give us ~~corner~~ optimum solⁿ.

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(HW) Q. Solve the following LPP graphically

(1) Maximize $z = 3000x + 7000y$

s.t $200x + 500y \leq 40,000$

$x \geq 10$

$y \geq 10$

$x - y \geq 0$

$x, y \geq 0$.

Ans:- $x = 175$

$y = 10$

$Z_{max} = 59500$

(2) Maximize $z = 9x + 20y$

s.t $4x + 6y \leq 1200$

$10x + 35y \leq 3500$

$x, y \geq 0$

Ans:- $x = 262$

$y = 25$

$Z_{max} = 2858$

§ SIMPLEX METHOD.

- **Requirements of Simplex Method:**

If not, make it
① All constraints (except non-negativity restriction) should be eqns.
② RHS constant must be non negative.
③ All decision variables are non negative.

- How to convert an inequation-type constraint into eqn :-

$\geq$ or $\leq$

Type - I :- $\leq$ type :-

eg :- $3x + 2y + 5z \leq 7$

[LHS $\leq$ RHS ⇒ If we add sth to LHS, it can become = RHS i.e LHS + s = RHS]

$\Rightarrow 3x + 2y + 5z + s_1 = 7$

s_1 : SLACK VARIABLE

Type - II :- $\geq$ type :-

eg :- $3x + 2y + 5z \geq 7$

$\Rightarrow 3x + 2y + 5z - s_2 = 7$

s_2 : SURPLUS VARIABLE

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eg :- $3x + 2y \leq -7$

$\times$ (-1) both sides

$\Rightarrow -3x - 2y \geq 7$

(type 2)

$\Rightarrow -3x - 2y - s_2 = 7$.

eg :- $2x - 5y \geq -2$

$\Rightarrow -2x + 5y \leq 2$.

(type - 1)

$\Rightarrow -2x + 5y + s_1 = 2$.

Q. Write the following LPP in eq^n form :-

Maximize :- $Z = 2x_1 + 3x_2 - x_3$

s.t $x_1 - 2x_2 + x_3 \leq 7$

$2x_1 - x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

Sol^n :- Maximize :- $z = 2x_1 + 3x_2 - x_3 + 0.s_1 + 0 \cdot s_2$

S1) Check if constraint has a -ve sign on RHS.

If not, continue as it is.

S2) s.t

$x_1 - 2x_2 + x_3 + s_1 = 7$

$2x_1 - x_3 - s_2 = 5$.

& $x_1, x_2, x_3, s_1, s_2 \geq 0$

Note :- s_1 : slack variable, s_2 : surplus variable.

☆ How to tackle variables which have no restriction in sign :-

eg : let LPP be :-

$$\text{Maximize} \quad Z = 2x_1 + 3x_2$$
$$\text{s.t} \quad x_1 + x_2 \leq 5$$
$$2x_1 + 5x_2 \leq 6$$
$$x_1 \geq 0 \; ; \; x_2 \text{ is unrestricted.}$$

Idea :-

Replace x_2 by x_3 & x_4.

s.t. $x_2 = x_3 - x_4$ s.t $x_3, x_4 \geq 0$

Why?

$\because$ It satisfies all values of x_2.

$$\left.\begin{array}{ll} x_2 = 0 \; ; & x_3 = x_4 \\ x_2 > 0 \; ; & x_3 > x_4 \\ x_2 < 0 \; ; & x_3 < x_4 \end{array}\right\} x_3, x_4 \geq 0$$

So, a 2 variable problem changes to 3 variable

Soln :- let $x_2 = x_3 - x_4$; $x_3, x_4 \geq 0$,

So, LPP :

$$Z = 2x_1 + 3x_3 - 3x_4$$
$$\text{st} \quad x_1 + x_3 - x_4 \leq 5$$
$$2x_1 + 5x_3 - 5x_4 \leq 6.$$
$$x_1, x_3, x_4 \geq 0.$$

Convert

$$Z = 2x_1 + 3x_3 - 3x_4 + 0s_1 + 0s_1'$$
$$\text{s.t} \quad x_1 + x_3 - x_4 + s_1 = 5 \quad ; \; x_1, x_3, x_4, s_1, s_1' \geq 0$$
$$2x_1 + 5x_3 - 5x_4 + s_1' = 6 \quad ; \; s_1, s_1' : \text{slack variable}$$

✲ How to convert a minimiz^n problem to maximiz^n problem.

Idea :- Minimizing $Z \Rightarrow$ Maximizing $-Z$.

$\Rightarrow$ Maximizing Z^*, say

$\hookrightarrow Z^* = -Z$.

eg :- Min $Z = 3x_1 - 5x_2$

$\Rightarrow$ Max $Z^* = -Z = -3x_1 + 5x_2$

→ Note :- In such questions, value of Z^* comes as −ve at times. We want value of Z, not Z^*. So, × by (-1) & tell.

✲ How to solve using SIMPLEX METHOD

S1) Convert given LPP into maximiz^n type & equality constraints.

S2) Construct the initial simplex tableau → many tables in 1 table

S3) Calculate $c_j - z_j$ for all variables.

S4) If all $c_j - z_j \leq 0$, then sol^n under test is optimum.
no further improvement req^d.

S5) If S4 doesn't get satisfied, i.e, atleast one of $c_j - z_j > 0$, then,
find entering variable (i.e, the variable corres-ponding to the max. +ve $c_j - z_j$)

S6) Find the departing or leaving variable i.e, the variable corresponding to the min. ratio.

S7) Revise the simplex table & go to step 3.

eg Maximize :- $Z = 2x_1 + 3x_2 + 0 \cdot S_1 + 0 \cdot S_2$

s.t $x_1 - 2x_2 + S_1 = 10$

$2x_1 + 5x_2 + S_2 = 20$

$x_1, x_2, S_1, S_2 \geq 0$

S_1, S_2 :- slack variables

Constructing Tableau

→ coeff. of x_1, x_2, S_1, S_2 in order

C_j		2	3	0	0		
	Basic	x_1	x_2	S_1	S_2	Solution	Ratio
0	S_1	1	-2	1	0	10	—
0	← S_2	2	5	0	1	20	4
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	2	3 ↑	0	0		

coeff. of constraints

coeff. of x_j in objective f^n.

coeff. of S_1, S_2 in objective f^n.

$0 \times 2 + 0 \times 1$

$0 \times 5 + 0 \times (-2)$

$2 - 0$ $\quad 3 - 0 \quad 0 - 0 \quad 0 - 0$

Entering variable :- variable corresponding to max value of $C_j - Z_j$

If +ve no. is there in $C_j - Z_j$, then sol^n can be improved.

The max +ve value is entering variable.

Taking ratio :- Don't take for −ve or zero element

Take $\dfrac{\text{Solution}}{x_2}$ = Ratio

Min ratio = Departing Variable.

- Intersection of entering & departing variable is called PIVOT element
- If ALL elements in the entering vector of LPP, in the simplex table are NEGATIVE, then, NO feasible solution exist.

☆ Cases :- If same values occur in :-

$C_j - Z_j$:- Choose any value (break the tie)

Ratio :- Choose any value. If solution is coming in LOOP, infinitely, then, that solution is called Degenerate solⁿ. Then, ratio is chosen by DEGENERACY

☆ Note :- ELEMENTARY MATRIX OPERATIONS ON ROW ONLY (not column) is possible in Simplex method.

☆ Making Identity matrix : $\begin{bmatrix} 1 & 0 & \times & \times \\ 0 & 1 & \times & \times \end{bmatrix} \rightarrow$ for x_1, x_2, s_1, s_2

Make the column with Pivot element as identity. i.e., Pivot element = 1, all other elements in that column = 0.

This makes the new simplex table

eg ① Solve the following LPP by simplex method

Maximize :- $Z = 9x_1 + 7x_2$

s.t :- $2x_1 + x_2 \leq 40$; $x_1, x_2 \geq 0$

$x_1 + 3x_2 \leq 30$

Introducing slack var. s_1, s_2 ; the given LPP can be written in following form :-

Maximize :- $Z = 9x_1 + 7x_2 + 0.s_1 + 0.s_2$

S.t $2x_1 + x_2 + s_1 = 40$

$x_1 + 3x_2 + s_2 = 30$

$x_1, x_2, s_1, s_2 \geq 0$

Constructing table :

pivot element

C_j		9	7	0	0		
	Basic	x_1	x_2	s_1	s_2	Sol^n	Ratio
0	$\boxed{s_1}$	2	1	1	0	40	20
0	s_2	1	3	0	1	30	30.
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	9	7	0	0	—	

$0 \times 1 + 0 \times 2$

Starting of table making :

∃ 2 basic variables :- s_1, s_2.

We assume initially, all non basic variables (x_1, x_2) are zero,

$\Rightarrow x_1, x_2 = 0$; $s_1 = 40$, $s_2 = 30$

Selecting entering variable ; max. value of $C_j - Z_j$ gives the column of entering variable.

But, entering done in place of S_1 or S_2? i.e., how to find the leaving variable?
↳ Find ratio

The min. ratio is selected & so, we get S_1 as leaving var. (here)

So, in next step, write x_1, instead of S_1

make col. of pivot element = 0

make pivot element = 1

$Row_1 \div 2$

C_j		9	7	0	0		
	Basic	x_1	x_2	S_1	S_2	Sol^n	Ratio
9	x_1	1	1/2	1/2	0	20	40
0 ← S_1	S_2	0	5/2	-1/2	1	10	4
	Z_j	9	9/2	9/2	0	180	
	$C_j - Z_j$	0	5/2	-9/2	0		

coeff. of x_1 in objective fn

+ve value ⟹ improvement possible

∃ min. ratio

& $Row_2 - Row_1$

5/2 is max. value & so, take ratio using 2nd column.

Min. ratio = 4.

So, in next table, change x_2 in place of S_2 & keep x_1 as it is.

C_j		9	7	0	0		
	Basic	x_1	x_2	s_1	x_2	Sol^n	Ratio
9	x_1	1	0	3/5	−1/5	18	
7	x_2	0	1	−1/5	2/5	14	
	Z_j	9	7	4	1	190	
	$C_j - Z_j$	0	0	−4	−1		

Step 1

Inference

Now, without changing Row_1, start making identity matrix. :-

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

So, $R2 \times (2/5)$ to make 2nd element in Row_2 = 1

So,

1	1/2	1/2	0	R_1
0	5/2	−1/2	1	R_2

$\times \frac{2}{5}$ → 0   1   −1/5   2/5

Now,

$$R_1 \longrightarrow R_1 - \frac{R_2}{5}$$

So,

$$\begin{cases} 1 & 0 & 3/5 & -1/5 \\ 0 & 1 & -1/5 & 2/5 \end{cases}$$

→ Put these in table.

In this (←) table, all $c_j - z_j \leq 0$.
So, solⁿ under test has now become optimum.

So, we find $x_1 = 18$, $x_2 = 4$ (had x_1 or x_2 not have been there in Basic column, their value is zero)

$Z_{max} = 190$ Ans

Q. Use simplex method to solve:-

Minimize :- $Z = 8x_1 - 2x_2$.
s.t $-4x_1 + 2x_2 \leq 1$;
$5x_1 - 4x_2 \leq 3$;
$x_1, x_2 \geq 0$.

Idea: To minimize z, maximize $\bar{Z}$ s.t $\bar{Z} = -Z$.
Note:- Change $\bar{Z}$ to Z in the end of result

So, let $\bar{Z} = -Z = -8x_1 + 2x_2$

So, now, we have

Maximize $\bar{Z} = -Z = -8x_1 + 2x_2$.
s.t $-4x_1 + 2x_2 + S_1 = 1$
$5x_1 - 4x_2 + S_2 = 3$
$x_1, x_2, S_1, S_2 \geq 0$

Now,

Constructing table

take ratio for -ve no

C_j		-8	2	0	0		
	Basic	x_1	x_2	s_1	s_2	Sol^n	Ratio
0	$\leftarrow s_1$	-4	2	1	0	1	1/2
0	s_2	5	-4	0	1	3	—
	$\bar{Z}_j$	0	0	0	0	0	
	$C_j - \bar{Z}_j$	-8	2	0	0		

↑ +ve value ⇒ Max. value can come
Only one +ve no. ⇒ it is entering variable

Now, make the pivot element has to made 1 & the col. = 0 for other elemts. So, $R_1 \to R_1/2$.

$R_2 \to R_2 + (R_1 \times 2)$

x_2 comes in place of s_1

2	x_2	-2	1	1/2	0	1/2	
0	s_2	-3	0	2	1	5	
	$\bar{Z}_j$	-4	2	1	0	1	
	$C_j - \bar{Z}_j$	-4	0	-1	0		

no +ve no.
⇒ sol^n under test is optimum

All $C_j - \bar{Z}_j \leq 0 \Rightarrow$ Sol^n is optimum.

$x_1 = 0$ (∵ not a basic variable)
$x_2 = 1/2$

Now,
$$Z_{min} = -\bar{Z}_{max} = -(1) = -1$$

[Aliter, put $x_1 = 0$, $x_2 = 1/2$ in original objective f^n]

HW Q Maximize $Z = 3x_1 + 5x_2$
s.t. $x_2 \leq 6$,
$3x_1 + 2x_2 \leq 18$
$x_1 \leq 5$
$x_1, x_2 \geq 0$

Converting to eq^n:-

Maximize $Z = 3x_1 + 5x_2 + 0.s_1 + 0.s_2 + 0.s_3$

s.t $x_2 + s_1 = 6$

$3x_1 + 2x_2 + s_2 = 18$

$x_1 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3 \geq 0$.

Table

C_j		3	5	0	0	0		
	Basic	x_1	x_2	s_1	s_2	s_3	Sol^n	Ratio
← 0	s_1	0	1	1	0	0	6	$6/1 = 6$
0	s_2	3	2	0	1	0	18	$18/2 = 9$
0	s_3	1	0	0	0	1	5	—
	Z_j	0	0	0	0	0		
	$C_j - Z_j$	3	5 ↑	0	0	0		

max. value possible ∴ no. ≥ 0 ∴ it is pivot element

$R_2 \to R_2 - 2R_1$

C_j	Basic	x_1	x_2	s_1	s_2	s_3	Sol^n	Ratio
5	x_2	0	1	1	0	0	6	—
← 0	s_2	3	0	−2	1	0	6	2
0	s_3	1	0	0	0	1	5	5
	Z_j	0	5	5	0	0	30	
	$C_j - Z_j$	3 ↑	0	−5	0	0		

$R_2 \to R_2/3$

$R_3 \to R_3 - R_2$

C_j	Basic	x_1	x_2	s_1	s_2	s_3	Sol^n
5	x_2	0	1	1	0	0	6
3	x_1	1	0	$-2/3$	$1/3$	0	2
0	s_3	0	0	$2/3$	$-1/3$	1	3
	Z_j	3	5	3	1	0	36
	$C_j - Z_j$	0	0	−3	−1	0	

no +ve. So, we got optimum sol^n

Sol^n:- $x_1 = 2$, $x_2 = 6$, $Z_{max} = 36$. Ans

IF ANY LPP CONSISTS OF ATLEAST ONE '=' TYPE AND/OR '$\geq$' CONSTRAINT, THEN WE CAN USE **BIG-M METHOD/PENALTY METHOD** TO SOLVE IT.

Steps

S1) Convert the problem into maximiz^n type (if not).

S2) Convert each constraint into eq^n by using the following rule:-

- Add a slack variable to each '$\leq$' type const.
- Subtract a surplus variable & add an additional variable to each '$\geq$' type constraint.
- Add an additional variable to each '=' type constraint. → Its final value = 0.

This add^{nal} var. is called Artificial variable.

eg. for constraint: $2x_1 - 3x_2 \geq 5$

$$2x_1 - 3x_2 - S_1 + A_1 = 5$$

eg 2 $\quad 2x_1 - 3x_2 = 5$.

$$\Rightarrow 2x_1 - 3x_2 + A_2 = 5$$

S3) In the initial table, take slack variables & artificial var$_s$ as initial basic variables. (don't take surplus variables)

Big M

In the objective f^n, coeff. of each artificial variable will be $-M$; M: large +ve no.
(& coeff. of slack & surplus var = 0)

eg 3 $\quad \text{Max } z = x_1 + 3x_2$

$$\text{s.t} \quad x_1 + 5x_2 \leq 2$$
$$2x_1 + x_2 \geq 5$$
$$x_1 + x_2 = 6$$
$$x_1, x_2 \geq 0$$

change

$$\text{st} \quad x_1 + 5x_2 + S_1 = 2$$
$$2x_1 + x_2 - S_2 + A_1 = 5$$
$$x_1 + x_2 + A_2 = 6$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

& $\text{Max}:- Z = x_1 + 3x_2 + 0.S_1 + 0.S_2 - MA_1 - MA_2$

S4) Solve this LPP by Simplex Method.

☆ Note :

If any artificial var. departs from basis (leaving basis), from next ~~tab~~ step (table), drop the column corresponding to that artificial var.

eg :-

	x_1	x_2	s_1	s_2	A_1	A_2
S_1						
← A_1						
A_2						

If this happens

remove this column

($\because$ Artificial var., once gone, cannot enter basis again)

Page

✲ Some Special Cases :-

1) Infeasible solⁿ :-

If in the final simplex table, any artificial variable (A) appears with a +ve value (in solⁿ column), then, given LPP has no feasible solⁿ.

→ valid only when ATLEAST ONE constraint is '=' type or '≥' type

2) Unbounded solⁿ :-

If, in any step of simplex method, all the elemts. in entering variable column are ≤ 0, then, given LPP has unbounded solⁿ.
(we cannot find ratio in that case).

✲ Note :-

Unbounded solⁿ is possible in maximizⁿ type LPP only.

3) Alternate Solⁿ :-

If, in the FINAL simplex table, '$c_j - z_j = 0$' for any non-basic variable, ~~then, given LPP~~ has '$c_j - Z_j = 0$', then, given LPP has alternate solⁿ. (we get 1 solⁿ)

After getting final table, see the column of non basic variables. If any of them is zero ($c_j - Z_j$ value) then, alternate solⁿ exists. Now, take that 0 as entering variable. You'll get 2nd solⁿ.

4) Degenerate sol^n.

If atleast one of the basic var. = 0 (in FINAL table). Then, sol^n is degenerate.

Q. Solve by Big M / penalty method :-

$$\text{Max. } Z = 2x_1 + 3x_2$$
$$\text{s.t} \quad 6x_1 + 9x_2 \leq 18$$
$$9x_1 + 3x_2 \geq 9$$
$$x_1, x_2 \geq 0$$

convert

$$\text{Max } Z = 2x_1 + 3x_2 + 0S_1 + 0.S_2 - MA_1$$
$$\text{s.t.} \quad 6x_1 + 9x_2 + S_1 = 18$$
$$9x_1 + 3x_2 - S_1 + A_1 = 9$$
$$x_1, x_2, S_1, S_2, A_1 \geq 0$$

slack   Surplus   Artificial.

Table

C_j		2	3	0	0	-M		
	Basic	x_1	x_2	s_1	s_2	A_1	Sol^n	Ratio
0	S_1	6	9	1	0	0	18	3
← -M	A_1	$\boxed{9}$	3	0	-1	1	9	1 min
	Z_j	-9M	-3M	0	M	-M		
	$C_j - Z_j$	$\underbrace{2+9M}_{\text{max value}}$	3+3M	0	-M	0		

↑ (9>3)

Now, $R_2/9$

$$R_1 \rightarrow R_1 - 6R_2$$

(i.e. Artificial var left basis) column dropped

C_j		2	3	0	0	-M		
	Basic	x_1	x_2	s_1	s_2	A_1	Sol^n	Ratio
← 0	s_1	0	7	1	2/3	–	12	12/7
← 2	x_1	1	1/3	0	-1/9	–	1	3
	Z_j	2	2/3	0	-2/9	–	2	
	$C_j - Z_j$	0	7/3 ↑	0	2/9	–	–	

Now, $R_1/7$

$R_2 \to R_2 - R_1/3$.

3	x_2	0	1	1/7	2/21	–	12/7
2	x_1	1	0	-1/21	-1/7	–	3/7
	Z_j	2	3	1/3	0	–	6
	$C_j - Z_j$	0	0	-1/3	0	–	

all -ve $C_j - Z_j$

→ non basic variable (s_2) Value = 0. So, Alternate sol^n exists.

Now, check basis col. If no artificial exists, we got optimum sol^n

So, sol^n :- $x_2 = \frac{12}{7}$, $x_1 = \frac{3}{7}$, sol^n = $Z_{max} = 6$

Ans

* Finding the alternate sol^n,

eg. done on next page

s_2 is entering variable.

x_2 goes out (see after solving)

So, we get new value $x_1 = (\ \ )$

$x_2 = '0'$

So, for these values of x_1, x_2 we get again $Z_{max} = 6$.

* which value of $c_j - z_j$ to take for entering variable?

This is what we do in maximiz^n problem ← ✓ Taking +ve value ⇒ optimum sol^n will increase

✓ Taking −ve value ⇒ optimum sol^n will decrease

✓ Taking 0 value ⇒ optimum sol^n won't change

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eg Final simplex table of an LPP is given below:

C_j		2	3	0	0		
	Basic	x_1	x_2	s_1	s_2	Sol^n	Ratio
← 3	x_2	0	1	1/7	2/21	12/7	18
2	x_1	1	0	−1/21	−1/7	3/7	—
	Z_j	2	3	1/3	0	6	
	$C_j - Z_j$	0	0	−1/3	0 ↑	—	

(a) What is the optimum sol^n?

(b) Is there any alternate sol^n? Justify

(c) If ans. is yes in part (b), find atleast one alternate sol^n.

(a) Optimum sol^n is: $x_1 = \frac{3}{7}$, $x_2 = \frac{12}{7}$, $Z_{max} = 6$

(b) Yes. Alternate sol^n exists ∵ one of the non basis variables in final table (s_2) has a zero in the row of $C_j - Z_j$.

(c) Finding one alternate sol^n → Take the column for which $C_j - Z_j = 0$ as the entering variable. ($C_j - Z_j = 0$ ⇒ optimum sol^n won't change)

Table contd

$R_1 \to R_1 \times \frac{21}{2}$	0	S_2	0	21/2	3/2	1	18
$R_2 \to R_2 + \frac{R_1}{7}$	2	x_1	1	3/2	1/6	0	3
		Z_j	2	3	1/3	0	6
		$C_j - Z_j$	0	0	−1/3	0	6

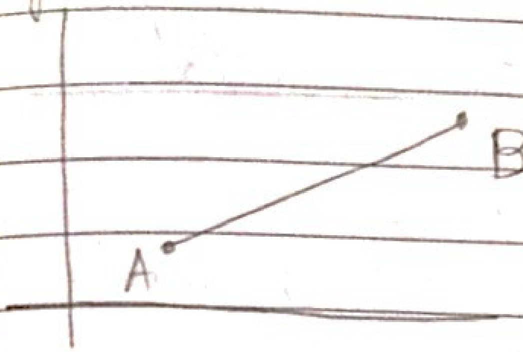
$\underbrace{0 \quad 0 \quad -1/3 \quad 0}_{\text{optimum sol}^n\text{ got}}$

So, new sol^n (alternate sol^n): $x_1 = 3$, $x_2 = 0$

$Z_{max} = 6$ (same)

* Suppose the alternate sol^n is $A(x_1, x_2)$
other sol^n is $B(y_1, y_2)$
Using pts. A & B, we can draw a line segment

Note



Now, whichever pt. we take b/w A & B is the optimum sol^n of given objective f^n.

Now, how to find any pt. in b/w?
Idea :- Take a variable α, $(0 \leq \alpha \leq 1)$.
Using Linear $Combin^n$:-

$$\alpha(x_1, x_2) + (1-\alpha)(y_1, y_2)$$

→ Take any value of α, solve to get a new pt. everytime

ex Max $Z = 2x_1 + 5x_2$
s.t $3x_1 + 2x_2 \geq 6$
$2x_1 + x_2 \leq 2$
$x_1, x_2 \geq 0$

Max $Z = 2x_1 + 5x_2 + 0.S_1 + 0.S_2 - MA_1$
$3x_1 + 2x_2 - S_1 + A_1 = 6$
$x_1 + x_2 + S_2 = 2$
$x_1, x_2, S_1, S_2, A_1 \geq 0$

Surplus Slack Artificial

Table :

	C_j	Basic	x_1	x_2	S_1	S_2	A_1	Sol^n	Ratio
			2	5	0	0	$-M$		
only slack & Artificial	$-M$	A_1	3	2	-1	0	1	6	2
←	0	S_2	2	1	0	1	0	2	1
		Z_j	$-3M$	$-2M$	M	0	$-M$	$-6M$	
		$C_j - Z_j$	$3M+2$ ↑	$2M+5$	$-M$	0	0	—	
$R_1 \to R_1 - 3R_2$	$-M$	A_1	0	$1/2$	-1	$-3/2$	1	3	6
$R_2/2$ ←	2	x_1	1	$1/2$	0	$1/2$	0	1	2
		Z_j	2	$-\frac{M}{2}+1$	M	$\frac{3M}{2}+1$	$-M$	$-3M+2$	
		$C_j - Z_j$	0	$\frac{M}{2}+4$ Max ↑	$-M$	$-\frac{3M}{2}-1$	0	—	
$R_1 \to R_1 - \frac{R_2}{2}$	$-M$	A_1	-1	0	-1	-2	1	2	
$R_2 \times 2$	5	x_2	2	1	0	1	0	2	
		Z_j	$M+10$	5	M	$2M+5$	$-M$	$-2M+10$	
		$C_j - Z_j$	$-M-8$	0	$-M$	$-2M-5$	0	—	

all $-$ve. So, no ~~optimum sol^n~~

See this { We have artificial var. in Basis column & its value in sol^n column is +ve.

$\because$ $C_j - Z_j < 0$; $\exists$ no feasible sol^n $\because$ artificial variable (A_1) appears with +ve value.

(If Artificial var. has 0 or $-$ve value, then optimum sol^n exists ✓)

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TWO PHASE METHOD

- another method if constraints are '=' or '≥' type
- ∃ 2 phases.

S1) Like Big M method, add slack/surplus/artificial variables to convert all the constraints into eq^{ns}

S2) PHASE-1:

Setup an LPP with above constraints, but, the objective f^n:

Maximize $w = -A_1 - A_2 \ldots - A_k$

; $A_1, A_2, \ldots A_k$ are artificial variables used in a problem.

Solve it by Simplex method.

After this, 2 possibilities:

(a) $W_{max} \neq 0 \Rightarrow$ given LPP has no feasible sol^n.

So, Stop at this point.

(b) $W_{max} = 0 \Rightarrow$ proceed to Phase-2.

PHASE-2:

In phase-2, take the optimum sol^n of phase-1 as starting basic feasible sol^n (BFS) & solve it the original LPP by simplex method.

Note :- We must drop all the artificial var., before proceeding to phase 2.

how!!

PTO

* how to drop artificial variables?

Rule 1 : Drop the columns of all non basic artificial variables (of the final table in phase 1)

now, how to drop an artificial variable present in the basis?

Rule 2 : If any of artificial var is present in basis with zero value,

eg

Basic	x_1	x_2	A_1	A_2	Sol^n
A_1	2	-1	3	4	0
x_2	2	1	5	3	2

Artificial var. has 0 value

S1) Consider A_1 as leaving var.

S2) See the non basic var. (x_1, A_2 → here).
See which their values (+ve or not)
Take the +ve value as pivot element
(here, take x_1(2) or A_2(4), any one)

S3) Then, solve similarly like done for Simplex method

ex Use 2 phase method to solve.

$$\text{Max } Z = x_1 + x_2$$
$$\text{S.t } 2x_1 + 4x_2 \geq 4$$
$$x_1 + 7x_2 \geq 8$$
$$x_1 + x_2 \geq 0.$$

(All $\geq$ type → only surplus var)

S1) : Converting to eqⁿˢ :-

$$\left.\begin{aligned} 2x_1 + 4x_2 - S_1 + A_1 &= 4 \\ x_1 + 7x_2 - S_2 + A_2 &= 8 \\ S_1, S_2, A_1, A_2 &\geq 0 \end{aligned}\right\} \text{①}$$

Phase - 1 :-

$$\text{Max} \quad W = -A_1 - A_2$$

($\because$ 2 artificial var. are used)

s.t $2x_1 + 4x_2 - S_1 + A_1 = 4$
$x_1 + 7x_2 - S_2 + A_2 = 8$ ← same
$S_1, S_2, A_1, A_2 \geq 0$.

Table -

C_j		0	0	0	0	-1	-1		
	Basic	x_1	x_2	S_1	S_2	A_1	A_2	Solⁿ	Ratio
-1	A_1	2	$\boxed{4}$	-1	0	1	0	4	1
-1	A_2	1	7	0	-1	0	1	8	8/7
	W_j	-3	-11	1	1	-1	-1	-12	
	$C_j - W_j$	3	11 ↑	-1	-1	0	0	—	
		$R_1 \to R_1/4$, $R_2 \to R_2 - \frac{7}{4}R_1$							
0	x_2	1/2	1	-1/4	0	1/4	0	1	—
-1	A_2	-5/2	0	$\boxed{7/4}$	-1	$-\frac{7}{4}$ ↓	1	1	4/7
	W_j	5/2	0	-7/4	1	7/4	-1	-1	
	$C_j - W_j$	-5/2	0	7/4 ↑	-1	-11/4	0	—	
						(Dropping column ($\because$ Artificial var. left basis)			
		$R_2 \to \frac{4}{7} \times R_2$, $R_1 \to R_1 + \frac{R_2}{7}$							
0	x_2	1/7	1	0	-1/7	—	—	8/7	
0	S_1	-10/7	0	1	-4/7	—	—	4/7	
	W_j	0	0	0	0	—	—	0	
	$C_j - W_j$	0	0	0	0	—	—	0	

$\exists$ no +ve no. $\Rightarrow$ we got optimum solⁿ.

The optimum solⁿ of phase (I) is

$A_1 = 0, A_2 = 0, \omega_{max} = 0$

(Note : Don't write anything about x_1, x_2 in solⁿ, because objective fn has A_1, A_2)

S3) Proceed to phase - II :- considering original objective fn

Last solⁿ of phase 1 is starting solⁿ of phase 2

C_j		1	1	0	0		
	Basic	x_1	x_2	S_1	S_2	Solⁿ	Ratio
1	x_2	$\boxed{1/7}$	1	0	$-1/7$	$8/7$	8
0	S_1	$-10/7$	0	1	$-4/7$	$4/7$	—
	Z_j	$1/7$	1	0	$-1/7$	$8/7$	
	$C_j - Z_j$	$6/7$	0	0	$1/7$		
		$R_1 \to 7R_1$					
1	x_1	1	7	0	-1		
0	S_1	0	10	1	-2		
	Z_j	1	7	0	-1		
	$C_j - Z_j$	0	-6	0	1		

All elements in this entering var. column are -ve $(-1, -2)$

So, ∃ UNBOUNDED SOLⁿ

Q) Solve by 2 phase method:-

Maximize $Z = 2x_1 + 5x_2$

s.t $3x_1 + 2x_2 \geq 6$

$2x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$.

Phase-I

Max $Z = -A_1$

$$\left.\begin{array}{l} \text{s.t} \quad 3x_1 + 2x_2 - S_1 + A_1 = 6 \\ 2x_1 + x_2 + S_2 = 2 \\ x_1, x_2, S_1, S_2, A_1 \geq 0 \end{array}\right\} (1)$$

C_j		0	0	0	0	-1		
	Basic	x_1	x_2	s_1	s_2	A_1	Soln	Ratio
-1	A_1	3	2	-1	0	1	6	2
← 0	S_2	$\boxed{2}$	1	0	1	0	2	1
	ω_j	-3	-2	1	0	-1	-6	
	$C_j - \omega_j$	3 ↑	2	-1	0	0	—	
		$R_2 \to R_2/2$.						
		$R_1 \to R_1 - 3(R_2)_{new}$						
-1	A_1	0	1/2	-1	-3/2	1	3	6
← 0	x_1	1	$\boxed{1/2}$	0	1/2	0	1	2
	ω_j	0	-1/2	1	3/2	-1	-3	
	$C_j - \omega_j$	0	1/2 ↑	-1	-3/2	0		
		$R_2 \to 2R_1$						
		$R_1 \to R_1 - (R_2)_{old}$						
-1	A_1	-1	0	-1	-2	—	2	
0	x_2	2	1	0	1	—	2	
	ω_j	1	0	1	2	—	-2	
	$C_j - \omega_j$	-1	0	-1	-2	—		

All $C_j - \omega_j \leq 0$, so we got optimum soln

The optimum of w from phase ① is :-

$W_{max} = -2$.

Now, $\because$ $W_{max} \neq 0$ so, $\exists$ no feasible solⁿ for given LPP

(can also be seen as :- In final table, value of artificial var. > 0 (= 2), so, $\exists$ no feasible solⁿ)

INTEGER PROGRAMMING PROBLEMS (IPP) → used only for LPP.

→ also abbreviated as ILP

Integer Linear Programming Problem

* ~~Wrong Idea: We need integer sol^{ns} in this problem. So, we can do is : solve the LPP & then round off the sol^n.~~

→ This won't work at all times ∵ sometimes constraints might be violated.

eg. if sol^n is $x_1 = 3$, $x_2 = \frac{1}{2}$

Constraint : $x_1 + 2x_2 \leq 4$

$3 + 1 = 4$. (satisfies constraint)

Rounding off $x_2 = 1$, say

$\Rightarrow 3 + 2 = 5 \; (\not\leq 4)$.

So, not satisfied.

☆ ILP

Pure ILP

When all decision variables are restricted to be Integers in an IIP (or ILP).

Mixed ILP

If some (not all) decision variables are restricted to be integers in an ILP.

Puffin
Date
Page

✫ METHODS TO SOLVE ILP.

(M1) Branch-and-Bound method (BB method)

not in course (M2) Cutting plane technique

(M1): Algorithm

S1) Ignore the ~~integral~~ integer condⁿ & solve the given LPP by simplex or graphical method.

eg (1)

Max $Z = 2x_1 + x_2$

s.t $x_1 + x_2 \geq 5$

$2x_1 + 5x_2 \leq 4$

$x_1, x_2 \geq 0$.

Not ILP problem

eg (2)

"

"

"

$x_1, x_2 \geq 0$

& are integers.

Pure ILP

eg (3)

"

"

"

$x_1, x_2 \geq 0$

& x_1 is integer

Mixed ILP

S2) If this solⁿ ~~self~~ satisfies integer condⁿ, it is optimum solⁿ. Stop at this point
If not, go to S3)

S3) Consider value of objective fⁿ obtained in S1) as the upper bound.
Obtain the lower bound by rounding off to the integral values of decision variables.
Note :- Value of Z got in S1) is max. possible value
If, after rounding off & putting values of solⁿ in Z, we get a value of Z > upper bound, then,
Rounding off was improper (constraints violated),
So, if you rounded up, round down instead.

* Representⁿ :- P_0 = original problem
SP_1 = Sub problem 1

S4) Let x_j^* be the value of x_j which is not integer
Then, P_0 (original problem) is divided as :

$$P_0 \begin{cases} SP_1 : P_0 \text{ with additional constraint } x_j^* \leq [x_j^*] \\ SP_2 : P_0 \quad '' \quad '' \quad x_j^* \geq [x_j^*]+1 \end{cases}$$

$[x_j^*]$ = Greatest integer $\leq x_j^*$

eg: if $x_j^* = 4.6$
$SP_1 :- x_j^* \leq 4$
$SP_2 :- x_j^* \geq 5$

eg (2) :- $x_1 = 0.25$
$SP_1 :- x_1 = 0$
$SP_2 :- x_1 \geq 1$
$\therefore x_1 < 0$ not possible

Solve these subproblems. (SP_1 & SP_2)

*If $\exists$ 2 variables : Both fractions,
eg: $x_1 = 2.57$, $x_2 = 7.21$.
For BRANCHING, choose the variable having larger fractional part as :-
Max $[0.57, 0.21] = 0.57$.
So, x_2 will be chosen for branching
If fractional part of both x_1 & x_2 are same, choose anyone

* How branching goes?
When Branching is done for P_0

$P_0 \to SP_1, SP_2$
SP_1: stop
$SP_2 \to (SP_2)_1, (SP_2)_2$

Choose anyone out of them & stop the other one & branch chosen one.

☆ KEY POINTS

✓ How to stop a branch?
- when $\mathbb{Z}$ solⁿ comes
- when infeasible solⁿ comes.

✓ If both branches are feasible & integer :-
Compare values of SP_1 & SP_2.
for maximizⁿ problem, choose max ($SP1$, SP_2)
" minimizⁿ " , min (SP_1, SP_2)

✓ Make a box at every point of branch when we get integer solⁿ.

✓ Rule for selecting a branch (SP) when both branches have non integer solⁿ.
Select that branch which has better Z value.

eg :-

	SP_1	SP_2
	$x_1 = 1.5$	$x_1 = 2.4$
	$x_2 = 0$	$x_2 = 1$
	$Z = 20$	$Z = 19$
for maximizⁿ	✓	X
minimizⁿ	X	✓

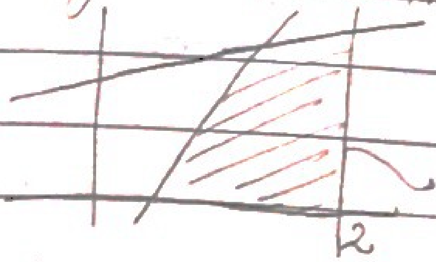
choose like that

✓ At every step, whatever value of Z that we get
for maximizⁿ problem : Its upper bound
for minimizⁿ problem : Its lower bound

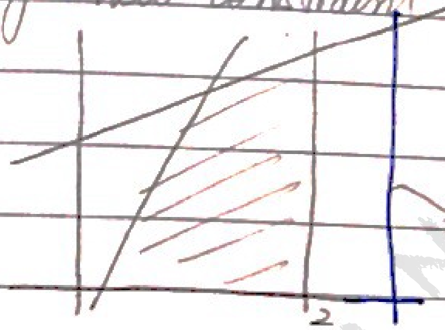
keeps changing at every step.

✓ If we add more constraints to an LPP, either size of feasible region either remains same or decreases. (NEVER increases)

eg:- If we had this region



If new constraint is $x_3 \leq 3$, graph changes as:



comes out of feasible region.
So, no common pt.
So, no feasible solⁿ

If new constraint is $x_4 \leq 1$, graph changes as:-

feasible region reduces

Q. Solve following LPP by Branch & bound method

Max :- $Z = 3x_1 + 2x_2$

s.t $2x_1 + 5x_2 \leq 9$; → ①

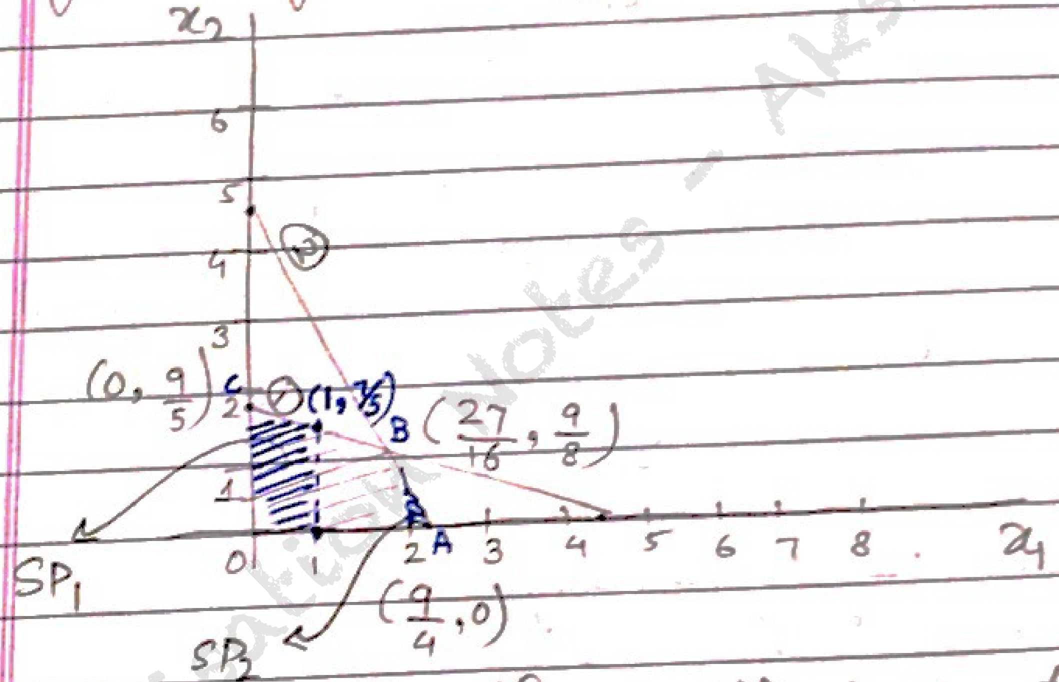
$4x_1 + 2x_2 \leq 9$ → ②

$x_1, x_2 \geq 0$ and are Integers

⇓

It's an IPP. Use branch & bound method

Solving graphically for feasible region :-



(way to write)

Let us consider the given LPP ignoring the integer cond^n & solving it graphically. Let this LPP be P_0 where :-

P_0 : max $Z = 3x_1 + 2x_2$

s.t - - - - -

After seeing max value of Z, we get

$\underbrace{x_1 = \frac{27}{16}, x_2 = \frac{9}{8}}_{\text{both } x_1 \text{ \& } x_2 \text{ are fractions.}}$, $Z = \frac{117}{16} = 7\frac{5}{16}$

RECORD → way of calling

Imp ✱ Make a box whenever you get integer solⁿ.
i.e, x_1, x_2 = integer.
Make a box around x_1, x_2 (not z)

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Choosing max. (x_1, x_2) for branching
$\Rightarrow \max\left(\frac{27}{16}, \frac{18}{16}\right)$.

So, x_1 is chosen.

greatest integer $\leq \frac{27}{16}$
$\leq 1\frac{11}{16}$
$= 1$

So, Branching:

P_0 → SP_1 : P_0 with additional constraint $x_1 \leq 1$
→ SP_2 : " " " $x_2 \geq 2$

From SP_1,
plotting $x_1 \leq 1$ on graph, as shown (in Blue color), the feasible region reduces giving the solution (max. → after solving ∀ points)
$x_1 = 1$, $x_2 = \frac{7}{5} = 1\frac{2}{5}$, $z = 5\frac{4}{5}$

From SP_2, again doing same thing & seeing max value of z, we get
$x_1 = 2$, $x_2 = 1/2$, $\boxed{z = 7}$ → Integer sol → new upper bound

its integer: so, no branching required

Seeing SP_1, SP_2, max. value of z is at SP_2 (= 7)
So, ignore SP_1.
Now further branching

SP_2 → SP_3 : SP_2 with additional constraint $x_2 = 0$
→ SP_4 : " " " $x_2 \geq 1$

Solⁿ of SP_3 (same way as prev) $x_1 = \frac{9}{4}$, $x_2 = 0$, $z = 6\frac{3}{4}$ → New upper bound

SP_4 :- Not feasible.

SP_3 → SP_5 : SP_3 with — — — $x_1 \le 2$

→ SP_6 :- " " $x_1 \ge 3$.

Integer solⁿ

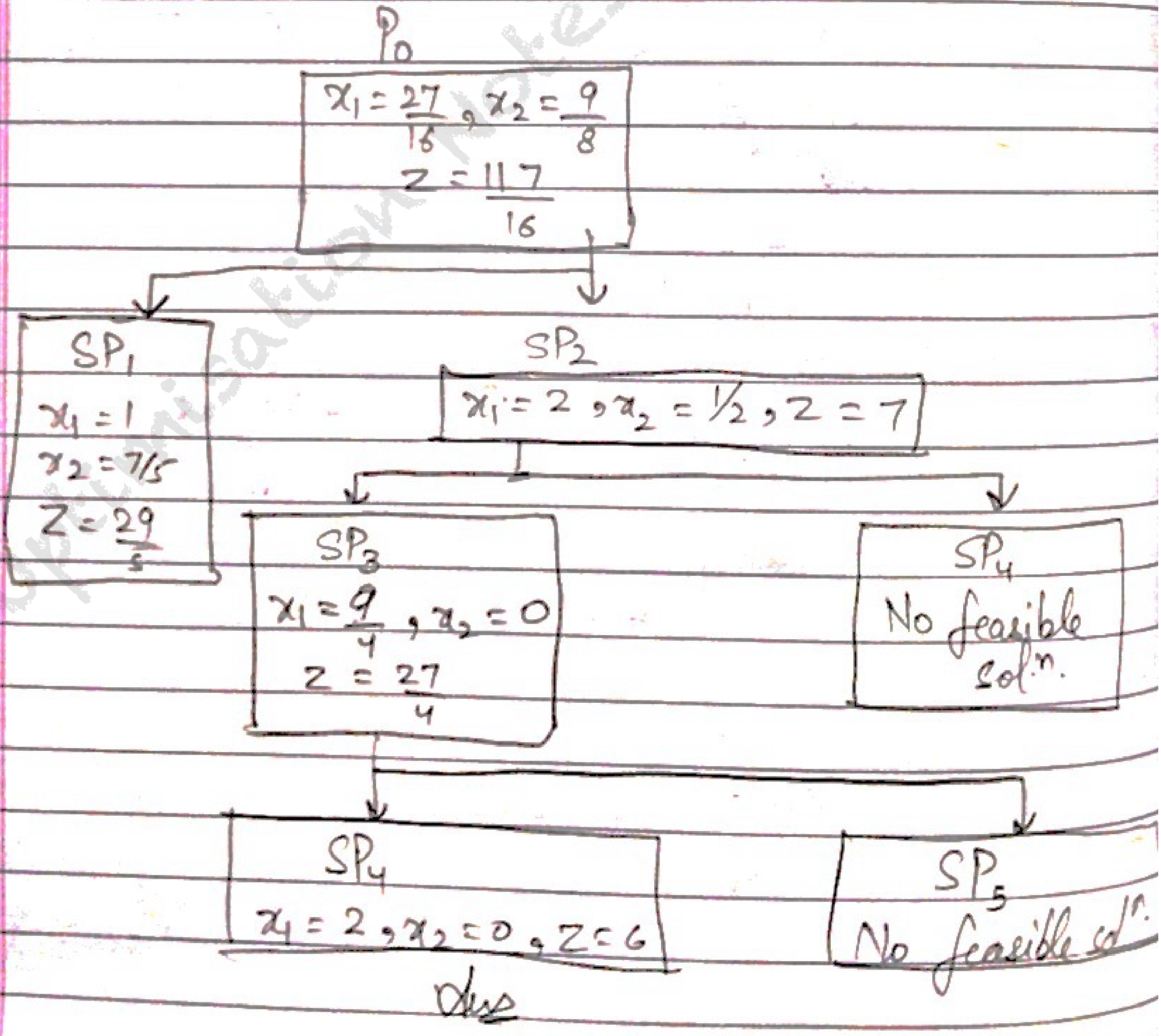
Solⁿ of SP_5 : $\boxed{x_1 = 2,\ x_2 = 0}$, $\boxed{z = 6}$ Integer

SP_6 : No feasible solⁿ

here, ∃ only one integer solⁿ. So, its optimum. & optimum solⁿ is $x_1 = 2$ & $x_2 = 0$.

$\underline{\underline{z = 6}}$

So, ans, written as :-



HW

Q. Solve the following ILP using branch & bound method:-

Minimize $z = 5x_1 + 4x_2$

s.t $3x_1 + 2x_2 \geq 5$

$2x_1 + 3x_2 \geq 7.$

& $x_1, x_2 \geq 0$ & are integers

Ans:- $x_1 = 0$

$x_2 = 3$

$Z_{min} = 12$

Ans:-

P_0 : $x_1 = 1/5$, $x_2 = 11/5$

$z = 49/5$

SP_1 : $x_1 = 0$, $x_2 = 5/2$

$z = 10$.

SP_2 : $x_1 = 1$, $x_2 = \frac{5}{3}$

$z = 35/3$

ignored, $\because$ $\min\{10, \frac{35}{3}\} = 10$

SP_3 : No feasible solution

SP_4 : $x_1 = 0$, $x_2 = 3$

$z = 12$

So, the optimum solⁿ is :- $x_1 = 0, x_2 = 3$ &

$Z_{min} = 12$

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Chapter-5

TRANSPORTATION PROBLEM

↳ Idea :- transport goods from factories at different locations to warehouses at different locⁿˢ. ∃ different capacities of the factories and warehouses.

Diagramatically :-

Sources	Destinations
○ F_1, a_1	○ W_1, b_1
○ F_2, a_2	○ W_2, b_2
○ F_3, a_3	○ W_3, b_3
	○ W_4, b_4

↳ capacity of storing goods

Let, ∃ 'm' sources and 'n' destinations.

Let 'c_{ij}' be the cost of transporting 1 unit from source 'i' to destination 'j'

eg :- $C_{25} = 4$

⇒ transportⁿ cost of transporting 1 unit of item from 2nd source to 5th dest. is 4 Rs.

a_i : amount available at source i $(i = 1, 2, \ldots, m)$

b_j : demand at destinⁿ j $(j = 1, 2, \ldots, n)$

x_{ij} : amount to be transported from source i to destinⁿ j $(i = 1, \ldots, m)$ $(j = 1, 2, \ldots, n)$

ex : x_{42} : amount of material transferred from 4th source to 2nd destinⁿ.

* The Cost Matrix in Tabular form

	D_1	D_2	- - -	D_n	Available
S_1	$x_{11}\ \boxed{c_{11}}$	$x_{12}\ \boxed{c_{12}}$	- - -	$x_{1n}\ \boxed{c_{1n}}$	a_1
S_2	$x_{21}\ \boxed{c_{21}}$	$x_{22}\ \boxed{c_{22}}$	- - -	$x_{2n}\ \boxed{c_{2n}}$	a_2
⋮	⋮	⋮		⋮	⋮
S_m	$x_{m1}\ \boxed{c_{m1}}$	$x_{m2}\ \boxed{c_{m2}}$	- - -	$x_{mn}\ \boxed{c_{mn}}$	a_m
Demand:	b_1	b_2	- - -	b_n	

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then, TP is called <u>balanced TP</u> (otherwise, unbalanced)

* SOLVING A BALANCED TP $\left(\sum a_i = \sum b_j \right)$

Objective f^n :- Minimize, $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

for rows' amount

s.t $\sum_{j=1}^{n} x_{ij} = a_i \quad ; \; i = 1, 2, ---, m$

for column's amount $\sum_{i=1}^{m} x_{ij} = b_j \quad ; \; j = 1, 2, ..., n$

$x_{ij} \geq 0 \quad \forall \; i, j$

§ Quiz Questions:

* Transport^n problem is a special kind of LPP.
* If there is no allocated amount in any cell of cost matrix, then, that cell is called EMPTY cell or vacant cell.
  If +ve allocation is there, then, cell is called Allocated cell or occupied cell.
* For a TP with m sources & n destin^ns, the max. no. of allocated cells = $m+n-1$
* If no. of allocated cells is exactly equal to $m+n-1$, then, sol^n is called non degenerate
  & its called degenerate sol^n if its less than $(m+n-1)$.
  $\Rightarrow$ Min. no. of empty cells = $mn-(m+n-1)$.
* If we add any constant with all C_{ij}'s, then, the optimum sol^n of TP will remain unchanged.

---

* SOLUTION METHOD (for Balanced TP)

→ Initial basic feasible sol^n

S1) Find a basic feasible sol^n (IBFS)
S2) Check whether its optimum. If yes, stop. Else, go to S3)
S3) Revise the sol^n & go to step 2.

* Methods to find IBFS.
  - M1) North West Corner Rule (NWCR)
  - M2) Least cost method (LCM) → or LCPM
  - M3) Vogel's approach method (VAM)

☆ How to convert an unbalanced TP to balanced.

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

eg: Convert the following TP to balanced.

Consider :-

	D_1	D_2	D_3	D_4	Available
S_1	20	10	10	15	25
S_2	10	5	20	25	35
S_3	30	31	25	35	50
Demand:	10	60	20	10	

Here, $\sum a_i = 110$

$\sum b_j = 100.$

When $\sum a_i > \sum b_j$

add a dummy column of zero cost & demand $= (\sum a_i - \sum b_j)$

Dummy column

So, it becomes

	D_1	D_2	D_3	D_4	D_5	
S_1	20	10	10	15	0	25
S_2	10	5	20	25	0	35
S_3	30	31	25	35	0	50
Demand	10	60	20	10	10	

||ly, when $\sum a_i < \sum b_j$,

add a dummy row of zero cost with the availability $(= \sum b_j - \sum a_i)$

✱ What to do for Maximiz^n Problem?

TP is for minimiz^n.
So, × all cost by −1.
So, all cost becomes −ve.
Now, take the most −ve cost & add ~~the~~ its +ve value to all costs to make cost +ve.

✱ North West Corner Rule (NWCR) to find BFS

→ first allocation that happens

eg: Consider

5 units left on S1 after allocating 15

	D_1	D_2	D_3	D_4	
S_1	[4] 15	[6] 5	[1] X	[7] X	20/5/0
S_2	[0] X	[2] 10	[1] 20	[5] X	30/20/0
S_3	[6] X	[2] X	[1] 5	[1] 35	40/35/0
	15/0	15/10/0	25/5/0	35/0	90 → balanced

North west corner all

See $\min(a_i, b_j)$

∵ D_1 doesn't require more amt.

no unit reqd to be transported to D_1

Idea :- starting from north west cell (x_{11}), go and fill up the table.

So, Solⁿ :- $x_{11} = 15$, $x_{12} = 5$
$x_{22} = 10$, $x_{23} = 20$
$x_{33} = 5$, $x_{34} = 35$
&

$$\text{cost} = (15\times 4) + (5\times 6) + (10\times 2) + (20\times 1) + (5\times 1) + (35\times 1) = 170$$

eg ② Consider

	D_1	D_2	D_3	D_4	
S_1	$\boxed{4}$ 15	$\boxed{6}$ ×	$\boxed{1}$ ×	$\boxed{7}$ ×	15/0
S_2	$\boxed{0}$ ×	$\boxed{1}$ 15	$\boxed{5}$ 15	$\boxed{5}$ ×	30/15/0
S_3	$\boxed{6}$ ×	$\boxed{2}$ ×	$\boxed{1}$ 10	$\boxed{1}$ 35	45/35/0.
	15/0	15/0	25/10/0	35/0	90.

Row & column got crossed
So, now, move diagonally & solve similarly.

allocate at that place, where cost is min.

Least Cost Method (LCM)

eg

	D_1	D_2	D_3	D_4	
S_1	[4] X	[6] X	[1] 20	[7] X	20/0
S_2	[0] 15	[2] 10	[1] 5	[5] X	30/15/10/0
S_3	[6] X	[2] 5	[1] X	[1] 35	40/5/0
	15/0	15/5/0	25/5	35/0	

* If min cost comes at 2 or more places, choose any one for allocation.

* After choosing min cost cell, start allocation for other cells just like North west corner method.
* After first allocation, select the next EMPTY cell where cost is min. In above problem, 1 is min cost (at 4 places). Take any place.

Final solⁿ:-

$x_{13} = 20,\ x_{21} = 15,\ x_{22} = 10,\ x_{23} = 5$
$x_{32} = 5,\ x_{34} = 35$

$$\text{Cost} = (20\times1) + (15\times0) + (10\times2) + (5\times1) + (5\times2) + (35\times1)$$
$$= 90$$

* We chose the same problem as before. But the cost = 90 here (earlier, we got 170).
  ⇒ This solⁿ is closer to optimum solⁿ. So, with this method, no. of steps reqd to go to corner solⁿ are lesser.

✯ Vogel's approximation method (VAM)

Steps

- we find penalty for each row & each column by Subtracting least & second least costs.
- enter through that row/col. where penalty is max.
- select that cell having least cost
- update value of penalty at every step.

eg

	W_1	W_2	W_3		Penalty
S_1	[0] 5	[2] ×	[1] 1	6/1/0	1 1 1
S_2	[2] ×	[1] 5	[5] 2	7/2	1 4 5
S_3	[2] ×	[4] ×	[3] 7	7/0	1 1 3
	5/0	5/0	10		
Penalty	2	1	2		
Revised Penalty	–	1	2		
Revised again	–	–	2		

* If only one column is left, allocate the amount that is left.

Basic feasible sol^n :-

$x_{11} = 5,\ x_{13} = 1,\ x_{22} = 5,\ x_{23} = 2,\ x_{33} = 7$

$$\text{Cost} = 0 + 1 + 5 + 10 + 21$$
$$= 37$$

Q. Trying above problem using NWCR.

	W_1	W_2	W_3	
S_1	5 $\boxed{0}$	1 $\boxed{2}$	X $\boxed{1}$	6/1/0
S_2	X $\boxed{2}$	4 $\boxed{1}$	3 $\boxed{5}$	7/3/0
S_3	X $\boxed{2}$	X $\boxed{4}$	7 $\boxed{3}$	7/0
	5/0	5/4/0	10/7/0	

Sol^n :- $x_{11} = 5,\ x_{12} = 1,\ x_{22} = 4,\ x_{23} = 3$
$x_{33} = 7.$

$$\text{Cost} = 0 + 2 + 4 + 15 + 21$$
$$= 42$$

(we got more cost in NWCR)

(VAM)

☆ Tie breaking situation: If $\exists$ max. penalty in a row & a column, then, see previous step. If we entered in prev. method using row, then, now enter using column & break the tie.

✯ Working rule to solve a TP

s1) Convert the TP into balanced one and minimiz^n type, if not.

s2) Find an IBFS by using any of the 3 methods.

s3) If this IBFS is degenerate, then, ~~convert it~~ resolve the degeneracy. If non degenerate, goto S4)

s4) Select row multipliers u_i and column multipliers & v_j for each row & each column, s.t.

$$C_{ij} = U_i + V_j \quad \text{for each occupied cell.}$$

s5) Calculate $\Delta_{ij} = C_{ij} - (u_i + V_j)$ $\forall$ empty cells.

u_i & V_j are costs.

10	6	2	2
	2		1
8	1		

empty cell so, it has $\Delta_{12} = 6 - (2+1) = 3$

s6) If all $\Delta_{ij} \geq 0$, sol^n under test is optimum. If not, goto S7)

s7) Find out ~~a~~ cell with most negative Δ_{ij}. Form a loop with this cell & some other occupied cells.

s8) Revise the sol^n & go to S4)

tie break. choose any one of the 2 3's

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Q. Solve the following TP :-
(↳ Solve completely. Don't stop till IBFS)

	D_1	D_2	D_3	D_4	a_i	Penalty			
O_1	[10] X	[7] X	[3] 3	[6] X	3/0	3	3	—	
O_2	[1] 3	[6] X	[8] X	[3] 2	5/2/0	2	3	3	3
O_2	[7] X	[4] 2	[5] 3	[3] 2	7/4/2/0	1	1	1	1
b_j	3/0	2	6/3/0	4	15				
Penalty	↑6	2	2	0					
	—	2	2	0					
	—	2	↑$\boxed{3}$	0					
	—	2	—	0					

S1) here, nothing is mentioned so, its minimiz^n problem.

Next, check if $\sum a_i = \sum b_j$

Checking degeneracy:

No. of occupied cells $= 6 = m + n - 1$

So, sol^n is non degenerate

Hence, goto S4)

Finding multiplier:-

Make table again & now, write only cost

Put * to occupied cell.

Now, start from row/column having max. allocation & take any value of u_i & v_j

After all u_i & v_j's are done, see if any is left or not.

all evaluation < 0 : reduction in Cost possible

				u_i
10	7	3 *	6	−2
1 *	6	8	3 *	0
7	4 *	5 *	3 *	0 → taken arbitrarily ← Start
1	4	5	3	

Now, make table again to find Δ_{ij} for empty cells.

Δ_{ij} − table
→ cell evaluation

11	5	·	5
·	2	3	·
6	·	·	·

Put · on occupied cells.
Find $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for all empty cells.

eg:- $\Delta_{11} = 10 - (-2+1)$

$\exists$ no negative Δ_{ij}. (All $\Delta_{ij} \geq 0$)
So, sol^n under test is optimum.
&, optimum sol^n is :- $x_{13} = 3$, $x_{21} = 3$, $x_{24} = 2$, $x_{32} = 2$
$x_{33} = 3$, $x_{34} = 2$. min. cost $= 47$
(always write) $(9+3+6+8+15+6)$

Q. Solve the following TP:-

	D_1	D_2	D_3	D_4	
F_1	19 / 5	30	50	10 / 2	7
F_2	70	30	40 / 7	60 / 2	9
F_3	40	8 / 8	70	20 / 10	18
	5	8	7	14	

self (solve using VAM)

Its non degenerate $6 = 4 + 3 - 1$

Now, finding u_i & v_j

19 *	30	50	10 *	10
70	30	40 *	60 *	60
40	8 *	70	20 *	20
9	-12	-20	0	

↑ start

Δ_{ij} - table.

.	32	60	.
1	-18	.	.
11	.	70	.

→ Δ_{ij} is -ve so, its possible for improvement

So, goto S7)

* Making a loop
  - a closed path → consisting of horizontal & vertical lines.
  - starts for one point, returns back there
  - in each row & column, ∃ even no. of cells in loop.

how looping is done?

•	32	60	•
1	-18	•	2
11	8	70	10

Start loop from this point.

(Compt - ve element is new allocated cell in revised sol^n)

Lets allocate θ amt at x_{22}.

So, we can allocate $\min\{x_{24}, x_{32}\}$
$= \min\{2, 8\}$
$= 2$.

So, if $\theta = 2$, $x_{24}' = x_{24} - 2 = 2 - 2 = 0$
$x_{32}' = x_{32} - 2 = 8 - 2 = 6$
$x_{34} = x_{34} + 2 = 10 + 2 = 12$

revised sol^n :-

19 5	30	50	10 2
70	30 2	40 7	60
40	8 6	70	20 12

Again: find u_i & v_j :

				u_i
19 *	30	50	10 *	-10
70	30 *	40 *	60	22
40	8 *	70	20 *	0
29	8	18	20	

Now, find Δij

·	32	42	·
19	·	·	18
11	·	52	·

All $\Delta ij > 0$. So, optimum sol^n ✓

Sol^n :-

$x_{11} = 5$, $x_{14} = 2$, $x_{22} = 2$, $x_{23} = 7$

$x_{32} = 6$, $x_{34} = 12$.

& min. cost = Rs 743.

revised sol^n :-

19 5	30	50	10 2
70	30 2	40 7	60
40	8 6	70	20 12

Again, find u_i & v_j

19 *	30	50	10 *	-10
70	30 *	40 *	60	22
40	8 *	70	20 *	0
29	8	18	20	

Now, find Δij

•	32	42	•
19	•	•	18
11	•	52	•

All $\Delta ij > 0$. So, optimum sol^n ✓

Sol^n :-

$x_{11} = 5$, $x_{14} = 2$, $x_{22} = 2$, $x_{23} = 7$

$x_{32} = 6$, $x_{34} = 12$.

& min. cost = Rs 743.

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Solve the following profit matrix of a TP.

	D_1	D_2	D_3	D_4	a_i
F_1	12	18	6	25	200
F_2	8	7	10	18	500
F_3	14	3	11	20	300
b_j	180	320	100	400	

Given :- Maximizn problem

To convert to minimizn problem, subtract all elements from largest element.

Corresponding minimizn problem :

	D_1	D_2	D_3	D_4	a_i
F_1	13	7	19	0	200
F_2	17	18	15	7	500
F_3	11	22	14	5	300
b_j	180	320	100	400	

Solving by IBFS : Vogel's approxim^n method we get

13	7 (200)	19	0
17	18 (100)	15	7 (400)
11 (180)	22 (20)	14 (100)	5

This sol^n is non degenerate sol^n

				u_i
13	7 ★	19	0	−15
17	18 ★	15	7 ★	−4
11 ★	22 ★	14 ★	5	0
v_j: 11	22	14	11	

17	.	20	4
10	$100+\theta$	5	$400-\theta$
.	$20-\theta$	.	-6 θ

let $\theta = 20$

(to determine θ, $\min(20, 400) = 20$)

$$\Delta_{ij} = c_{ij} - (u_{ij} + v_{ij})$$

	200		
	120		380
180		100	20

				u_i
13	7 ★	19	0	−9
17	18 ★	15	7 ★	2
11 ★	22	14 ★	5 ★	0
11	16	14	5	

Note:- Multiply the values of x_{ij} with original table's cost, NOT the transformed table.

11	·	14	4
4	·	θ −1	$380-\theta$
·	6	$100-\theta$	$20+\theta$

$\underline{\theta = 100}$

	200		
	120	100	280
	180		120

					u_i
	$\boxed{13}$	$\boxed{7}$ *	$\boxed{19}$	$\boxed{0}$	11
	$\boxed{17}$	$\boxed{18}$ *	$\boxed{15}$ *	$\boxed{7}$ *	0
	$\boxed{11}$ *	$\boxed{22}$	$\boxed{14}$	$\boxed{5}$ *	−2
v_j	13	18	15	7	

$x_{12} = 200$
$x_{22} = 120$
$x_{23} = 100$

11	·	15	4
4	·	·	·
·	6	1	·

$x_{24} = 280$
$x_{32} = 180$
$x_{34} = 120$.

& max. profit = 1540#

All $\Delta_{ij} > 0$: So, solⁿ under test is optimum.

- how to resolve degeneracy?

If occupied cells $< m+n-1 \Rightarrow$ Degenerate solⁿ

Now, allocate (0) (amt) to that cell which cannot form loop with any of the occupied cells (independent cell)

$\Rightarrow$ depends whether to allocate it to 1 cell or more than one cell.

Original IBFS non degenerate, but, during iterations solⁿ maybe degenerate.

Assignment Problems. (AP)

m : workers
n : jobs
Total assignment cost or time required is the least.

An assignment problem is said to be balanced if the cost matrix is a square matrix, otherwise its unbalanced.

	J_1	J_2	J_3
W_1	2	5	7
W_2	1	3	9
W_3	2	1	2

Above is a minimiz^n table.

* Converting Unbalanced to Balanced :

It is a special type of transportation problem.

eg Given:

	J_1	J_2	J_3
W_1	2	5	7
W_2	1	3	9

It is unbalanced.
Now, converting to balanced (as done in T.P.)

→ denotes assignment of a job

∴ No. of assignment = order of matrix.

→ by adding Dummy row/col

	J_1	J_2	J_3
W_1	2	5	7
W_2	1	3	9
W_3	0	0	0

Why balance?
↳ 2 or more jobs cannot be assigned to a single person. Illy, 2 or more persons cannot be assigned for a single job.

HUNGARIAL ALGORITHM

Step 1) Convert the AP to minimiz^n type & balance it, if not.

S2) Subtract the least element of each row from all elements of that row.

S3) In the revised matrix, subtract the least element of each column from all elements of that column. This matrix is called STARTING REDUCED COST MATRIX.

Step 4) ~~Step~~ Start from 1st row and check all the rows sequentially whether any row contains a single zero. If any row contains a single zero, enclose that zero by a box and cross off (X) all zeros in the corresponding column.

eg :-

only 1 zero in the row. So, put a box on it

then, if ∃ any zero for the boxed element, cross off it

$$\left.\begin{matrix} 0 & 1 & 2 & 0 \\ 2 & \boxed{0} & 1 & 3 \\ 2 & 1 & 0 & 0 \\ \boxed{0} & \cancel{0} & 2 & 1 \end{matrix}\right\} \text{checking row}$$

S5) Start from 1st column and check all columns sequentially whether any ~~row~~ column contains a single zero. If any column contains a single zero, enclose that zero by a box & cross off all the zeros in corresponding row.

Repeat steps 4) & 5) untill all zeroes are marked.

Continuing above for column checking.

$$\begin{matrix} \cancel{0} & 1 & 2 & \boxed{0} \\ 2 & \boxed{0} & 1 & 3 \\ 2 & 1 & \boxed{0} & \cancel{0} \\ \boxed{0} & \cancel{0} & 2 & 1 \end{matrix}$$

S6) Count the no. of boxes. If no. of boxes = order of ~~a~~ cost matrix, then, optimum assignment is obtained & boxes represent the assignments.

If the no. of boxes is less than order of cost matrix, go to S7)

S7) Revise the solⁿ (how? – told later) & go to S4).

eg: Consider a TP with rows = columns

	D_1	D_2	D_3	
S_1	1	×	×	1/0
S_2	×	1	×	1/0
S_3	×	×	1	1/0
	1/0	1/0	1/0	

If seen as the no. of boxes, ∃ 3 1's. That is equal to order of TP.

So, in some terms, Assignment problem can be seen as special case of TP

eg Solve the following assignment problem:-

Workers \ Jobs →	I	II	III	IV
A	10	5	13	15
B	3	9	18	3
C	10	7	3	2
D	5	11	9	7

Cost matrix

S1) check for minimiz^n / maximiz^n
& check for balanced / unbalanced
→ Given AP is square matrix . So, balanced.
→ Nothing is mentioned . So, minimiz^n .

S2) Subtract the min element in any row from all elements of that row.

	I	II	III	IV
A	5	0	8	10
B	0	6	15	0
C	8	5	1	0
D	0	6	4	2

S3) Column reduction & row checking (S4)

	I	II	III	IV
A	5	$\boxed{0}$	7	10
B	~~0~~	6	14	0
C	8	5	0	0
D	$\boxed{0}$	6	3	2

S5) Column checking

	I	II	III	IV
A	5	$\boxed{0}$	7	10
B	~~0~~	6	14	$\boxed{0}$
C	8	5	$\boxed{0}$	~~0~~
D	$\boxed{0}$	6	3	2

worker D is assigned Job I

Assignment :-

A → II , B → IV , C → III , D → I ,

min cost = 5 + 3 + 3 + 5 = 16

↳ original matrix value for A → II

Note how to do row checking & column checking when:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{0} & \cancel{0} \\ \cancel{0} & \boxed{0} \end{bmatrix} \text{ or } \begin{bmatrix} \cancel{0} & \boxed{0} \\ \boxed{0} & \cancel{0} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \boxed{0} & \cancel{0} \end{bmatrix}$$

---

eg Solve the following APs

M_1 M_2 M_3 → Machine

Job	M_1	M_2	M_3
J_1	12	24	15
J_2	23	18	24
J_3	30	14	28

S1) Row reduction

	M_1	M_2	M_3	
J_1	0	12	3	→ ( ) − 12
J_2	5	0	6	→ ( ) − 18
J_3	16	0	14	→ ( ) − 14

S2) Column red^n

	M_1	M_2	M_3	
J_1	0	12	0	→ leave it for row checking ∵ ∃ 2 zeros.
J_2	5	$\boxed{0}$	3	
J_3	16	$\cancel{0}$	11	
	↓ no change	↓ no change	↓ ( ) − 3	

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$$\begin{array}{c|ccc|} & M_1 & M_2 & M_3 \\ \hline J_1 & \boxed{0} & 12 & \cancel{0} \\ J_2 & 5 & \boxed{0} & 3 \\ J_3 & 16 & \cancel{0} & 11 \\ \hline \end{array} \quad \checkmark$$

no. of box = 2, order = 3
So, not revised solⁿ.

* how to revise solⁿ :-

S1) Mark the row not having any assignment

$$\begin{bmatrix} \boxed{0} & 12 & \cancel{0} \\ 5 & \boxed{0} & 3 \\ 16 & \cancel{0} & 11 \end{bmatrix} \checkmark$$

S2) Tick those columns which have zeros in the ticked row.

S3) Tick rows having assignment in ticked column

S4) Draw horizontal lines through unticked rows & ~~ticked columns~~. Vertical lines through ticked columns

Junction

S4)

$$\begin{array}{|ccc|c} \boxed{0} & 12 & \cancel{0} & \\ 5 & \boxed{0} & 3 & \checkmark \rightarrow S3 \\ 16 & \cancel{0} & 11 & \checkmark \rightarrow S1 \\ & \checkmark & & \end{array}$$

→ S2

These lines are called covering lines.
Basically, draw min. no. of lines to cover all the zeros.
Now, for all unmarked elements. See the min element. (3, here)

S5) Subtract the least element ~~of~~ ~~fro~~ out of all uncovered elements from uncovered elements. Add this element at junction of horizontal & vertical lines. Rest all elements are unchanged.

So,

12+3

$\boxed{0}$	15	$\cancel{0}$
2	$\cancel{0}$	$\boxed{0}$
13	$\boxed{0}$	8

Revised matrix

Proceed again for row & column checking (S4, S5) we get 3 boxes = order. So, revised ✓

Hence, $J_1 \rightarrow M1$, $J_2 \rightarrow M_3$, $J_3 \rightarrow M_2$

min. cost = 12 + 24 + 14 = 50

☆ Note :-

	J_1	J_2	J_3
P_1			∞
P_2			
P_3			

In practical situation, sometimes, a person can't do a particular job.

eg: If P_1 can't do J_1, then corresponding element is ∞ (represented as — sometimes)

HW Q. Solve the following AP

Ans :-

min. cost = 21

Jobs.

		I	II	III	IV
Workers	A	1	4	6	3
	B	9	7	10	9
	C	4	5	11	7
	D	8	7	8	5

S1) Square matrix ⇒ balanced.
nothing given ⇒ minimizⁿ type.

S2)

$$\begin{bmatrix} 0 & 3 & 5 & 2 \\ 2 & 0 & 3 & 2 \\ 0 & 1 & 7 & 3 \\ 3 & 2 & 3 & 0 \end{bmatrix}$$

S3), S4), S5)

$$\begin{bmatrix} \boxed{0} & 3 & 2 & 2 \\ 2 & \boxed{0} & \cancel{0} & 2 \\ \cancel{0} & 1 & 4 & 3 \\ 3 & 2 & \boxed{0} & \cancel{0} \end{bmatrix}$$

no. of box ≠ order.
So, not resolved.

$$\begin{bmatrix} \boxed{0} & 3 & 2 & 2 \\ 2 & \boxed{0} & \cancel{0} & 2 \\ \cancel{0} & 1 & 4 & 3 \\ 3 & 2 & \boxed{0} & \cancel{0} \end{bmatrix}$$

✓ (iii)

✓ (i)

✓ (ii)

Revised matrix :-

$\boxed{0}$	2	1	1
3	$\cancel{0}$	$\boxed{0}$	2
$\cancel{0}$	$\boxed{0}$	3	2
4	2	$\cancel{0}$	$\boxed{0}$

& applying S4) & S5)

↳ 4 box = order. So, revised ✓

Assignment :-

A → I , B → III , C → II , D → IV

min. cost = 1 + 10 + 5 + 5 = 21

Q. Solve the following AP

→ Machine 2 cannot do job C

	A	B	C	D	E
M_1	9	11	15	10	11
M_2	12	9	–	10	9
M_3	–	11	14	11	7
M_4	14	8	12	7	8

machine 3 ⇒ M_3 cannot do this job A.

Replace blanks by ∞.

	A	B	C	D	E
M_1	9	11	15	10	11
M_2	12	9	∞	10	9
M_3	∞	11	14	11	7
M_4	14	8	12	7	8
M_5	0	0	0	0	0

→ Converting to Balanced (making square matrix)

* Nothing given
⇓
minimiz^n type

S2)

	A	B	C	D	E
M_1	0	2	6	1	2
M_2	3	0	∞	1	0
M_3	∞	4	7	4	0
M_4	7	1	5	0	1
M_5	0	0	0	0	0

Column reduction not reqd (∵ last row: all 0s)

Row & column checking:

	A	B	C	D	E
M_1	$\boxed{0}$	2	6	1	2
M_2	3	$\boxed{0}$	∞	1	~~0~~
M_3	∞	4	7	4	$\boxed{0}$
M_4	7	1	5	$\boxed{0}$	1
M_5	~~0~~	~~0~~	$\boxed{0}$	~~0~~	~~0~~

last unmarked zero. Mark after repeating process

No. of boxes = 5 = order of matrix

So, Assignment:

$M_1 \to A$, $M_2 \to B$, $M_3 \to E$, $M_4 \to D$

min. cost :- $9 + 9 + 7 + 7 = 32$

★ eg.:-

	A	B	C
I	2	7	8
II	1	5	9
III	2	1	5

If the values given in this are PROFITS (not costs), then, its Maximizn problem. Convert to minimizn (Subtract largest elmt (9) from all elements)

9-2	7	2	1
9-1	8	4	0
9-2	7	8	4

This is minimizn problem.

After solving:

3	$\boxed{0}$	~~0~~
5	3	$\boxed{0}$
$\boxed{0}$	3	~~0~~

Assignment:
I → B, II → C, III → A.

Max Profit = 7+9+2 = 18

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Game Theory

• Game : A competitive activity s.t., each one of them try to maximise their gain or minimise its loss.
↳ If only 2 persons are there : 2 person game
n " : n person game.

Types of games

Zero Sum game
s.t gains + loss = 0
(total gain & loss)
If only 2 persons are involved,
its called 2 person zero sum game.

non zero sum game
(not in syllabus)

• Strategy:
A particular activity that a player adopts during a game
→ Pure strategy
Fixed / predefined strategy. eg. If I do this, I'll get 10 gain & 5 loss. So, ∄ no probability
↳ Mixed strategy :
Where probabilities are involved.

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- **Pay off matrix:**
  A matrix which represents the payoffs of a player x against different strategies of other player. y

eg: If player X has strategies: x_1, x_2, x_3
If player Y has strategies y_1, y_2, y_3
Payoff matrix of player X

$$
\begin{array}{cc|cccc|}
 & & \multicolumn{4}{c}{\text{Player } Y} \\
 & & y_1 & y_2 & y_3 & y_4 \\
\text{Player} & X_1 & x_{11} & x_{12} & x_{13} & x_{14} \\
X & X_2 & x_{21} & x_{22} & x_{23} & x_{24} \\
 & X_3 & x_{31} & x_{32} & x_{33} & x_{34} \\
\end{array}
$$

value +ve ⇒ Strategy x_3 wins over y_1
-ve ⇒ Strategy x_3 loses over y_1
(with some amount)

Solⁿ ⇒ the optimum strategy of X & Y

★ **Optimum solⁿ of 2-person zero sum game with saddle point:**

eg: consider payoff of player A

MAXIMIN STRATEGY of A

$$
\begin{array}{c|cccc|c}
 & \multicolumn{4}{c}{\text{Player } B} & \\
 & B_1 & B_2 & B_3 & B_4 & \text{Row min} \\
A_1 & a_{11} & a_{12} & a_{13} & a_{14} & \checkmark \\
A_2 & a_{21} & a_{22} & a_{23} & a_{24} & \checkmark \\
A_3 & a_{31} & a_{32} & a_{33} & a_{34} & \checkmark \\
\text{Column max} & \checkmark & \checkmark & \checkmark & \checkmark &
\end{array}
$$

now, find max(row min) = $\underline{v}$

find min(col. max) = $\overline{v}$

Row min : The min. gain of player A with that strategy

Now, find the max. value among the min. gains for all rows, i.e

max(row min) = maximin value $= \underline{v}$

Column max : The max loss of player B with that strategy

Find min. of the max. losses $\forall$ columns. i.e

min(column max) = minimax value $= \overline{v}$

If $\exists$ a point where $\underline{v}$ & $\overline{v}$ coincide, its a SADDLE POINT

↳ Indicates optimum strategies of A & B both.

Idea : Put a □ over every element for row min

Put a ○ over every element for col. max.

If $\exists$ any element having s.t like $\boxed{a_{11}}$, then its saddle point

Now, $v = \underline{v} = \overline{v}$ = value of game.

* If value of game $= \begin{cases} 0 & : \text{fair game} \\ \neq 0 & : \text{unfair game} \end{cases}$

If $v \neq 0$, then $\underline{v} \leq v \leq \overline{v}$

eg. Given a payoff matrix of player A for 2 player zero sum game

$$
\begin{array}{ll|cccc|}
 & & B_1 & B_2 & B_3 & B_4 \\
\hline
\text{Player} & A_1 & 1 & 7 & 3 & 4 \\
A & A_2 & 5 & 6 & 4 & 5 \\
 & A_3 & 7 & 2 & 0 & 3 \\
\hline
\end{array}
$$

Determine : (a) Saddle point
(b) Optimum strategies of A & B
(c) Value of game
(d) Is the game fair ?

Now, start finding row min & col. max
($\square$) ($\bigcirc$)

$$
\begin{array}{l|cccc|}
 & B_1 & B_2 & B_3 & B_4 \\
\hline
A_1 & \boxed{1} & 7 & 3 & 4 \\
A_2 & 5 & 6 & \boxed{4} & 5 \\
A_3 & 7 & 2 & \boxed{0} & 3 \\
\hline
\end{array}
$$

(a) Clearly, saddle pt. is (A_2, B_3) or $(2,3)$

(b) Optimum strategies of A : A_2
B : B_3

(c) Value of game $= V = 4$.

(d) $V \neq 0$. So, game is unfair

Q. Determine saddle point, the associated pure strategies & value of game when payoff matrix of A is

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 8 & 6 & \boxed{2} & 8 \\ A_2 & 8 & 9 & \boxed{4} & 5 \\ A_3 & 7 & 5 & \boxed{3} & 5 \end{array}$$

Saddle pt :- (2, 3)
Value of game = 4
Unfair.
Pure strategies A : A_2
B : B_3

Q. Payoff matrix for A

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \text{Row min} \\ \hline A_1 & 1 & 9 & 6 & \boxed{0} & 0 \\ A_2 & \boxed{2} & 3 & 8 & 4 & 2 \\ A_3 & \boxed{-5} & -2 & 10 & -3 & -5 \\ A_4 & 7 & 4 & -2 & \boxed{-5} & -5 \\ \hline \text{Col. max} & 7 & 9 & 10 & 4 & \end{array}$$

Find the range for value of game
$\Rightarrow$ $\exists$ no saddle point

Clearly, $\exists$ no saddle point So, now find

maximin: $\underline{v}$ = max(row min) = max(0, 2, -5, -5) = 2

minimax: $\overline{v}$ = min(col. max) = min(7, 9, 10, 4) = 4

So, range = $\underline{v} \leq v \leq \overline{v}$ = $2 \leq v \leq 4$ Ans

SOLUTION OF MIXED STRATEGY Problems

Graphical Method (used for $2\times n$ or $n\times 2$)

(Linear ~~mat~~ Programming method: not in course)

for $2\times n$.

$$
\begin{array}{ccc}
 & & B \\
 & & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\
A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}
\end{array}
$$

when one of P_1 or $P_2 = 0$. AND $P_1 + P_2 = 1$. then, it is called Pure Strategy.

let the mixed strategy of A be

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$$

B's pure strategy	A's expected payoff
B_1	$E_1 = a_{11}p_1 + a_{21}p_2$
B_2	$E_2 = a_{12}p_1 + a_{22}p_2$
$\vdots$	$\vdots$
B_n	$E_n = a_{1n}p_1 + a_{2n}p_2$

How to plot ?

let $p_1 = x$, $p_2 = 1 - x$

($\because$ Total probability = 1)

So, suppose

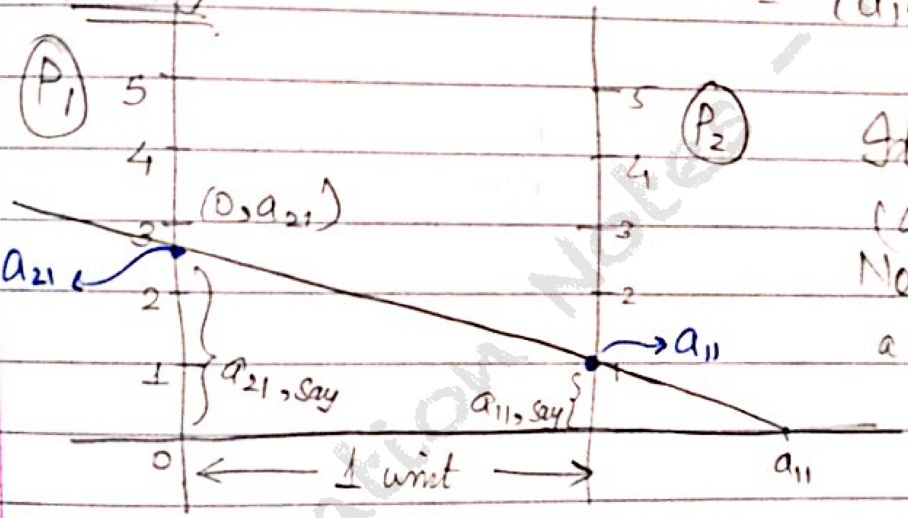
$$y = a_{11}x + a_{21}(1-x)$$

$\hookrightarrow$ eqn of st. line (can be plotted easily)

Take $(x, 0)$ & $(0, 1)$

$= (a_{11}, 0), (0, a_{21})$

Plotting



Idea :-

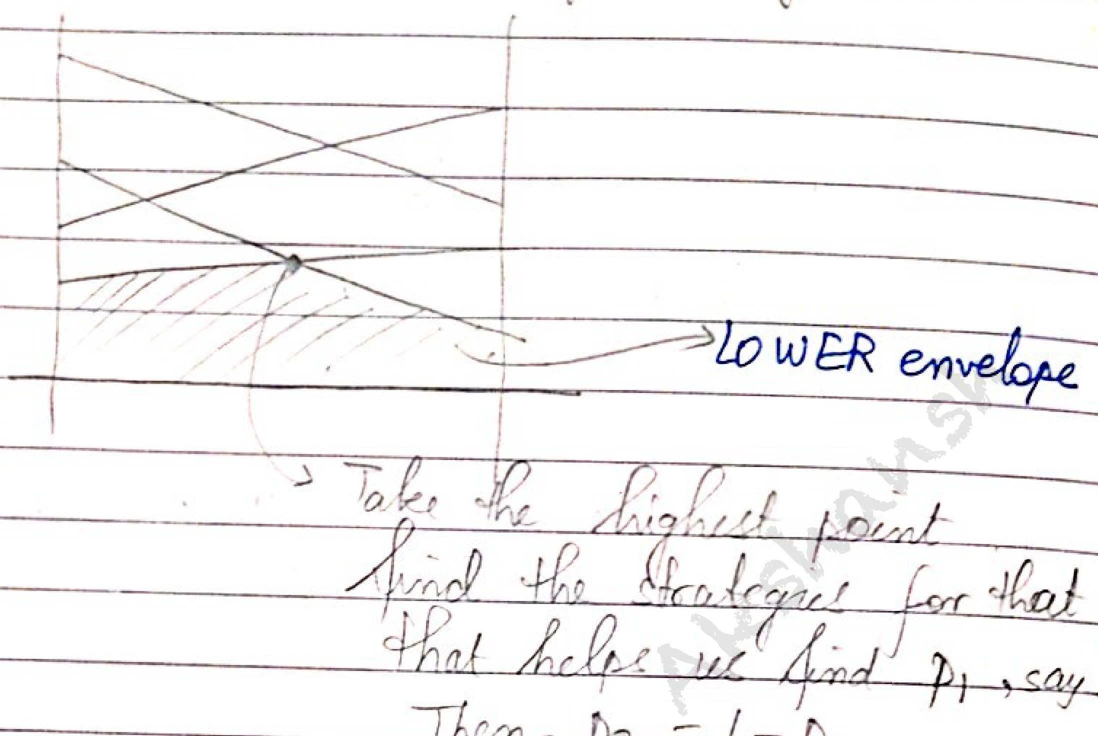
$(a_{11}, 0)$ is on x-axis. Now, $(1, a_{11})$ is also a pt.

why 2 lines ? $\longrightarrow$ $\because$ we are concerned only inside that region (probability from 0 to 1)

Basic idea of doing this is, we don't need to plot every eqn by taking values. Put $x = 0$, find y, Put $y = 0$, find x.

Hence, plot it

So, for E_1, E_2 ..... we get many lines



Now, with strategies known, find/make probability of all other strategies (not at highest pt.) as zero.

Q. Use graphical method to solve following gain problem :

		B_1	B_2	B_3
A	A_1	1	3	11
	A_2	8	5	2

S1) Check for saddle pt

	B_1	B_2	B_3
A_1	$\boxed{1}$	3	11
A_2	8	5	$\boxed{2}$

Clearly, $\nexists$ no saddle pt. of payoff matrix.

Now, solving the mixed strategy problem by Graphical method :-

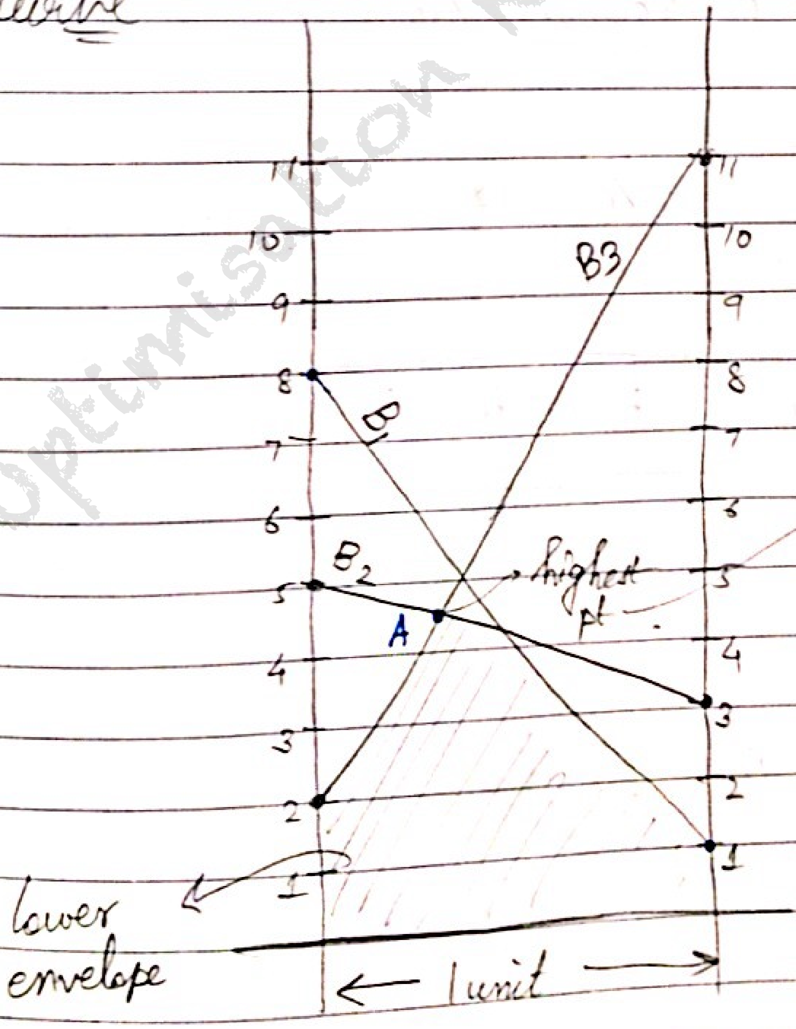
Let the mixed strategy for A be

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad ; \ p_1, p_2 \geq 0 \quad ; \ p_1 + p_2 = 1$$

B's pure strategy	A's expected payoff
B_1	$E_1 = p_1 + 8p_2$
B_2	$E_2 = 3p_1 + 5p_2$
B_3	$E_3 = 11p_1 + 2p_2$

Plot them

Curve



Intersection of B_2 & B_3 strategies.

So, now, finding p_1 & p_2.

finding p_1 & p_2

for B_2 : expected profit : $3p_1 + 5p_2$
B_3 : " " " : $11p_1 + 2p_2$

Now, $3p_1 + 5p_2 = 11p_1 + 2p_2$
(gain of A at B_2 = its gain at B_3)
(B_2 & B_3 at same pt)

$\Rightarrow 3p_1 + 5(1-p_1) = 11p_1 + 2(1-p_1)$

$\Rightarrow 8p_1 = 3 - 3p_1$

$\Rightarrow p_1 = \frac{3}{11}$

$\Rightarrow p_2 = \frac{8}{11}$ $(1-p_1)$

So, mixed strategy for player A,

$$S_A = \begin{pmatrix} A_1 & A_2 \\ 3/11 & 8/11 \end{pmatrix}$$

Let mixed strategy for player B,

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & q_1 & q_2 \end{pmatrix}$$

∴ highest pt. has B_2 & B_3 only.

finding them ↗.

Revised payoff matrix

$$\begin{array}{c|cc|} & B_2 \; q_1 & B_3 \; q_2 \\ \hline A_1 & 3 & 11 \\ A_2 & 5 & 2 \\ \hline \end{array}$$

B_1 goes off ∴ probability = 0.

Now, In similar terms,

(gain for A at B's strategy B_2

= gain for A at B's strategy B_3)

done before

Illy,

loss of B at B_2 = loss at B_3

$\Rightarrow$

$3q_1 + 11q_2 = 5q_1 + 2q_2$

$\Rightarrow 3q_1 + 11(1-q_1) = 5q_1 + 2(1-q_1)$

$\Rightarrow q_1 = \frac{9}{11}$

$\Rightarrow q_2 = \frac{2}{11}$

So, $S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & 9/11 & 2/11 \end{pmatrix}$

Lastly, finding value of game :

in above matrix

Take any eqn :- $3q_1 + 11(1-q_1)$ or $5q_1 + 2(1-q_1)$ } using rows

or $3p_1 + 5p_2$ or $11p_1 + 2p_2$. } using columns

Put values : $V = 3p_1 + 5p_2 = \frac{9}{11} + \frac{40}{11} = \frac{49}{11}$ Ans

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Q. Solve the following game graphically:-

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_1	0	5	$\boxed{-2}$	3
	A_2	2	3	4	$\boxed{1}$

S1) Checking for saddle point :-

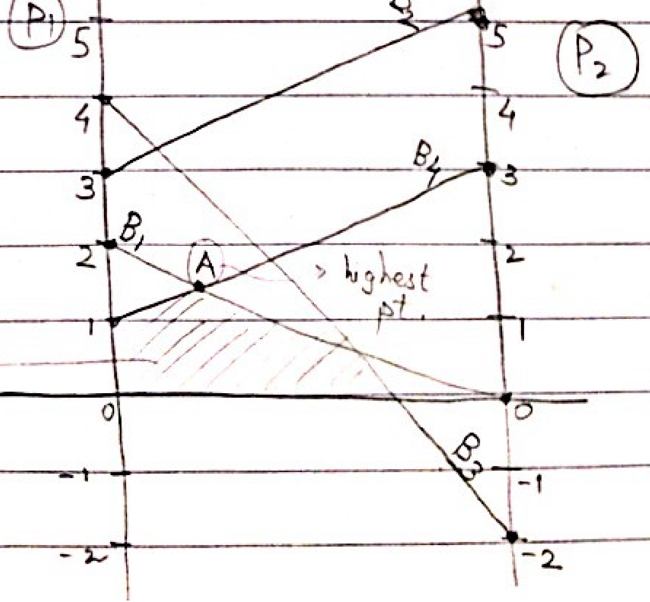
Clearly, from row min □ & column max O, there is no saddle point (∵ no common point). for given payoff matrix
Solving mixed strategy problem by graphical method:

let $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$; $p_1, p_2 \geq 0$
$p_1 + p_2 = 1$

B is pure strategy	A is expected payoff
B_1	$E_1 = 0 \cdot p_1 + 2p_2$
B_2	$E_2 = 5p_1 + 3p_2$
B_3	$E_3 = -2p_1 + 4p_2$
B_4	$E_4 = 3p_1 + p_2$

S2) Plotting them

lower envelope



Highest pt. of A is intersection of B_1 & B_4.

S3) Now, finding p_1 & p_2.

for B_1 : expected profit of A $= 2p_2$

B_3 : " " $= 3p_1 + p_2$

Now,

$2p_2 = 3p_1 + p_2$

$\Rightarrow 3p_1 = p_2$

Now, $p_2 = 1 - p_1 \Rightarrow 3p_1 = 1 - p_1$

$\Rightarrow p_1 = 1/4 \Rightarrow p_2 = 3/4$

So, $S_A = \begin{pmatrix} A_1 & A_2 \\ 1/4 & 3/4 \end{pmatrix}$

S4) Now, finding S_B.

$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ q_1 & 0 & 0 & q_2 \end{pmatrix}$ $q_1, q_2 \geq 0$, $q_1 + q_2 = 1$

$\because$ highest pt. only has B_1 & B_4. So, probability of other strategies $= 0$

Now, Revised payoff matrix of A:

	B_1 (q_1)	B_4 (q_2)
A_1	0	3
A_2	2	1

Loss of B at strategy B_1 = Loss at strategy B_4

$\Rightarrow 0q_1 + 3q_2 = 2q_1 + q_2$

$\Rightarrow 2q_1 = 2q_2$

$\Rightarrow q_1 = q_2 = 1/2$.

So, $S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$

S5) finding value of game;

$0.q_1 + 3(1-q_1) = \text{value}$
or, $2.q_1 + 3(1-q_1) = \text{Value}$ } using rows

$\Rightarrow 3\left(\frac{1}{2}\right) = \text{value of game}$ (or, use columns from final payoff matrix)

$\Rightarrow \text{Value}^{(V)} = \underline{1.5}$ Ans

☆ Note

Above, we had $2 \times n$.

↳ If we have $m \times 2$ order,

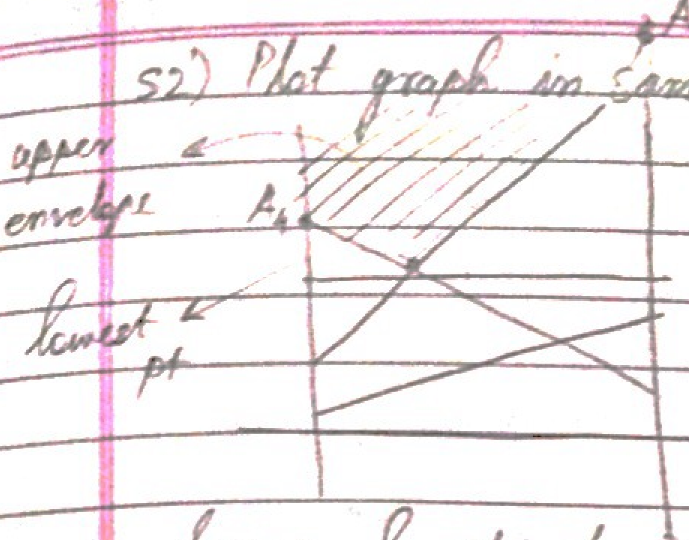
eg :-

$$\begin{array}{c|cc} & B_1 & B_2 \\ A_1 & \checkmark & - \\ A_2 & - & - \\ , & - - & - - \\ , & - - & - - \\ A_m & - & - \end{array}$$

S1) let $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$ $q_1, q_2 \geq 0$
$q_1 + q_2 = 1$.

A's pure strategy	B's expected loss
A_1	E_1 - - -
,	E_2 - -
,	

S2) Plot graph in same way



Idea: We have to plot the expected loss of player for each pure strategy of A
for that, take lowest pt. in upper envelope.

Suppose lowest pt. is corresponding to A_1 & A_4.

So,

S3) finding q_1, q_2 $(eq^n(A_1) = eq^n(A_4))$

S4) finding $S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & \cdots & A_m \\ p_1 & 0 & 0 & p_2 & \cdots & 0 \end{pmatrix}$

S5) find value.

Q Solve the game graphically.

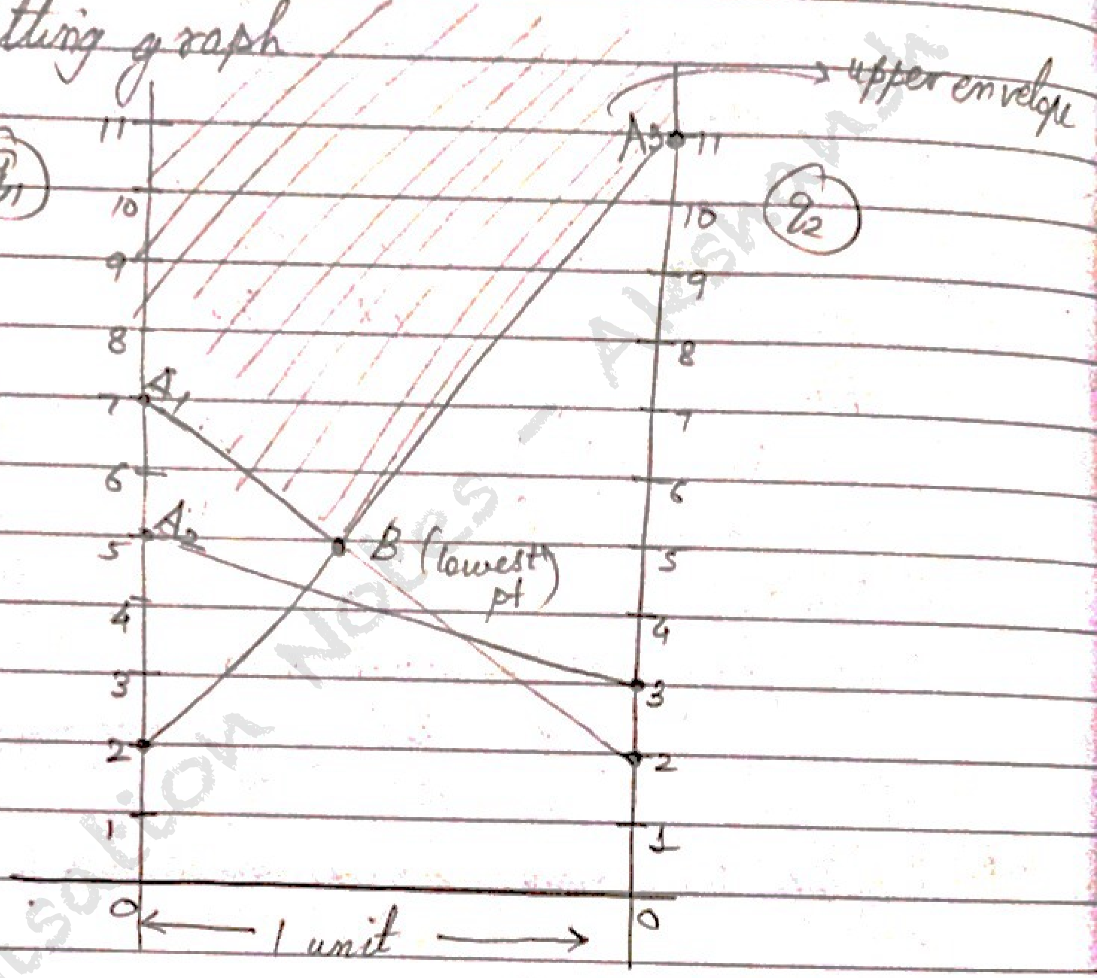
$$\begin{array}{c c} & \quad B \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ A \quad \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} \boxed{2} & 7 \\ \boxed{3} & 5 \\ 11 & \boxed{2} \end{bmatrix} \end{array}$$

S1) Checking for saddle pt
Clearly, no saddle point.

Let $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$; $q_1, q_2 \geq 0$
$q_1 + q_2 = 1$.

A's pure strategy	B's expected loss
A_1	$E_1 = 2q_1 + 7q_2$
A_2	$E_2 = 3q_1 + 5q_2$
A_3	$E_3 = 11q_1 + 2q_2$

s2) plotting graph



take lowest pt

★ We are plotting loss of B against strategies of A, who wants to maximise his profit.

take upper envelope

lowest pt. corresponds to A_1 & A_3.

for A_1, expected loss of B $= 2q_1 + 7q_2$

A_3, " " $= 11q_1 + 2q_2$.

Now, loss is same for B in both cases

$\Rightarrow \quad 2q_1 + 7q_2 = 11q_1 + 2q_2$

$\Rightarrow \quad 9q_1 = 5q_2$

$\Rightarrow \quad q_1 = \frac{5}{14}$

$\& \; q_2 = \frac{9}{14}.$

So, now

$$S_B = \begin{pmatrix} B_1 & B_2 \\ 5/14 & 9/14 \end{pmatrix}$$

Let

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & 0 & p_2 \end{pmatrix} \quad p_1, p_2 \geq 0 \quad p_1 + p_2 = 1$$

Now,

from reduced matrix

$$\begin{array}{c} \\ A_1 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 2 & 7 \\ 11 & 2 \end{bmatrix} \end{array}$$

So, $2p_1 + 11p_2 = 7p_1 + 2p_2$

$\Rightarrow p_1 = 9/14$

$p_2 = 5/14.$

So, $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 9/14 & 0 & 5/14 \end{pmatrix}$

Value of game :-

$V = 2q_1 + 7q_2$

$\Rightarrow V = \frac{73}{14}$ Ans

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DUALITY IN LPP

↳ ∃ 2 LPP's — PRIMAL & DUAL.

✓ Knowing final simplex table of one LPP, we can get final table for second LPP.

✓ They are closely related.

eg: Suppose condⁿ $2x_1 + 5x_2 \leq 4000$

x_1: toy 1, x_2: toy type 2, 4000: profit earned on selling them.

Primal :- finding values of x_1, x_2

Dual :- If value of condⁿ(4000) is increased by 1 unit, whether ∃ profit/loss

(basically, analysing a dual means, analysing if ∃ profit/loss if resource is added)

Idea : Given LPP, find Dual.

* Properties

P1) If primal is of maximizⁿ type, its dual is of ~~max~~ minimizⁿ type.

P2) Dual of a dual is primal.

eg: for an LPP, let primal = P
dual = D.

So, finding dual of D, we get P.

P3) No. of dual variables (represented by $y_1, y_2, \ldots$) = No. of primal constraints.

eg: If for an LPP, constraints = 2, (not including non -ve restriction)

no. of dual var = 2 (y_1, y_2)

P4) No. of dual constraints = No. of primal variables.

* If an LPP is given ⇒ we are given primal.

* Represent^n : Primal constraints or Dual variables } $y_1, y_2, \ldots$

Dual constraints or Primal variables } $x_1, x_2, \ldots$

* Now, seeing signs of $x_1, x_2, \ldots$ & $y_1, y_2, \ldots$

Maximiz^n type		Minimiz^n type
1) i^{th} constraint	→	i^{th} variable
$\geq$ type		≤ 0
$\leq$ type		≥ 0
= type		unrestricted

eg. If a primal constraint is $2x_1 + 3x_2 \geq 0$, it is ≥ 0 type. Corresponding dual var is $y_1 \leq 0$

2) i^{th} variable	→	i^{th} constraint
≥ 0		$\geq$ type
≤ 0		$\leq$ type
unrestricted		= type.

eg. If a primal var. $x_1 \geq 0$, This will help in making constraint for dual.

Q. Write dual of following primal.

(1) maximize $Z = 5x_1 + 12x_2 + 4x_3$

$\hookrightarrow$ ⇒ maximiz^n type

⇒ primal is given

s.t $x_1 + 2x_2 + x_3 \leq 10 \rightarrow y_1$

$2x_1 - x_2 + 3x_3 = 8 \rightarrow y_2$

$x_1, x_2, x_3 \geq 0$

$\hookrightarrow$ ⇒ 2 constraints are there in primal.

Let y_1, y_2 be dual variables.

Dual is

minimize $W = 10\,y_1 + 8\,y_2$

∵ primal is maximiz^n type (P1)

don't use Z

(Take RHS from primal constraints)

s.t.

$1.\,y_1 + 2.\,y_2 \geq 5$ → coeff. of x_1 in primal Z_{max}

coeff. of x_1 in primal constraints.

($x_1 \geq 0$). So, constraint is $\geq$ type

$2.\,y_1 - 1.\,y_2 \geq 12$ → coeff. of x_2 in Z

coeff of x_2 in primal constraint

$y_1 + 3y_3 \geq 4$

Non -ve restriction $y_1 \geq 0$ → ∵ primal's 1st restriction is $\leq$ type.

y_2 is unrestricted ∵ primal's second restriction is = type.

Q. Write dual of following LPP

(1) max. $z = 5x_1 + 6x_2 \rightarrow (1)$

s.t $x_1 + 2x_2 = 5 \rightarrow (2)$ (y_1)

$-x_1 + 5x_2 \geq 3 \rightarrow (3)$ (y_2)

$4x_1 + 7x_2 \leq 8 \rightarrow (4)$ (y_3)

x_1 is unrestricted $\rightarrow (5)$

$x_2 \geq 0. \rightarrow (6)$

Writing dual

minimize $(w) = 5y_1 + 3y_2 + 8y_3$ (from (1) (2) (3))

s.t $y_1 - y_2 + 4y_3 = 5$ $\rightarrow$ $\because x_1$ is unrestricted

$2y_1 + 5y_2 + 7y_3 \geq 6$ $\because x_2 \geq 0.$

y_1 is unrestricted (from (2)).

$y_2 \leq 0$ (from (3))

$y_3 \geq 0$ (from (4))

Q. Find dual of following LPP.

minimize $z = 1x_1 + 2x_2 + 0.x_3$

s.t $x_1 - x_2 + 3x_3 \leq -4$ (y_1)

$2x_2 - x_3 \geq 1$ (y_2)

$3x_1 + 2x_3 \leq 1$ (y_3)

$x_1, x_2, x_3 \geq 0$

Dual max. $w = -4y_1 + y_2 + y_3$

s.t $y_1 + 0.y_2 + 3.y_3 \leq 1$ $y_1 \leq 0$

$-1.y_1 + 2y_2 + 0.y_3 \leq 2$ $y_2 \geq 0$

$3y_1 - y_2 + 2y_3 \leq 0.$ $y_3 \leq 0$

Finding sign of constraints of dual

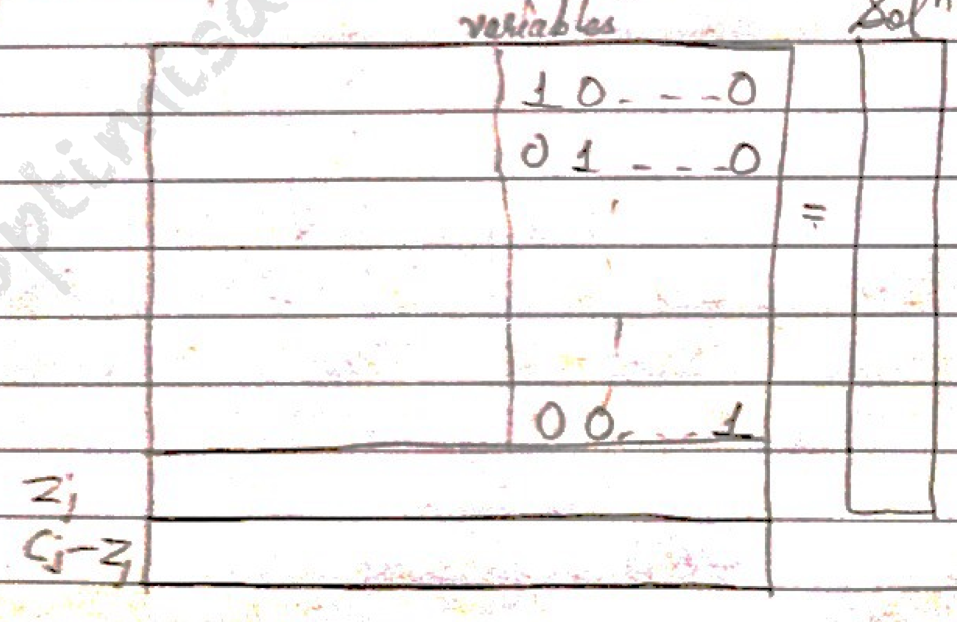
↳ we saw minimiz^n column (from table),
$x_1, x_2, x_3 \geq 0$ so ≥ 0. correspondingly it'll be $\leq$ type. Hence, all constraints are $\leq$ type for dual. (from ①)

Finding sign of variable of dual

↳ we saw minimiz^n column.
$x_1 - 2x_2 + 3x_3 \leq -4$. i.e $\leq$ type.
So, correspondingly, y_1 will be ≤ 0
(from ②)

✱ Forming a DUAL ~~matrix~~ simplex table from final simplex table of Primal (without solving dual)

Initial general format :- Starting basic variables

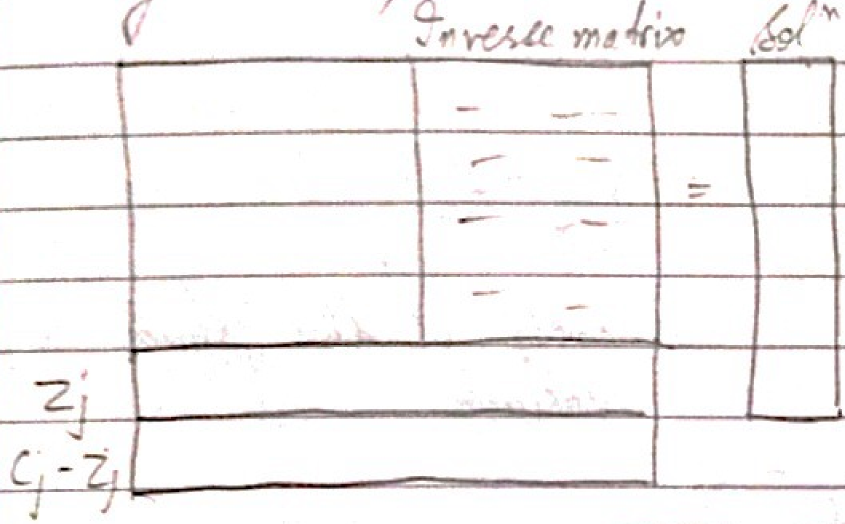


$y_i = -(c_j - z_j) +$ coeff. of slack/surplus var. in constraint

$= z_j$

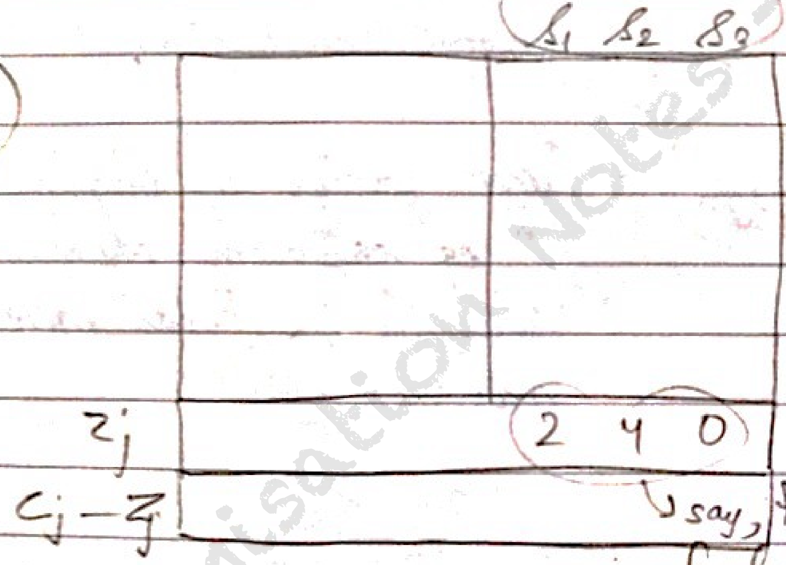
After solving by simplex method:

Final general format:



→ 3 slack/surplus variables
↓
3 dual variables

eg



y_1 is associated with s_1
y_2 〃 s_2
y_3 〃 s_3

say, these are final values of s_1 & s_2 & s_3 column.

So, $y_1 = 2$, $y_2 = 4$, $y_3 = 0$ is sol^n (values of Z_j)

☆ Note :- If we are given only inverse matrix at any Stage of simplex table, then, we can use it to generate complete table.

☆ Method to get optimal solⁿ of ~~so~~ Dual from final simplex table of primal.

Let y_j be the j^{th} dual variable corresponding to j^{th} primal constraint

METHOD 1 : Using Z_j row.

$y_j = -(c_j - Z_j)$ of corresponding slack or surplus variable

c_j ← + coeff. of this slack/surplus variable in the objective f^n

$\Rightarrow y_j = Z_j$

METHOD 2 : Using inverse of matrix.

$(y_1, y_2, \ldots, y_n)$ = Row vector of original objective f^n of optimum primal basic variables × Final inverse matrix.

Q Find dual of following primal. Solve the primal by simplex method & use the final simplex table of primal to find optimum solⁿ of dual

Given :- Minimize $Z = 15x_1 + 12x_2$

s.t. $x_1 + 2x_2 \geq 3$ (y_1)

$2x_1 - 4x_2 \leq 5$ (y_2)

$x_1, x_2 \geq 0$.

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Dual :- Maximize $w = 3y_1 + 5y_2$

s.t $y_1 + 2y_2 \leq 15$.
$2y_1 - 4y_2 \leq 12$

$y_1 \geq 0$, $y_2 \leq 0$.

Now, solving (by simplex method) for Primal:

$\Rightarrow$ Maximize $Z' = (-Z) = -15x_1 - 12x_2$

s.t $x_1 + 2x_2 - S_1 + A_1 = 3$
$2x_1 - 4x_2 + S_2 = 5$
$x_1, x_2, S_1, S_2, A_1 \geq 0$.

Now, solving by simplex method:

$\rightarrow$ Initial identity matrix

C_j		-15	-12	0	-M	0		
	Basic	x_1	x_2	S_1	A_1	S_2	Sol^n	Ratio
$\leftarrow$ -M	A_1	1	2	-1	1	0	3	3/2
0	S_2	2	-4	0	0	1	5	—
	Z_j	-M	-2M	M	-M	0	-3M	
	$C_j - Z_j$	M-15	2M-12 $\uparrow$	-M	0	0	—	

$R_1 \rightarrow R_1 \div 2$

$(R_2)_{new} = R_2 + 4(R_1)_{new}$ $\rightarrow$ Inverse matrix

-12	x_2	1/2	1	-1/2	1/2	0	3/2
0	S_2	4	0	-2	2	1	11
	Z_j	-6	-12	6	-6	0	-18
	$C_j - Z_j$	-9	0	-6	-M+6	0	

All $C_j - Z_j \leq 0$

So, optimum sol^n of primal is $x_1 = 0$, $x_2 = 3/2$

$Z_{min} = -Z'_{max} = -(-18)$

$\Rightarrow Z_{min} = 18$

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Now, finding solⁿ of dual

Using M1 $y_1 = 6$ → value of $(s_1)_{z_j}$ in final table

$y_2 = 0$ → value of $(s_2)_{z_j}$ in final table

$\omega_{max} = 18$. (after putting values of y_1 & y_2)

***

Using M2 $$(y_1 \;\; y_2) = \begin{array}{c} x_2 \quad s_2 \\ \begin{pmatrix} 12 & 0 \end{pmatrix} \end{array} \times \begin{pmatrix} 1/2 & 0 \\ 2 & 1 \end{pmatrix}$$

→ left non basic var

→ values in * original initial objective fⁿ

→ Inverse matrix

$$= (6 \quad 0)$$ → final solⁿ after solving

---

eg. Pg-155 Q.5 Write dual of following Primal. Solve primal using simplex method & use it to find optimum solⁿ of dual

Primal :- Maximize $z = 2x_1 + 4x_2 + 4x_3 - 3x_4$

s.t
$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + 4x_2 + x_4 = 8 \end{cases}$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

→ all = type. So, no need for slack/surplus variable.

5) Write dual (self)

So, now y_1 is in relation with x_3
y_2 " " x_4

So, solving simplex table of primal, final simplex table is given below.

C_j		2	4	4	-3	
	Basic	x_1	x_2	x_3	x_4	Sol^n
4	x_3	3/4	0	1	-1/4	2
4	x_2	1/4	1	0	1/4	2
	Z_j	4	4	4	0	16
	$C_j - Z_j$	-2	0	0	-3	—
		all ≤0 So, final table				

The optimum sol^n of primal is :

$x_1 = 0$, $x_2 = 2$, $x_3 = 2$, $x_4 = 0$

$Z_{max} = 16$

Method 1 :

$y_1 = 4$
$y_2 = 0$ values of Z_j

$\omega_{min} = 4y_1 + 8y_2$
$= 16 + 0 = 16$

Same

Method (2)

$$(y_1 \;\; y_2) = \begin{matrix} x_3 & x_2 \\ (4 & 4) \end{matrix} \times \begin{pmatrix} 1 & -1/4 \\ 0 & 1/4 \end{pmatrix} = (4 \;\; 0)$$

The basic variables in final table

$\Rightarrow (y_1 \cdot y_2) = (4 \;\; 0)$ (Same sol^n)

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* Property:
Let P be any feasible solⁿ of Primal
& D " " " " " Dual.
Then, $P \leq Z_{max} \leq D$; when Primal is max type
$D \leq Z_{min} \leq P$; when Primal is min type
↳ Z: objective fⁿ of primal

Q.8(a) Primal: Min. $Z = 5x_1 + 2x_2$
s.t. $x_1 - x_2 \geq 3$ (y_1)
$2x_1 + 3x_2 \geq 5$ (y_2)
$x_1, x_2 \geq 0$.

(i) Write dual of the primal.
(ii) Find one feasible solⁿ of primal & dual by inspection
(iii) find range of Z_{min}.

(i) Dual: Max $w = 3y_1 + 5y_2$
s.t. $y_1 + 2y_2 \leq 5$.
$-y_1 + 3y_2 \leq 2$
$y_1, y_2 \geq 0$

(ii) Take any value of x_1, x_2 satisfying its constraints. Now, put that in objective fⁿ. That is one feasible solⁿ.
Say, $x_1 = 5$, $x_2 = 2$ So, $5 - 2 \geq 3$ ✓
& $2(5) + 3(2) \geq 5$
Now, put in Z. $= 16 \geq 5$ ✓
$Z = 5(5) + 2(2) = 29$ is one feasible solⁿ.
$\Rightarrow P = Z = 29$

||ly ; for dual, take

$y_1 = 3,\ y_2 = 1$

we get $\omega = 14 = D$.

So,

(iii) Range :- $D \leq Z_{min} \leq P$

$\Rightarrow 14 \leq Z_{min} \leq 29$

Q.8(b) Primal :- Max $Z = x_1 + 5x_2 + 3x_3$

s.t $x_1 + 2x_2 + x_3 = 3$ (y_1)

$2x_1 - x_2 + 0x_3 = 4$ (y_2)

$x_1, x_2, x_3 \geq 0$

(i) Dual :- Min :- $\omega = 3y_1 + 4y_2$

s.t $y_1 + 2y_2 \geq 1$

$2y_1 - y_2 \geq 5$

$y_1 \geq 3$

✗ y_1, y_2 is unrestricted

✓ y_2 is unrestricted (∵ we got $y_1 \geq 3$, so, its restricted)

(ii)

A feasible solⁿ of primal is :

let $x_1 = 2,\ x_2 = 0,\ x_3 = 1$

(It was the case of 2 eqⁿs 3 unknowns so, take any var, say $x_2 = 0$ & solve for others)

So, $Z = 2 + 3 = 5 = P$

||ly, let $y_1 = 3,\ y_2 = 0$.

So, $\omega = 9 = D$.

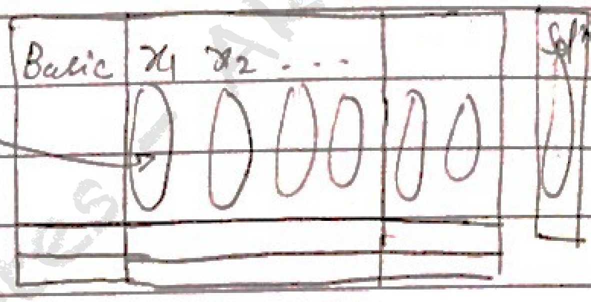
(iii) So, range is $5 \leq Z_{max} \leq 9$

☆ SIMPLEX TABLE COMPUTATION AT ANY ITERATION

Given: (a) Original data
or
(what's reqd to find table)
(b) Inverse of that iteration
(c) dual problem
(d) Basic variables in that iterations.

Following 2 things are reqd to be calculated:

1. Constraint columns



2. $C_j - Z_j$ row. (Z_j-row)

1. for Constraint Columns:

In any iteration i, the LHS & RHS of the constraint is calculated by using the following formula:

$$\left(\text{Constraint column in } i^{th} \text{ iteration}\right) = \left(\text{Inverse in } i^{th} \text{ iteration}\right) \times \left(\text{Corresponding original constraint column.}\right)$$

2. formula for $C_j - Z_j$

$$C_j - Z_j \text{ element} = \left(\text{RHS of } j^{th} \text{ dual constraint}\right) - \left(\text{LHS value of that dual constraint}\right)$$

★ Note: value of $c_j - z_j$ ∀ basic variables is ALWAYS zero. So, no need to calculate $c_j - z_j$ values for them.

Q.4 / Pg-157 Consider the following LPP

$$\max. \ Z = 2x_1 + x_2$$

$$\text{s.t } \ 3x_1 + x_2 - x_3 = 3$$

$$4x_1 + 3x_2 \quad - x_4 = 6$$

$$x_1 + 2x_2 \qquad + x_5 = 3$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Compute the entire simplex table associated with the following basic solⁿ. Also check the feasibility & optimality.

Given:-

(a) Basic variables :- (x_1, x_2, x_5)

(b) Inverse :- $\begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

S1) x_1 column = (Inverse) × (original x_1 column)

$$= \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Illy, x_2 column $= \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

x_3 column $= \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/5 \\ 4/5 \\ -1 \end{pmatrix}$

x_4 column $= \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/5 \\ -3/5 \\ 1 \end{pmatrix}$

x_5 column $= \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Analysis :- all basic variables gave us $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as expected

RHS $= \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$

S2) Writing $c_j - z_j$

finding solⁿ of dual first

Let y_1, y_2, y_3 be dual variables corresponding to 1st 2nd & 3rd primal constraints

So,

$$\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} = \overset{x_1 \quad x_2 \quad x_5 \;\to\; \text{basic variables}}{\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}} \times \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & 0 \end{pmatrix}$$

$\Rightarrow y_1 = 2/5,\ y_2 = 1/5,\ y_3 = 0$

(not optimum solⁿ. ∵ table here is not final table)

Now, ∵ ∃ $z = 2x_1 + x_2$

$$c_1 - z_1 = 2 - (3y_1 + 4y_2 + y_3)$$
$$= 2 - \left(3\left(\frac{2}{5}\right) + 4\left(\frac{1}{5}\right) + 0\right) = 0$$

$\because z = 2x_1 + 1 x_2$

$C_2 - Z_2 = 1 - (y_1 + 3y_2 + 2y_3) = 0$

$C_3 - Z_3 = 0 - (-y_1 + 0 + 0) = \frac{2}{5}$

$C_4 - Z_4 = 0 - (0 - y_2 + 0.y_3) = \frac{1}{5}$

$C_5 - Z_5 = 0 - (0 + 0 + 1.y_5) = 0$

0

So, simplex table is:-

C_j		2	1	0	0	0	
	Basic	x_1	x_2	x_3	x_4	x_5	sol^n
2	x_1	1	0	-3/5	1/5	0	3/5
1	x_2	0	1	4/5	-3/5	0	6/5
0	x_5	0	0	-1	1	1	0
	$C_j - Z_j$	0	0	2/5	1/5	0	

x_1 column

$C_1 - Z_1$ $C_5 - Z_5$

At this stage, basic feasible sol^n is

$x_1 = 3/5$

$x_2 = 6/5$

non basic variables $\begin{cases} x_3 = 0 \\ x_4 = 0 \\ x_5 = 0 \end{cases}$

The sol^n is not optimum, as, $C_j - Z_j \geq 0$

Type

Q. Given inverse, name of basic variables, original problem.

Find sol^n. Is it optimum?

Q2) Consider following LPP

Max $z = 4x_1 + 14x_2$

s.t. $2x_1 + 2x_2 + x_3 = 21$

$7x_1 + 2x_2 + x_4 = 21$

$x_1, x_2, x_3, x_4 \geq 0$

(1) Check the optimality & feasibility of following basic solns

(a) Basic variables: (x_2, x_4) & Inverse: $\begin{pmatrix} 1/7 & 0 \\ -2/7 & 1 \end{pmatrix}$

(b) Basic var: (x_2, x_3) & Inverse: $\begin{pmatrix} 0 & 1/2 \\ 1 & -7/2 \end{pmatrix}$

(a) Idea: Simplex table not reqd. We need only RHS column.

For checking optimality, find $c_j - z_j$ row.

Now

$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$ = (Inverse) × (RHS column)

$= \begin{pmatrix} 1/7 & 0 \\ -2/7 & 1 \end{pmatrix} \begin{pmatrix} 21 \\ 21 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 15 \end{pmatrix}$

∴ Basic soln is $x_1 = 0, x_2 = 3, x_3 = 0, x_4 = 15$

Now,

(check if any value of Basic soln is -ve. If not, the soln is feasible. If even 1 soln is -ve, its infeasible soln)

Here, all values +ve.

So, feasible ✓

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(i) Optimality condⁿ: finding $C_j - Z_j$ row.
Corresponding to all Basic variables, $C_j - Z_j = 0$.
So, for x_2, x_4, $C_2 - Z_2 = 0$
& $C_4 - Z_4 = 0$.

Now, for the other 2:
let y_1, y_2 be the dual variables

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{array}{cc} x_2 & x_4 \\ \end{array} \begin{pmatrix} 14 & 0 \end{pmatrix} \begin{pmatrix} 1/7 & 0 \\ -2/7 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \end{pmatrix}$$

$\Rightarrow y_1 = 2, \; y_2 = 0$

Now, $C_1 - Z_1 = 4 - (2y_1 + 7y_2)$

coeff of x_1 in objective fⁿ

$= 4 - ((2\times2) + 7(0))$

$\Rightarrow C_1 - Z_1 = 0$.

$C_3 - Z_3 = 0 - (1 \cdot y_1 + 0 \cdot y_2)$

coeff. of x_3 in objective fⁿ

$\Rightarrow C_3 - Z_3 = -2$.

here, all $C_j - Z_j \leq 0$. So, optimality condⁿ is satisfied.

(b) $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1 & -7/2 \end{pmatrix} \begin{pmatrix} 21 \\ 21 \end{pmatrix}$

$= \begin{pmatrix} \frac{21}{2} \\ -105/2 \end{pmatrix}$

$\Rightarrow x_2 = \frac{21}{2}, \; x_3 = \frac{-105}{2}$ & $x_1 = x_4 = 0$

It is not feasible but Basic ($\because x_3 = \frac{(-)105}{2}$) -ve

Optimality:
$C_2 - Z_2 = 0$ & $C_3 - Z_3 = 0$ (for Basic var.)
Finding y_1 & y_2,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{array}{cc} x_2 & x_3 \end{array} \begin{pmatrix} 14 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 \\ 1 & -7/2 \end{pmatrix} = \begin{pmatrix} 0 & 7 \end{pmatrix}$$

$\Rightarrow y_1 = 0, \; y_2 = 7.$

$C_1 - Z_1 = 4 - (2y_1 + 7y_2)$
$= -45$

$C_4 - Z_4 = 0 - (0 \cdot y_1 + 1 \cdot y_2)$
$= -7$

all -ve. So, optimum ✓.

§ DUAL SIMPLEX METHOD

Idea: start with a value outside the region bounded by graphical method (infeasible sol^n).
But, everytime, optimality $cond^n$ is satisfied. This infeasible optimal sol^n moves to feasible optimal sol^n.

Here, the RHS can be -ve. If ∃ any ≥ type constraint in SIMPLEX method, we needed slack variable & artificial var. (2 var. needed). Now, just × (-1) & any ≥ type can be converted to ≤ type & only one surplus var. is needed now (instead of 2 var.).

Method :-

S1) Convert it to maximiz^n type, if not.

S2) Multiply (-1) on both sides of all '$\geq$' type constraints, make them '$\leq$' type.

S3) Construct the initial simplex table.

S4) Calculate $c_j - z_j$

There are 3 possibilities for $c_j - z_j$

a. All $c_j - z_j \leq 0$ & all RHS elements (elements of sol^n column) are ≥ 0.
In this case, sol^n under test is optimum.

b. Atleast one $c_j - z_j \geq 0$.
This method is not applicable in this case.
($\because$ optimality cond^n got dissatisfied)

c. $c_j - z_j \leq 0$ & atleast one element in sol^n column < 0 (i.e., infeasible sol^n)
In this case, go to S5).

S5) Basic variable corresponding to most -ve element in the sol^n column is departing (leaving) variable

		sol^n
x_4		-2
departing vector $\leftarrow x_2$	α_1 α_2 α_3 α_4	-5 $\rightarrow$ departing

$\rightarrow$ If all entries in the departing vector row ≥ 0, then $\nexists$ no optimal sol^n of given LPP. $\boxed{\text{Stop}}$.

If any (atleast one) value of $\alpha_{ij} < 0$, then, proceed as :-

Calculate $\dfrac{c_j - z_j}{\alpha_{ij}}$

Then, pivot element is $\min\left(\dfrac{c_j - z_j}{\alpha_{ij}}\right)$'s column

PTO

Becomes pivot element

					Sol^n
x_1					-2
x_2	$\boxed{-2}$	1	-1	3	-5
$C_j - Z_j$	$\boxed{-2}$	0	-3	0	
$\frac{C_j - Z_j}{\alpha_{ij}}$	1	—	3	—	

min. value

take it as entering var.

After knowing the pivot element, solve by normal simplex method and proceed to S4).

keep doing it till we get all elements in sol^n column (+ve) → feasible sol^n.

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ex 1) Solve by Dual simplex method:

minimize: $Z = 3x_1 + x_2$

s.t. $x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$.

s1) Convert to maximizⁿ type

s2) Convert $\geq$ type to $\leq$ type.

So, we have

Max. $\bar{Z} = -Z = -3x_1 - x_2$

s.t $-x_1 - x_2 \leq -1$

$-2x_1 - 3x_2 \leq -2$

$x_1, x_2 \geq 0$.

S3) Add slack var. for each $\leq$ type constraint.

So, we have max $\bar{Z} = -3x_1 - x_2$

s.t $-x_1 - x_2 + S_1 = -1$

$-2x_1 - 3x_2 + S_2 = -2$

$x_1, x_2, S_1, S_2 \geq 0$, S_1, S_2: slack var.

S4) Construct table of Dual simplex method

Infeasible solⁿ (all -ve)

C_j		-3	-1	0	0		
	Basic	x_1	x_2	S_1	S_2	Solⁿ	Ratio
0	S_1	-1	-1	1	0	-1	
0	S_2	α_{21} -2	α_{22} $\boxed{-3}$ pivot	α_{23} 0	α_{24} 1	-2	→ most -ve → Departing var
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	-3	-1	0	0	—	
	$C_j - Z_j$ / α_{ij}	-3 / -2	-1 / -3	—	—	—	

→ find only for -ve value of α_{ij}

entering var

(all $c_j - z_j$ are ≤ 0, continue ✓ otherwise this method is not applicable)

$R_2 \to -R_2/3$

$R_1 \to \frac{R_1}{3} + (R_2)_{new}$

Pivot

Revised solⁿ:

infeasible again

C_j		-3	-1	0	0	
	Basic	x_1	x_2	s_1	s_2	Solⁿ
0	s_1	-1/3	0	1	$\boxed{-1/3}$	-1/3
-1	x_2	-2/3	1	0	-1/3	2/3
	Z_j	-2/3	-1	0	1/3	-2/3
	$C_j - Z_j$	-7/3	0	0	-1/3	—
	$\frac{C_j - Z_j}{\alpha_{ij}}$	7	—	—	1	

-ve no. So, departing vector

all -ve So, optimal. So, proceed.

Taking min. element

$R_1 \to -3R_1$

$R_2 \to R_2 - (R_1)_{old}$

	s_2	1	0	-3	1	1
	x_2	1	1	-1	0	1
	Z_j	-1	-1	1	0	-1
	$C_j - Z_j$	-2	0	-1	0	—

all +ve So, Feasible

all -ve So, optimum.

So, we got optimal solⁿ

Solⁿ: $x_1 = 0, \; x_2 = 1$

$$Z_{min} = -Z_{max} = -(-1) = 1$$

Q) Solve the following LPP by dual simplex method

Minimize $Z = 2x_1 + x_2$

s.t $3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 < 3$

$x_1, x_2 \geq 0.$

s_1, s_2) :-

Max. $\bar{Z} = -2x_1 - x_2$ $\quad (= -Z)$

$-3x_1 - x_2 + s_1 = -3$ $\quad (\times(-1)$ & $+ s_1)$

$-4x_1 - 3x_2 + s_2 = -6$ $\quad (\times(-1)$ & $+ s_2)$

$x_1 + 2x_2 + s_3 = 3$ $\quad (+ s_3)$

$x_1, x_2, s_1, s_2, s_3 \geq 0$.

C_j		-2	-1	0	0	0	
	Basic	x_1	x_2	s_1	s_2	s_3	b
0	s_1	-3	-1	1	0	0	-3
0	s_2	-4	$\boxed{-3}$	0	1	0	-6 → -ve so infeasible
0	s_3	1	2	0	0	1	3
	Z_j	0	0	0	0	0	0
	$C_j - Z_j$	-2	-1	0	0	0	—
	$\frac{C_j - Z_j}{\alpha_{ij}}$	$\frac{-2}{-4}$	$\frac{-1}{-3}$	—	—	—	—

Taking most -ve soln

→ optimum So take

Take min value

$R_2 \rightarrow -R_2/3$

$R_1 \rightarrow R_1 + (R_2)_{new}$

$R_3 \rightarrow R_3 - 2(R_2)_{new}$

C_j	Basic	x_1	x_2	s_1	s_2	s_3	b
0	s_1	$\boxed{-5/3}$	0	1	$-1/3$	0	-1
-1	x_2	$4/3$	1	0	$-1/3$	0	2
0	s_3	$-5/3$	0	0	$2/3$	1	-1
	Z_j	$-4/3$	-1	0	$1/3$	0	-2
	$C_j - Z_j$	$-2/3$	0	0	$-1/3$	0	—
	$\frac{C_j - Z_j}{\alpha_{ij}}$	$\frac{2}{5}$	—	—	1	—	—

-ve, so infeasible

Both solns equally -ve. Take any one as departing var.

Take min value

$R_1 \rightarrow -\frac{3}{5} R_1$ $\qquad R_3 \rightarrow R_3 - (R_1)_{old}$

$R_2 \rightarrow \frac{3}{4} R_2 - (R_1)_{new}$

C_j		-2	-1	0	0	0	
	Basic	x_1	x_2	s_1	s_2	s_3	sol^n
-2	x_1	1	0	-3/5	1/5	0	3/5
-1	x_2	0	1	4/5	-3/5	0	6/5
0	s_3	0	0	-1	1	1	0
	Z_j	-2	-1	2/5	+1/5	0	-12/5
	$C_j - Z_j$	0	0	-2/5	-1/5	0	

+ve so feasible

all -ve
So optimum.

Optimum sol^n:-

$x_1 = 3/5$

$x_2 = 6/5$

$Z_{min} = -Z_{max} = \frac{12}{5}$ Ans

In future, if some other condns are added, how to get revised soln.

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POST OPTIMAL ANALYSIS

⇒ It can be carried out in the following cases:

Case 1. Making changes in the RHS constants of constraints.

Case 2. Making changes in the objective coeffs.

eg:- $\text{Max } z = 5x_1 + 3x_2 + 4x_3$

$5 \to c_1$, $3 \to c_2$, $4 \to c_3$

These c_j are objective coeffs.

Case 3. Adding a new constraint

Case 4. Adding a new decision variable.

CASE-I

Changes in RHS constants of constraints

how to get final revised soln from final simplex table

Formula:

$$\left(\text{New RHS column}\right) = \left(\begin{matrix}\text{Inverse of final}\\ \text{simplex table}\end{matrix}\right) \times \left(\begin{matrix}\text{New RHS}\\ \text{Constants}\end{matrix}\right)$$

↳ we get new RHS column

↳ Soln column in final table.

If all elements in soln column of final simplex table are ≥ 0, then this column gives Revised Optimised soln.

If atleast one element in RHS column is -ve (infeasible), apply dual simplex method to get revised optimum solⁿ.

Will come in exam

(ex-1) Consider the following LPP.

Max.

$$Z = 6x_1 + 8x_2$$

$$\text{s.t } 5x_1 + 10x_2 \leq 60;$$

$$4x_1 + 4x_2 \leq 40;$$

$$x_1, x_2 \geq 0.$$

The final simplex table is

→ Inverse in final table

C_j		6	8	0	0	
	Basic	x_1	x_2	S_1	S_2	Sol^n
8	x_2	0	1	1/5	-1/4	2
6	x_1	1	0	-1/5	1/2	8
	Z_j	6	8	2/5	1	64
	$C_j - Z_j$	0	0	-2/5	-1	—

Find: revised optimum sol^n if RHS constts of constraints are changed

(a) from $\begin{bmatrix} 60 \\ 40 \end{bmatrix}$ to $\begin{bmatrix} 40 \\ 20 \end{bmatrix}$

(b) from $\begin{bmatrix} 60 \\ 40 \end{bmatrix}$ to $\begin{bmatrix} 20 \\ 40 \end{bmatrix}$: Meaning: (supplier 1 has reduced supply from 60 to 20 & supplier 2 has kept supply constant)

Inverse of final table :

$$\begin{pmatrix} 1/5 & -1/4 \\ -1/5 & 1/2 \end{pmatrix}$$

Basic variables : (x_2, x_1)

(a) New RHS elements :

$$\begin{pmatrix} x_{2\,new} \\ x_{1\,new} \end{pmatrix} = \begin{pmatrix} 1/5 & -1/4 \\ -1/5 & 1/2 \end{pmatrix} \begin{pmatrix} 40 \\ 20 \end{pmatrix}$$

→ revised constraint constants.

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

both +ve So, sol^n is still feasible & optimum.

$\therefore$ Revised optimum sol^n is :-

$$x_1 = 2, \quad x_2 = 3$$

$$Z_{max} = 6(2) + 8(3) = 36$$

(b)

$$\begin{pmatrix} x_{2\,new} \\ x_{1\,new} \end{pmatrix} = \begin{pmatrix} +1/5 & -1/4 \\ -1/5 & 1/2 \end{pmatrix} \begin{pmatrix} 20 \\ 40 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 16 \end{pmatrix}$$

→ one -ve value ⇒ sol^n column has a -ve value. So, sol^n is infeasible for $x_2 = -6$.

$\therefore$ $C_j - Z_j$ is still all -ve. (optimum)

So, using dual simplex method.

Table

	C_j	6	8	0	0	
	Basic	x_1	x_2	s_1	s_2	sol^n
8	x_2	0	1	1/5	$\boxed{-1/4}$	-6
6	x_1	1	0	1/5	1/2	16
	Z_i	6	8	2/5	1	48
	$c_j - z_j$	0	0	-2/5	-1	
	$(c_j - z_j)/\alpha_i$	—	—	—	$(-1) \div (-1/4) = 4$	

departing var. ← Corresponding to -ve no.

→ Change these 2 things

Treat this as the initial table of dual simplex

$R_1 \rightarrow (-4)R_1$
$R_2 \rightarrow R_2 + (2R_1)_{old}$.

C_j		6	8	0	0	
	Basic	x_1	x_2	S_1	S_2	Solⁿ
0	S_2	0	-4	-4/5	1	24
6	x_1	1	2	1/5	0	4
	Z_j	6	12	6/5	0	24
	$C_j - Z_j$	0	-4	-6/5	0	—

→ all +ve so, feasible

→ all -ve. So, still optimum.

hence, optimum solⁿ is:-
$x_1 = 4$
$x_2 = 0$
$Z_{max} = 24$.

---

§ CASE-II

Changes in the objective coeff$_s$.

↳ C_j is being changed. So, $C_j - Z_j$ will change. So, now $C_j - Z_j$ may not remain (all ≤ 0) so, now perform further steps of simplex table & get optimum solⁿ.

S1) Compute the revised values of $C_j - Z_j$.
2 possibilities - The C_j that is changed
is basic : all values change.
or non basic variable : only one value of $C_j - Z_j$ changes

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S2) If all $c_j - Z_j \leq 0$ then the solⁿ (optimum values of decision variables) will remain unchanged.
If atleast one $c_j - Z_j \geq 0$, we have to perform some addnl steps of simplex method to get revised solⁿ.

ex Consider the LPP

Maximize $Z = 10x_1 + 15x_2 + 20x_3$

s.t. $2x_1 + 4x_2 + 6x_3 \leq 24$

$3x_1 + 9x_2 + 6x_3 \leq 30$

$x_1, x_2, x_3 \geq 0$

The final simplex table of given LPP is:

C_j		10	15	20	0	0	
	Basic	x_1	x_2	x_3	s_1	s_2	solⁿ
20	x_3	0	-1	1	1/2	-1/3	2
10	x_1	1	5	0	-1	+1	6
	Z_j	10	30	20	0	10/3	100
	$C_j - Z_j$	0	-15	0	0	-10/3	

all -ve so, its final simplex table (Re-verificⁿ) - (optimized)

Now, (a)
Find:- Range of objective coeff. c_1 (of x_1) s.t optimality remains unaffected.
(b) Do same for coeff c_2 (of x_2)
(c) If (c_1, c_2, c_3) are changed from (10, 15, 20) to (7, 14, 15) : find the revised solⁿ.

(M1) Modify the objective fn using new coeff
(Long) Solve by normal method of simplex table & solve (like done before)

(M2) Directly evaluating from final simplex table
(Short)

(a) We want range of c_1 (coeff of x_1)
See if c_1 is basic variable or not
Here, it is a basic variable (x_1)
So, all $c_j - z_j$ are changed.
So, calculating $(c_j - z_j)$ revised $\forall$ non basic variables
↳ (∵ for basic var., it's = 0 (always))

here, c_1 (x_1) and c_3 (x_3) are basic variables. So,

$$c_1 - z_1 = 0 \;\&\; c_3 - z_3 = 0$$

Hence, find $\underset{(x_2)}{c_2 - z_2}$, $\underset{(s_1)}{c_4 - z_4}$, $\underset{(s_2)}{c_5 - z_5}$

$$c_2 - z_2 = c_2 - \begin{array}{c} \begin{matrix} x_3 & x_4 \end{matrix} \\ \begin{bmatrix} c_3 & c_1 \end{bmatrix} \end{array} \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

→ writing coeff. of basic var. in order

*

$$= 15 - \begin{bmatrix} 20 & c_1 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

→ constraint column for x_2

$$= 15 - [-20 + 5c_1]$$

$$\Rightarrow c_2 - z_2 = 35 - 5c_1 \rightarrow (1)$$

$$c_4 - z_4 = c_4 - \begin{bmatrix} c_3 & c_1 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$$

$$= 0 - \begin{bmatrix} 20 & c_1 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1 \end{bmatrix} = -10 + c_1 = c_1 - 10$$

$$\Rightarrow c_4 - z_4 = c_1 - 10 \rightarrow (2)$$

$$c_5 - z_5 = c_5 - [c_3 \;\; c_1]\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

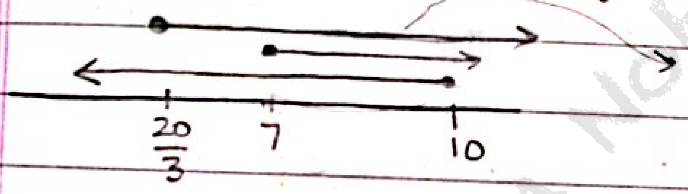
$$= 0 - [20 \;\; c_1]\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

$$\Rightarrow c_5 - z_5 = \frac{20}{3} - c_1 \rightarrow (3)$$

Now, for optimality to be satisfied,
eq^{ns} (1), (2), (3) ≤ 0

So, $35 - 5c_1 \leq 0$, $\quad c_1 - 10 \leq 0$, $\quad \frac{20}{3} - c_1 \leq 0$
$\Rightarrow c_1 \geq 7$, $\quad \Rightarrow c_1 \leq 10$, $\quad \Rightarrow c_1 \geq \frac{20}{3}$

Taking intersection



$\underline{\underline{7 \leq c_1 \leq 10}}$ range.

(b) x_2 is non basic var. in final simplex table.
So, calculate only revised value of $c_2 - z_2$.
Hence, revised $c_2 - z_2 = c_2 - [c_3 \;\; c_1]\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$$= c_2 - [20 \;\; 10]\begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$$= c_2 - 30$$

For optimality, $c_2 - 30 \leq 0$
$\Rightarrow \underline{\underline{c_2 \leq 30}}$.

(c) All 3 objective coeff's are changed.

C_j		7	14	15	0	0	
	Basic	x_1	x_2	x_3	S_1	S_2	Sol^n.
15	x_3	0	-1	1	1/2	-1/2	2
7	x_1	1	5	0	-1	1	6
	Z_j						100
	$C_j - Z_j$						

$(C_1)_{new} = 7$, $(C_2)_{new} = 14$, $(C_3)_{new} = 15$

Revised $c_1 - Z_1 = C_1 - \begin{array}{c} \begin{matrix} x_3 & x_1 \end{matrix} \\ \begin{bmatrix} C_3 & C_1 \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= 7 - \begin{bmatrix} 15 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$= 0$. (Checked → verific^n. x_1 is basic variable, so, $C_1 - Z_1 = 0$, always)

Revised $C_2 - Z_2 = C_2 - \begin{bmatrix} C_3 & C_1 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$= 14 - \begin{bmatrix} 15 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$= 14 - 20 = -6$

$\Rightarrow C_2 - Z_2 = -6 \rightarrow (4)$

Revised $C_3 - Z_3 = 0$ (Basic var.)

Revised $C_4 - Z_4 = C_4 - \begin{bmatrix} C_3 & C_1 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$

$= 0 - \begin{bmatrix} 15 & 7 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1 \end{bmatrix} = -1/2$

$\Rightarrow C_4 - Z_4 = -1/2 \rightarrow (5)$

Revised $c_5 - z_5 = c_5 - [C_3 \;\; c_1]\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$

$= 0 - [15 \;\; 7]\begin{bmatrix} -1/3 \\ 1 \end{bmatrix} = -2$

$\Rightarrow c_5 - z_5 = -2 \rightarrow (6)$

From (4), (5), (6),

all $c_j - z_j \leq 0$.

So, optimality will remain unaffected.

So, optimal solⁿ is :-

$x_1 = 6,\; x_2 = 0,\; x_3 = 2$

Revised $Z_{max} = (c_1)_{new} x_1 + (c_2)_{new} x_2 + (c_3)_{new} x_3$

$= 7x_1 + 14x_2 + 15x_3$

$= 7(6) + 14(0) + 15(2)$

$\Rightarrow (Z_{max})_{new} = 72$

☆ Note :-

If atleast one revised $c_j - z_j \geq 0$, then, we have to apply simplex method to find optimum solⁿ (by putting revised values of $c_j - z_j$ in final simplex table)

☆ Adding a new constraint decreases / remains same the value of Z_{max}.

§ CASE – III

Adding a new constraint.

If a new constraint is added, then, either the optimum solⁿ will remain unchanged, or, ∃ change in optimum solⁿ.

To test whether ∃ no change in optimum solⁿ:

Substitute values of the decision vars. in new constraint > If these values satisfy new constraint, then, solⁿ is unchanged.

eg:- If final solⁿ is $x_1 = 5, x_2 = 7$.
Suppose I add new constraint $2x_1 + 3x_2 \le 500$
Put these values $2(5) + 3(7) = 31 \le 500$
So, solⁿ is unchanged.

> If these values doesn't satisfy new constraint, ∃ change in optimum solⁿ.
In this case, we have to add an additional row in the final simplex table & also an additional column corresponding to new slack var. Then, using elementary row operⁿˢ, convert elements in this new row to zero (corresponding to basic var). Then, perform addⁿᵃˡ steps of simplex/dual simplex method to find revised optimum solⁿ.

* Final simplex table sow
Simplex Method = Dual Simplex Method

Ex Consider the LPP

Maximize $Z = 6x_1 + 8x_2$

s.t. $5x_1 + 10x_2 \le 60$;

$4x_1 + 4x_2 \le 40$;

$x_1, x_2 \ge 0$.

The optimum / final simplex table is :-

C_j		6	8	0	0	
	Basic	x_1	x_2	S_1	S_2	Sol^n
8	x_2	0	1	1/5	-1/4	2
6	x_1	1	0	-1/5	1/2	8
	Z_j	6	8	2/5	1	64
	$C_j - Z_j$	0	0	-2/5	-1	

Check whether the add^n of given constraint will affect the optimum sol^n. If yes, find the revised optimum sol^n. The added constraint :

(a) $7x_1 + 2x_2 \le 65$

(b) $6x_1 + 3x_2 \le 48$

Sol The optimum sol^n of original problem is :-

$x_1 = 8$, $x_2 = 2$, $Z_{max} = 64$.

(a) The new constraint is

$7x_1 + 2x_2 \le 65$.

Substituting the values of x_1 & x_2 in new constraint

$\Rightarrow 7(8) + 2(2) = 60 \le 65$. So, original optimum sol^n ($x_1 = 8$, $x_2 = 2$) satisfies given constraint

So, the optimum solⁿ remains UNCHANGED

So, optimum sol^n remains $x_1 = 8,\ x_2 = 2,\ Z_{max} = 64$

(b) New constraint is :-

$$6x_1 + 3x_2 \leq 48$$

Putting value of $x_1 = 8,\ x_2 = 2$.

$$\Rightarrow 6(8) + 3(2) = 54 \ \cancel{\leq 48}$$

So, $x_1 = 8,\ x_2 = 2$ do not satisfy new constraint. So, optimal sol^n is being affected. Hence, sol^n will be revised.

We have to add this constraint.

The new constraint with slack variable s_3 can be written as :-

new row

$$\text{①}\ 6x_1 + 3x_2 + s_3 = 48$$

Writing old rows of final table

$$\text{②}\ x_2 + \frac{1}{5}s_1 - \frac{1}{4}s_2 = 2$$

$$\text{③}\ x_1 - \frac{1}{5}s_1 + \frac{1}{2}s_2 = 8$$

eq^n set Ⓐ

Note :

When we do any simplex table problem, the final table's basic variables form a unit matrix.

Now, we are adding new constraint (variable s_3).

So, the values should come as

s_3	1	0	0
x_2	0	1	0
x_1	0	0	1

So, make x_1 coeff $= 0$, x_2 coeff $= 0$, $s_3 = 1$.

Applying the transform^n on eq^n set (A),

(1) − { (2) × 3 + (3) × 6 }

we get

$$0.x_1 + 0.x_2 + \frac{3}{5}S_1 - \frac{9}{4}S_2 + S_3 = -6$$

So, simplex table modifies as:

C_j		6	8	0	0	0	
	Basic	x_1	x_2	S_1	S_2	S_3	Sol^n
8	x_2	0	1	1/5	−1/4	0	2
6	x_1	1	0	−1/5	1/2	0	8
0	S_3	0	0	3/5	$\boxed{-9/4}$	1	−6 →
	Z_j	6	8	2/5	1	0	64
	$C_j - Z_j$	0	0	−2/5	−1	0	
	$\frac{C_j - Z_j}{\alpha_{ij}}$	−	−	−	4/9	−	

−ve. unfeasible

α_{ij} row

Sol^n column is −ve. Not feasible. So, use dual simplex

8	x_2	0	1	2/15	0	−1/9	8/3
6	x_1	1	0	−1/15	0	2/9	20/3
0	S_2	0	0	−4/6	1	−4/9	8/3
	Z_j	6	8	2/3	0	4/9	184/3
	$C_j - Z_j$	0	0	−2/3	0	−4/9	

all +ve so, feasible

all −ve. So, optimised

Optimized $x_1 = \frac{20}{3}$, $x_2 = 8/3$, $Z_{max} = \frac{184}{3}$.

Case IV

Adding a new decision variable.

Let x_k be a new decision variable added in the LPP

c_k : objective coeff. of x_k

& $\alpha_1, \alpha_2, \alpha_3$ ---- are constraint coeff of x_k

eg : Suppose we have

$Z_{max} = c_1 x_1 + c_2 x_2$

If we add a new decision variable x_k.

So, new objective f^n, $Z_{max} = c_1 x_1 + c_2 x_2 + c_k x_k$

where, for each constraint, the coeff. of $x_k = \alpha_1, \alpha_2, \alpha_3$ (for 3 constraints, say)

In this situation, we find 2 things :

(i) Constraint coeff. of x_k in final simplex table or revised simplex table

(ii) $c_k - Z_k$ i.e, $c_j - Z_j$ corresponding to new variable

Formula :

(Constraint coeff. of x_k) = (Inverse) × (Original constraint coeff. of x_k)

eg : Consider the LPP:

Max. $Z = 6x_1 + 8x_2$

s.t $5x_1 + 10x_2 \leq 60$

$4x_1 + 4x_2 \leq 40$

$x_1, x_2 \geq 0$.

Final simplex table is :-

Inverse

C_j		6	8	0	0	
	Basic	x_1	x_2	S_1	S_2	Sol^n
8	x_2	0	1	$1/5$	$-1/4$	2
6	x_1	1	0	$-1/5$	$1/2$	8
	Z_j	6	8	$2/5$	1	64
	$C_j - Z_j$	0	0	$-2/5$	-1	

A new decision variable, x_3 is added to this LPP. The objective coeff. of x_3 is 20 & constraint coeff. are 6 & 5 for 1st & 2nd constraint resp.

Check whether inclusion of this new variable will affect optimality.

Idea : The column of x_3 has to be filled using what is given
$C_j - Z_j$ row has to be modified using formula

Now,

$$(\text{Constraint coeff. of } x_3) = \begin{pmatrix} 1/5 & -1/4 \\ -1/5 & 1/2 \end{pmatrix} \times \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -1/20 \\ 13/10 \end{pmatrix}$$

Now,

$$C_3 - Z_3 = C_3 - \begin{array}{c} \begin{array}{cc} x_2 & x_1 \end{array} \\ \begin{bmatrix} C_2 & C_1 \end{bmatrix} \end{array} \begin{pmatrix} 1/5 & -1/4 \\ -1/5 & 1/2 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

$$= 20 - \begin{bmatrix} 8 & 6 \end{bmatrix} \begin{pmatrix} 1/5 & -1/4 \\ -1/5 & 1/2 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

$$\Rightarrow C_3 - Z_3 = \frac{63}{5}$$

The revised simplex table is :-

C_j	Basic	x_1	x_2	x_3	s_1	s_2	Solⁿ
8	x_2	0	1	$-1/20$	$1/5$	$-1/4$	2
6	x_1	1	0	$13/10$	$-1/5$	$1/2$	8
	Z_j	6	8	$37/5$	$2/5$	1	64
	$C_j - Z_j$	0	0	$63/5$	$-2/5$	-1	

find

not optimum.

So, applying simplex method.

After solving, final solⁿ is

$$x_1 = x_2 = 0, \quad x_3 = 8, \quad Z_{max} = 160$$

So, basically, I am getting max profit if I don't produce any unit of x_1 & x_2 & produce 8 units of x_3

Always : $(+) d^-$
$(-) d^+$

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Chapter - 8
GOAL PROGRAMMING PROBLEM

applicable when no. of objectives (goals) > 1.

here, we have flexible constraints.
eg: $2x_1 + 3x_2 \geq 5$.
we'll try to make it true. Its fine even if it doesn't happen.

$$2x_1 + 3x_2 + d^- - d^+ = 5$$

under achievement   over achievement

DEVIATIONAL Variables

↳ d^- & d^+ cannot be simultaneously +ve.

~~Objective fn~~

for $\geq$ type $\Rightarrow$ over achievement can come
So, d^+ term can be removed.
for $\leq$ type $\Rightarrow$ under achievement is permissible
So, d^- term can be dropped.

To convert a flexible constraint into eqⁿ :-
(Introduce deviational vars.)

If constraint is '=' type :- add d^- & sub d^+
If constraint is '≥' type :- add d^- & (drop)
If constraint is '≤' type :- (drop) & sub d^+

eg : Constraints :

$$x_1 + x_2 \geq 4 \Rightarrow x_1 + x_2 + d_1^- = 4$$
$$2x_1 - x_2 \leq 6 \Rightarrow 2x_1 - x_2 - d_2^+ = 6$$
$$5x_1 + x_2 = 7 \Rightarrow 5x_1 + x_2 + d_3^- - d_3^+ = 7$$
$$\hookrightarrow x_1, x_2, d_1^-, d_2^+, d_3^-, d_3^+ \geq 0$$

Objective fnˢ :- Minimize $G_1 = d_1^-$
Minimize $G_2 = d_2^+$
Minimize $G_3 = d_3^- + d_3^+$

* ∃ 2 methods to solve Goal Programming Problem:

M1. Weight method
M2. Preemptive method

(M1) Assign some weights to goals $G_1, G_2 \ldots$ (any random integer value)
As from above, let weight for $G_1 \rightarrow \omega_1$
" $G_2 \rightarrow \omega_2$
$G_3 \rightarrow \omega_3$

Then, take weighted sum of goals.
i.e, minimize $Z = \omega_1 G_1 + \omega_2 G_2 + \omega_3 G_3$
(Linear)

In this chapter, mainly what comes —
make the goal (formulation only)
Won't come to solve the problems

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So, 3 goals have become a single objective
This can be solved by Simplex method.

(M2) Arrange the goals (G_1, G_2, ——) in order of their importance

eg, Say importance $G_2 > G_3 > G_1$

So, G_2
G_3
G_1

Now, solve the top goal.

$\text{Min } G_2 = d_2^+$

This gives value of d_2^+. Put in eq^n & reduce the variable d_2^+.

Then solve G_3.

$\text{Min } G_3 = d_3^- + d_3^+$

$\text{Min } G_1 = d_1^-$

Ex- 8.18

Q2) Shopping mall conducts events to attract people: teenagers (T), young & middle age (Y/M), seniors (S)

Band concert: \$1500
Art show: \$3000.

Total budget \$15000
→ This is strict constraint.

The attendance is seen as follows:

PTO

	No. attending per present^n		
Event	T	Y/M	S
Band Concert	200	100	0
Art Show	0	400	250

Manager has set goals of 1000 : T
1200 : Y/M
800 : S

Formulate problem as Goal Programming.

Sol^n :- Identify decision var.

x_1 : no. of present^ns of band concert per year
x_2 : no. of present^n of art show per year.

Flexible constraints are :-

$$200x_1 + 0.x_2 \geq 1000$$
$$100x_1 + 400x_2 \geq 1200$$
$$0.x_1 + 250x_2 \geq 800$$

Strict constraints :-

$$1500x_1 + 3000x_2 \leq 15000 \longrightarrow (1)$$

↳ constraint similar to LPP constraint (can be solved)

Introducing deviational var. (for flexible)
slack var (for strict).

$$\Rightarrow 200x_1 + d_1^- = 1000$$
$$100x_1 + 400x_2 + d_2^- = 1200$$
$$250x_2 + d_3^- = 800$$
$$x_1, x_2, d_1^-, d_2^-, d_3^- \geq 0$$

Objectives are :-

Minimize $G_1 = d_1^-$
Minimize $G_2 = d_2^-$
Minimize $G_3 = d_3^-$ (B)

If the importance of G2 is twice of other weights, then giving weights as (1, 2, 1),

Ans.

Q5) Mentel produce toy carriage needing :
4 wheels
2 seats.

∃ 3 shifts for production

Shift	Units produced per hour: Wheels	Seats
1	500	300
2	[illegible]	280
3	[illegible]	360

Goal : In each shift, how many production hrs. ? so that imbalance is minimized
Ideally, no. of wheels produce = 2× (no. of seats)
But, its difficult to maintain this balance.

Strict restrictions :
no. of production hrs.: $4 \leq$ Shift 1 ≤ 5
$10 \leq$ Shift 2 ≤ 20
$3 \leq$ Shift 3 ≤ 5

Formulate this problem as a GPP.

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Let x_j be the no. of production hours in shift j ($j = 1, 2, 3$)

My goal is to minimize balance in production no.
So, make deviation = 0.

In shift 1:
deviation per run = $500 - 2(300) = -100$.
(Seats = 300
Ideally, we should have got
wheels = $2 \times 300 = 600$.
But, its given 500. So, $500 - 600 =$
-100 deviation.)

In shift 2, similarly
deviation/run = $600 - 2(280) = 40$.
Shift 3,
deviation/run = $640 - 2(360) = -80$.

Total deviation = $-100x_1 + 40x_2 - 80x_3$.
We want to make total deviation = 0.
$\Rightarrow -100x_1 + 40x_2 - 80x_3 = 0$.
Using/Introducing deviational vars, we get,

$$-100x_1 + 40x_2 - 80x_3 + d_1^- - d_1^+ = 0$$

d_1^+ → overachievement
d_1^- → underachievement

Given:-
$4 \leq x_1 \leq 5$
$10 \leq x_2 \leq 20$
$3 \leq x_3 \leq 5$.

here, both can happen. So, included both.

The objective fn is :-
Minimize :- $G = d_1^- + d_1^+$

Q-7) A firm produces 2 types of items : x & y
x sells @ profit (net) $\rightarrow$ ₹ 80 / unit
y sells @ net profit of ₹ 40 / unit

The goal of the problem is to get a profit of ₹ 900
Another goal is : the sale volume of x & y :
sale volume of $x \rightarrow 17$
$y \rightarrow 15$
→ very close to

Decision vars : no. of units of product $x = x_1$
" " " $y = x_2$

Profit goal :- $80x_1 + 40x_2 = 900$
Sale volume goal :- $x_1 \geq 17$ (assumed ($\geq$). Take
$x_2 \geq 15$. $\leq$ or $=$ if wanted.)

Introducing deviation :-
$80x_1 + 40x_2 + d_1^- - d_1^+ = 900$

$x_1 + d_2^- - d_2^+ = 17$
$x_2 + d_3^- - d_3^+$
→ X : ignore them ∵ over achievement is okay with me, as I've taken $\geq$ type

$x_1, x_2, d_1^-, d_1^+, d_2^-, d_3^- \geq 0$.

Goals :
$$\left.\begin{aligned} \text{Min } G_1 &= d_1^- + d_1^+ \\ \text{Min } G_2 &= d_2^- \\ \text{Min } G_3 &= d_3^- \end{aligned}\right\} (B)$$

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Q. A manufacturer produces 2 kinds of products X & Y. Each unit of type X & Y requires 2 hrs of production capacity of plant. Plant has max. production capacity of 20 hrs.
The plant manager has following goals :-
1) To avoid underutiliz^n of plant capacity & Avoid overtime oper^n.
2) To limit overtime hrs to 4 hrs per week.
Formulate it as a GPP.

Decision vari: No. of units of type X : x_1
" " " Y : x_2.

Now, $2x_1 + 2x_2 \geq 20$
To reduce underutiliz^n.

& $2x_1 + 2x_2 \leq 20$
To minimize overtime

So, First goal constraint : $2x_1 + 2x_2 = 20$.

Clearly, any overtime is actually (in this case) the overachievement of the facility.
Hence, for the objective fn :- (Introducing deviations)
$2x_1 + 2x_2 + d_1^- - d_1^+ = 20$.

→ where $d_1^+ \leq 4$. : Constraint.
So, for this, adding deviation ⇒
$d_1^+ + \cancel{d_1^-} - d_2^+ = 4$.

no need. ∵ It is underachievement & is satisfying "≤". So, its not deviation.

$x_1, x_2, d_1^+, d_1^-, d_2^+ \geq 0$

So, Goals :- Min $G_1 = d_1^- + d_1^+$

Min $G_2 = d_2^+$

* DP is used when many routes are there & shortest route has to be found.

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Chapter - 10

DYNAMIC PROGRAMMING (DP)

✓ ∃ a single problem with many var.s (n, say). It is broken n subproblems (called as stages). Each subproblem contains a single var. For i^{th} stage, variable will be called x_i. The stages are solved one after the other for final solⁿ.

✓ BELLMAN'S Principle

An optimal policy has the property that whatever the initial state & decision are, the remaining decision must constitute an optimal policy with regard to the state resulting from 1st decision.

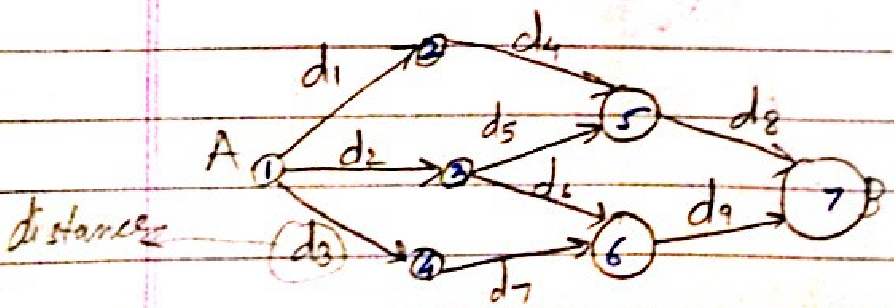
- each stage is related with prev. stage by a recurrence relⁿ

* To find shortest path in a network

M1) → Forward Recursive Approach

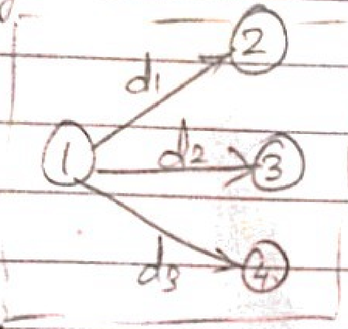
Consider a network:

Idea :- Go from A to B via min. distance



Break this into stages. So, for 1st stage, it consists of only 1 jump / move

» Stage 1:



Let the distance in a jump $= f_{①}$ → for 1st stage.

So, my $f^n = f_1(x_1)$

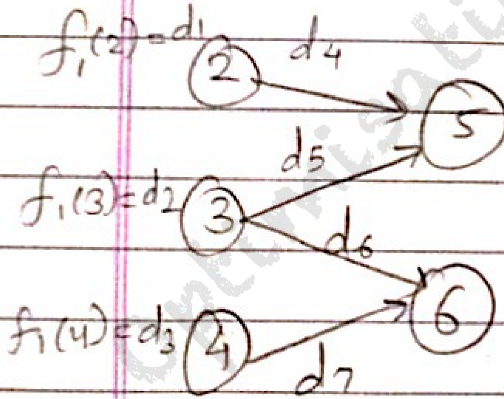
This move is called x_1. ∃ 3 values of x_1, depending upon if it goes to node 1, 2 or 3

Here, $f_1(2) = d_1$
$f_1(3) = d_2$
$f_1(4) = d_3$

Also, $f_0(1) := 0$, (assume if not given)
→ 0th jump, i.e. coming to 1st node

Now,

» Stage 2:



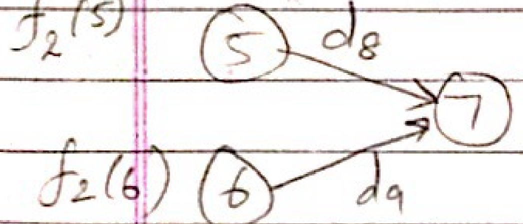
Possible values of $x_{②} = 5$ or 6. → 2nd stage

f^n: $f_2(x_2)$

Now, $f_2(5) = \min[d_1 + d_4, d_2 + d_5]$

$f_2(6) = \min[d_2 + d_6, d_3 + d_7]$

» Stage 3:



Here, $x_3 = 7$

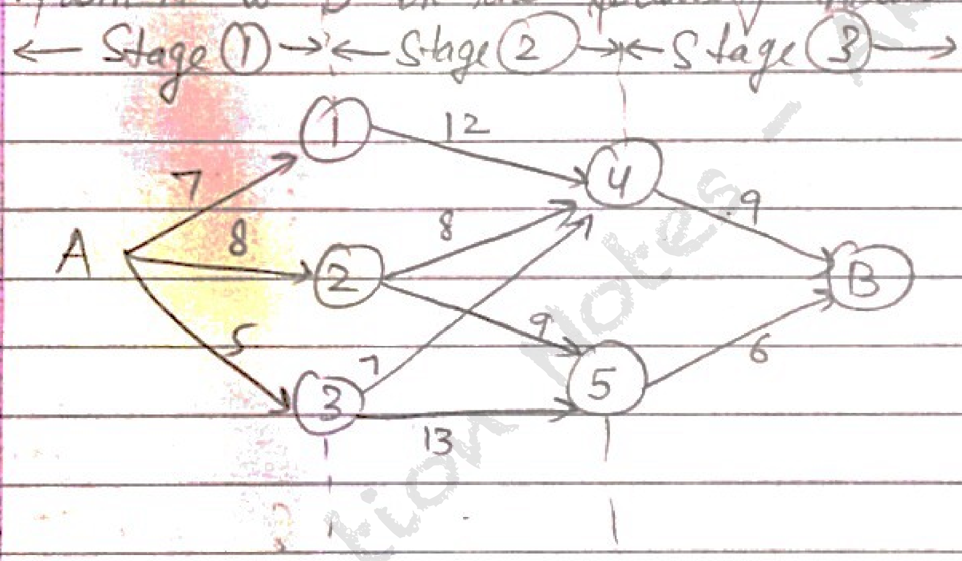
$f_3(7) = \min[f_2(5) + d_8, f_2(6) + d_9]$

Recurrence rel^n :-

$$f_i(x_i) = \min.\left[f_{i-1}(x_{i-1}) + d_{i-1,i} \right]$$

distance from i-1 to i

Q. Use DP approach to find the shortest route from A to B in the following network :



Let x_i be the state var. in state i, $(i = 1, 2, 3 \ldots)$

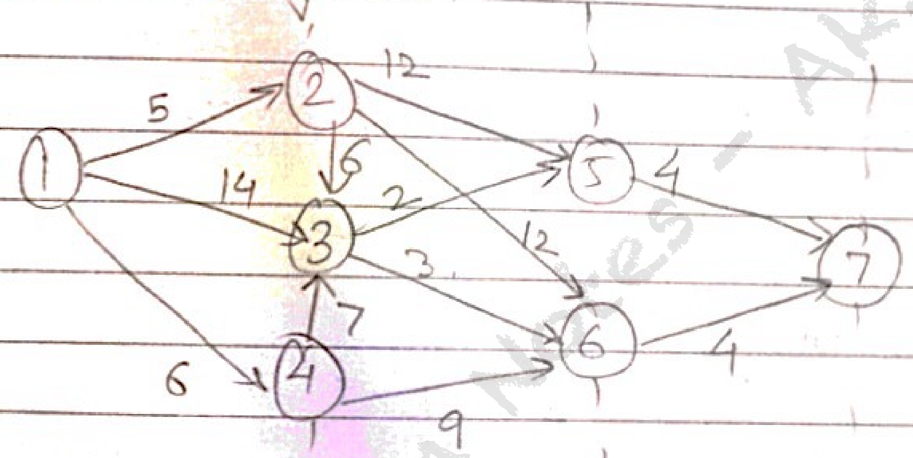
Stage (0)	Stage (1)	Stage (2)	Stage (3)
$x_0 = A$	$x_1 = 1, 2, 3$	$x_2 = 4, 5$	$x_3 = B$
$f_0(x_0) = f_0(A) = 0$	Finding $f_1(x_1)$	Finding $f_2(x_2)$	Finding $f_3(x_3)$
	$f_1(1) = 7$ (A→1)	$f_2(4) = \min(\underbrace{7}_{f_1(1)} + \underbrace{12}_{d_{1,4}}, 8+8, 5+7)$	$f_3(B) = \min(f_2(4)+9, f_2(5)+6)$
	$f_1(2) = 8$ (A→2)	$\Rightarrow f_2(4) = 12$ (3→4)	$= \min(12+9, 17+6)$
	$f_1(3) = 5$ (A→3)	$f_2(5) = \min(8+9, 5+13)$	$= 21$ (4→B)
		$f_2(5) = 17$ (2→5)	

∴ Min distance from $A \to B$ = 21 units

Now, taking backward approach, starting from B,

$A \to 3 \to 4 \to B$

Q. Find shortest route & shortest distance of following network using DP.



Stage 1 | Stage 2 | Stage 3

Stage ①	Stage ②	Stage ③
$x_1 = 2, 3, 4$	$x_2 = 5, 6$	$x_3 = 7$
Finding :- $\underline{\underline{f_1(x_1)}}$	$\underline{\underline{f_2(x_2)}}$	$\underline{\underline{f_3(x_3)}}$
$f_1(2) = 5$ $(1 \to 2)$	$f_2(5) = \min[5+12, 11+2]$ $= 13$ $(3 \to 5)$	$f_3(7) = \min.$ $[13+4, 14+4]$
$f_1(3) = \min[5+6, 14, 6+7]$ $= 11$ $(1 \to 2 \to 3)$	$f_2(6) = \min[5+12, 11+3, 6+9]$	$= 17$ $(5 \to 7)$
$f_1(4) = 6$	$= 14$ $(3 \to 6)$	

So, min. distance from $A \rightarrow B = \frac{17}{\text{units}}$ $(f_3(7))$

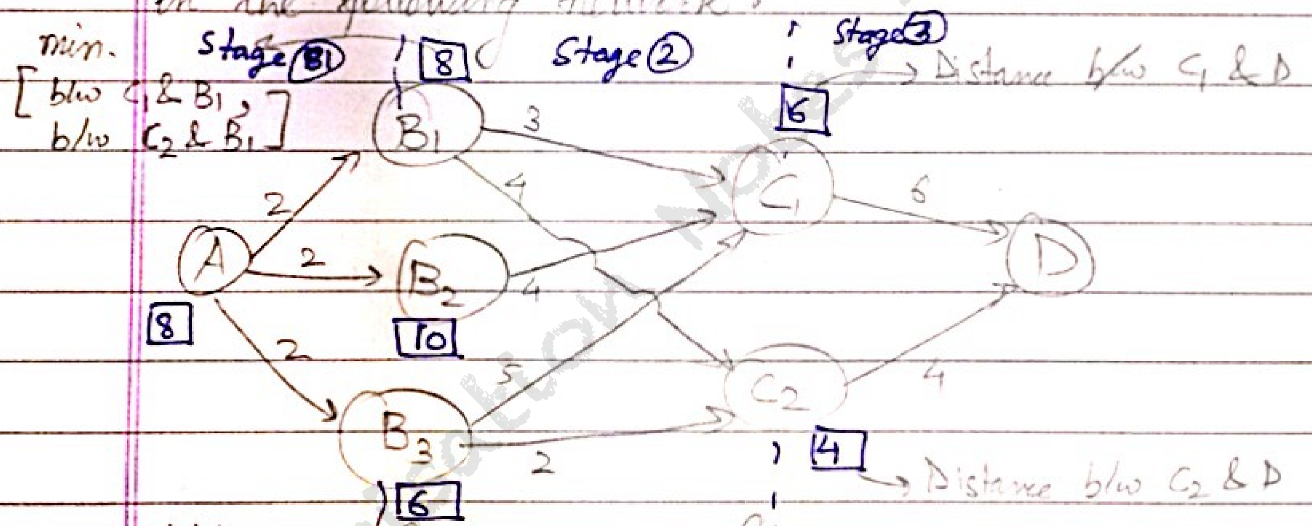
& path (Seeing path of $f_3(7)$, then $f_2(5)$, then $f_1(3)$)

Path $= 1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 7$

Path of $f_1(3)$ Path of $f_2(5)$ Path of $f_3(7)$

Ans

eg Use backward recursion to find shortest route/path in the following network:



Using Backward approach:

Starting from D.

Make box above every circle. That box contains the shortest path

For B_3, we can have connection b/w B_3 & C_1 & B_3 & C_2

$\min(B_3 \rightarrow C_1,\ B_3 \rightarrow C_2) = \min(6+5,\ 4+2)$
$= 6$

$\Rightarrow B_3 = 6$

So, $A \rightarrow B_3 \rightarrow C_2 \rightarrow D$ is the shortest path.

Stage ③ — Stage ② — Stage ①

$f_3(C_1) = 6$ ($C_1 \to D$ ✓, $D \to C_1$ ✗)

$f_3(C_2) = 4$ ($D \to C_2$ ✗, $C_2 \to D$ ✓)

$f_2(B_1) = \min(6+3, 4+4)$
$= 8$ ($B_1 \to C_2$)

$f_2(B_2) = 6+4 = 10$ ($B_2 \to C_1$)

$f_2(B_3) = \min(6+6, 2+4)$
$= 6$ ($B_3 \to C_2$)

$f_1(A) = \min(8+2, 10+2, 6+2)$
$= 8$ ($A \to B_3$)

Now, see from

Stage ① — Stage ② — Stage ③

$A \to B_3 \longrightarrow C_2 \longrightarrow D$

$f_1(A)$ — $f_2(B_3)$ — $f_3(C_2)$

min. distance $= 8 + 6 + 4 = 18$ units

☆ APPLICATION OF DP TO NON LINER PROG. PROBLEMS :-

Model 1

When objective f^n is multiplicative & constraint is additive :

eg: Max. $Z = U_1 U_2 U_3$ → non linear ($\because$ multiplicative)

s.t $U_1 + U_2 + U_3 = 10$

$U_1, U_2, U_3 \geq 0$

Here, no. of decision $\text{var}_s = 3$. So, we have to break it into 3 stages.

Let x_1, x_2, x_3 be the state vars. of these 3 stages & are defined by:

Start from last stage

State var :- x_3 = constraint consisting u_1, u_2, u_3.

So, $x_3 = u_1 + u_2 + u_3 = 10$. $\longrightarrow$ (1)

Now,

x_2 = Same constraint, but not taking u_3.

$\Rightarrow x_2 = u_1 + u_2$

$\Rightarrow x_2 = x_3 - u_3$ (don't use '10' now)

x_1 = All terms containing u_1 only

$\Rightarrow x_1 = u_1 = x_2 - u_2$

(Suppose constraint $\rightarrow 2u_1 + u_2 + 3u_3$

So, $x_3 = 2u_1 + u_2 + 3u_3$

$x_2 = 2u_1 + u_2$

$x_1 = 2u_1$)

Now, using recurrence relⁿ:

Here, it'll be defined as:

× : ∵ its multiplicative objective fⁿ

$$f_j(x_j) = \max_{u_j} \left(u_j \times f_{j-1}(x_{j-1})\right)$$

Now, $f_1(x_1) = \max_{u_1}(u_1)$

$$f_2(x_2) = \max_{u_2}\left(u_2\, f_1(x_1)\right)$$

$$f_3(x_3) = \max_{u_3}\left(u_3 \cdot f_2(x_2)\right)$$

Now, $f_1(x_1) = U_1 = x_2 - U_2$

$$f_2(x_2) = \underset{U_2}{Max}\left\{U_2(x_2 - U_2)\right\}$$

↳ Finding $max_{(U_2)}$

let $g = U_2(x_2 - U_2)$

$$\frac{dg}{dU_2} = x_2 - 2U_2 = 0.$$

$$\Rightarrow \quad U_2 = \frac{x_2}{2} \rightarrow \text{extrema}$$

(minima/maxima)

It can be verified that is max/min:

$$\frac{d^2 g}{dU_2^2} = -2 < 0$$

So, maximum.

$$\Rightarrow f_2(x_2) = \frac{x_2}{2}\left(x_2 - \frac{x_2}{2}\right)$$

$$= \frac{x_2^2}{4} = \frac{(x_3 - U_3)^2}{4}.$$

$$f_3(x_3) = \underset{U_3}{Max}\left\{U_3\left(\frac{x_3 - U_3}{4}\right)^2\right\}$$

↳ $h = U_3\left(\frac{x_3 - U_3}{4}\right)^2$

$$\frac{dh}{dU_3} = 0. \Rightarrow U_3 = x_3, \frac{x_3}{3}$$

$$\left.\frac{d^2 h}{dU_3^2}\right|_{x_3} > 0 \;;\; \left.\frac{d^2 h}{dU_3^2}\right|_{\frac{x_3}{3}} < 0$$

So, ✗ So, ✓

$$\Rightarrow f_3(x_3) = \frac{x_3}{3}\left(\frac{x_3 - x_3/3}{4}\right)^2 = \frac{x_3^3}{27.} = \frac{(10)^3}{27} \quad (\text{from ①})$$

Puffin
Date
Page

$\Rightarrow f_3(x_3) = \frac{1000}{27}$

$\Rightarrow Z_{max} = \frac{1000}{27}$

Now, finding values of decision vars:

$U_3 = \frac{x_3}{3} = \frac{10}{3}$

$U_2 = \frac{x_2}{2} = \frac{(x_3 - U_3)}{2} = \frac{\left(10 - \frac{10}{3}\right)}{2} = \frac{10}{3}$

$U_1 = x_2 - U_2 = x_2 - \frac{x_2}{2} = \frac{x_2}{2} = \frac{(x_3 - U_3)}{2}$

$= \frac{\left(10 - \frac{10}{3}\right)}{2}$

$\Rightarrow U_1 = \frac{10}{3}$

So, optimum sol^n: $U_1 = U_2 = U_3 = \frac{10}{3}$

& $Z_{max} = \frac{1000}{27}$

Model 2 Objective f^n is additive & constraint is also additive :-

eg Use DP approach to solve :-

Min. $Z = y_1^2 + y_2^2 + y_3^2$

s.t $y_1 + y_2 + y_3 \geq 30$

$y_1, y_2, y_3 \geq 0$

No. of decision vars = 3 $\Rightarrow \exists$ 3 state vars.

let state vars be x_1, x_2, x_3.

Defining vars :-

$$x_3 = y_1 + y_2 + y_3 \geq 30$$
$$x_2 = y_1 + y_2 = x_3 - y_3$$
$$x_1 = y_1 = x_2 - y_2$$

Recurrence relⁿ :-

$$f_1(x_1) = \min_{y_1} \{y_1^2\} = y_1^2$$ → ∵ Min. value of $y_1^2 = y_1^2$

Taking only y_1^2 term in Z_{min}.

$$f_2(x_2) = \min_{y_2} \{ y_2^2 + f_1(x_1) \}$$
$$f_3(x_3) = \min_{y_3} \{ y_3^2 + f_2(x_2) \}$$

Now

$$f_1(x_1) = y_1^2 = (x_2 - y_2)^2$$
$$f_2(x_2) = \min_{y_2} \{ y_2^2 + (x_2 - y_2)^2 \}$$

$$g = y_2^2 + (x_2 - y_2)^2$$
$$\frac{dg}{dy_2} = 2y_2 - 2(x_2 - y_2)$$
$$= 4y_2 - 2x_2$$
$$= 0.$$
$$\Rightarrow y_2 = x_2/2$$
$$\frac{d^2 g}{dy_2^2} > 0 \text{ (no need to check)}$$

$$\Rightarrow f_2(x_2) = \left(\frac{x_2^2}{4} + \left(x_2 - \frac{x_2}{2}\right)^2 \right) = \frac{x_2^2}{2}$$

→ giving optimum value

$$f_3(x_3) = \underset{y_3}{\text{Min}}\left(y_3^2 + f_2(x_2)\right)$$

$$= \underset{y_3}{\text{Min}}\left(y_3^2 + \frac{x_2^2}{2}\right)$$

$$= \underset{y_3}{\text{Min}}\left(y_3^2 + \frac{1}{2}(x_3 - y_3)^2\right)$$

$$\hookrightarrow h = y_3^2 + \frac{1}{2}(x_3 - y_3)^2$$

$$\frac{dh}{dy_3} = 0 \Rightarrow y_3 = \frac{x_3}{3}.$$

$$\left(\frac{d^2h}{dy_3^2} > 0, \text{ got}\right)$$

$$\Rightarrow f_3(x_3) = \left(\left(\frac{x_3}{3}\right)^2 + \frac{1}{2}\left(x_3 - \frac{x_3}{3}\right)^2\right)$$

$$\Rightarrow f_3(x_3) = \frac{x_3^2}{3}$$

↳ Now

$x_3 = y_1 + y_2 + y_3 \geq 30$.

So, $x_3 \geq 30$

So, min. value $= 30$

↳ Minimiz^n problem

$$\Rightarrow Z_{min} = \frac{30^2}{3} = 300.$$

Var: :- $y_3 = \frac{x_3}{3} = \frac{30}{3} = 10$

$y_2 = \frac{1}{2}(30 - 10) = 10$

$y_1 = x_2 - y_2 = (x_3 - y_3) - y_2 = 30 - 10 - 10$
$= 10$,

Optimum sol^n : $y_1 = y_2 = y_3 = 10$; $Z_{min} = 300$

Model 3 : Objective f^n is additive & constraint is multiplicative :-

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Q. Use DP to solve :-

$$\text{Min. } z = y_1^2 + y_2^2 + y_3^2$$
$$\text{s.t. } y_1 y_2 y_3 = 27 ;$$
$$y_1, y_2, y_3 \geq 0$$

Let no. of decision var$_s$ = 3 $\Rightarrow \exists$ 3 state var$_s$.
Let state var$_s$ be x_1, x_2, x_3
So,

$$x_3 = y_1 y_2 y_3 \quad (= 27)$$
$$x_2 = y_1 y_2 = x_3 / y_3$$
$$x_1 = y_1 = x_2 / y_2$$

Recurrence rel^n :-

$$f_1(x_1) = \underset{y_1}{\text{Min}} \{ y_1^2 \} = y_1^2$$

$$f_2(x_2) = \underset{y_2}{\text{Min}} \{ y_2^2 + f_1(x_1) \}$$

$$f_3(x_3) = \underset{y_3}{\text{Min}} \{ y_3^2 + f_2(x_2) \}$$

Now,

$$f_1(x_1) = y_1^2 = \left(\frac{x_2}{y_2}\right)^2$$

$$f_2(x_2) = \underset{y_2}{\text{Min.}} \left\{ y_2^2 + \left(\frac{x_2}{y_2}\right)^2 \right\}$$

$$g = y_2^2 + \left(\frac{x_2}{y_2}\right)^2$$

$$\frac{dg}{dy_2} = 2y_2 - \frac{2x_2^2}{y_2^3} = 0 \Rightarrow y_2^4 = x_2^2$$
$$\Rightarrow y_2 = \sqrt{x_2}$$

$$\frac{d^2 g}{dy_2^2} > 0 \text{ (verificⁿ for min value)}$$

$$\Rightarrow y_2 = \sqrt{x_2}$$

$$\Rightarrow f_2(x_2) = x_2 + \frac{x_2^2}{x_2} = 2x_2 = 2\left(\frac{x_3}{y_3}\right)$$

Now,

$$f_3(x_3) = \min_{y_3}\left\{y_3^2 + f_2(x_2)\right\}$$

$$= \min_{y_3}\left\{\underbrace{y_3^2 + \frac{2x_3}{y_3}}_{h}\right\}$$

$$\frac{dh}{dy_3} = 0 \Rightarrow 2y_3 - \frac{2x_3}{y_3^2}$$

$$\Rightarrow 2y_3^3 - 2x_3 = 0$$

$$\Rightarrow y_3 = (x_3)^{1/3}$$

So,

$$f_3(x_3) = (x_3)^{2/3} + \frac{2x_3}{(x_3)^{1/3}}$$

$$= 3x_3^{2/3}$$

$$= 3(27)^{2/3}$$

$$\Rightarrow f_3(x_3) = 27$$

So, min. value of objective $f^n = 27 = Z_{min}$

$$y_3 = (x_3)^{1/3} = (27)^{1/3} = 3$$

$$y_2 = \sqrt{x_2} = \left(\frac{x_3}{y_3}\right)^{1/2} = \left(\frac{27}{3}\right)^{1/2} = 3.$$

$$y_1 = \frac{x_2}{y_2} = \frac{x_3}{y_3 \cdot y_2} = \frac{27}{3 \times 3} = 3 \Rightarrow \underline{y_1 = y_2 = y_3 = 3}$$

Chapter - 6

PROJECT MANAGEMENT WITH PERT & CPM

- PERT :

  Project / Program Evaluation & Review Technique.
- CPM :

  Critical Path Method

» We schedule & manage activities - which comes when

- deterministic : CPM

  probabilistic : PERT

» Drawing network diagram

Notation : Activity : $① \xrightarrow[A]{5} ②$

(shows dirⁿ of activity)
(Name of activity : A)
(Time reqd to complete : 5)
(Starting & ending pts called as NODES)

★ • A network must fulfil the following cond^{ns} :-

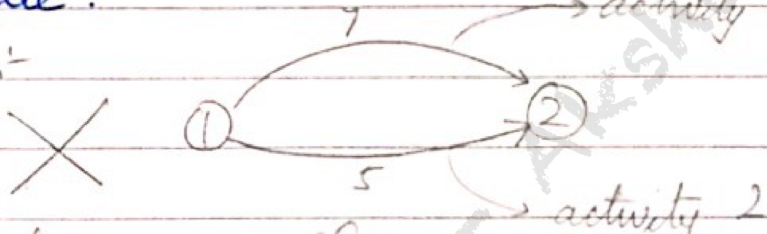
(1) Each activity will be represented by ONE & only one arrow.

||ly, each arrow tells only one activity

(2) Each activity must be identified by 2 distinct nodes.

(3) No 2 activities have same initial node & ending node.

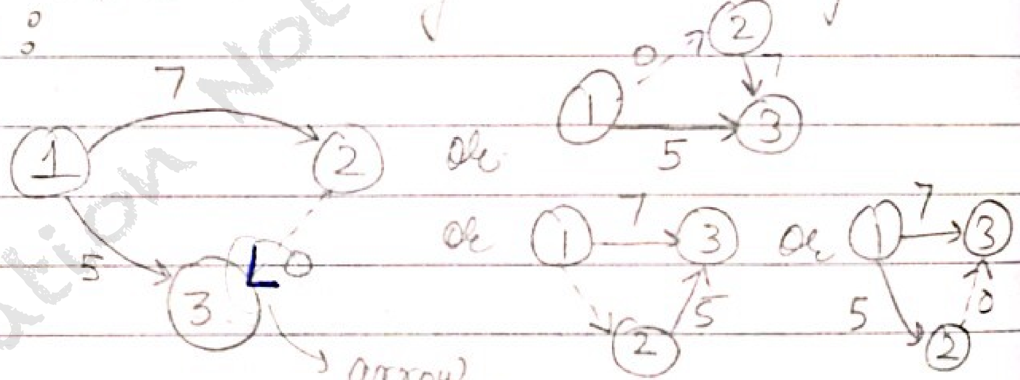
eg :-



Above isn't allowed.

So, use a dummy node & activity :

Possibilities :



arrow always towards node with largest no. (ascending order)

2 ≡ 3). So, $1 \to 3 \equiv 1 \to 2$

dummy

$1 \to 2 \to 3 \equiv 1 \to 2$

So, we are going from $1 \to 2$ in both cases.

(4) Arrow heads shouldn't form a closed loop.

(i.e, for a project, I will reach at some end).

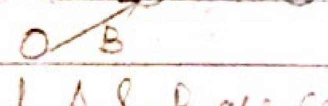
(5) Starting & ending nodes should be unique
⇓
converging at one pt.

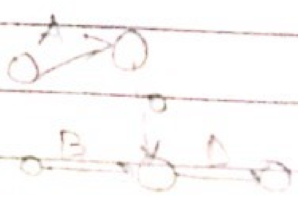
✯ To draw correct network, following 3 info. are very imp :-
(1) What activities must immediately preceed the current activity.
(2) What activities must follow the current activity.
(3) What activities occur concurrently.

eg. Suppose we write a book
Activity 1 Write manuscript
2 Edit it
3 Print it
↓ order matters

✯ How to draw Network Diagram of a Project ?

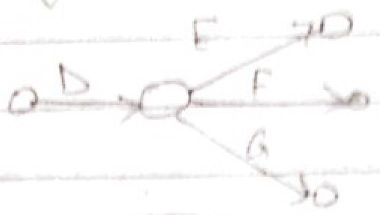
→ We need these info :-
1. Predecessor / successor of each activity.
2. Starting and ending activities.
✯ If A & B are immediate predecessors of D, they will merge at a pt.
i.e. :





If Starting N. of A & B are same
Use dummy activity

If E, F, G activities have immediate predecessor D, i.e., only one predecessor ∀ activities:-

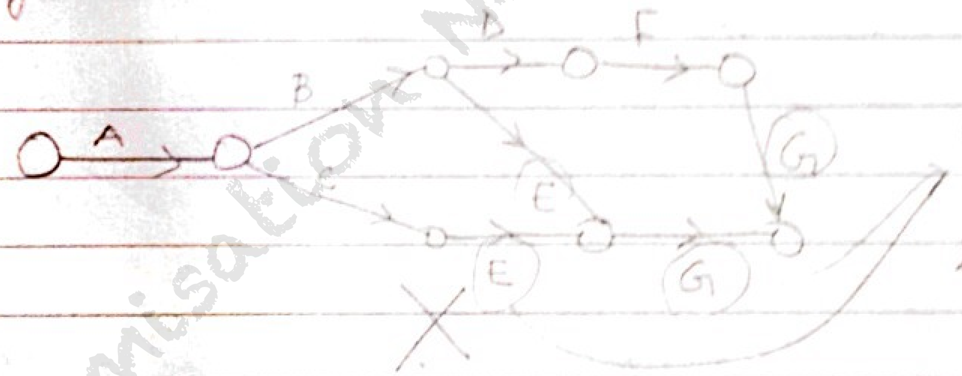


Q. Draw network diagram of a project using following info:-

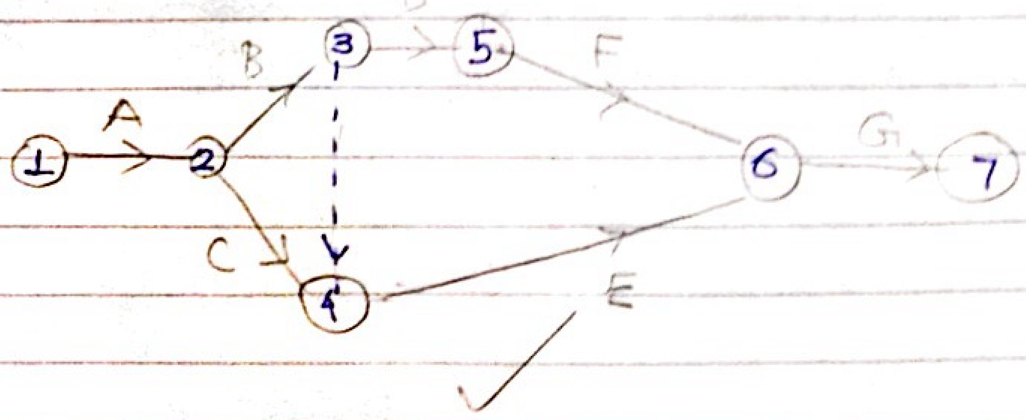
Activity :	A	B	C	D	E	F	G
Immediate predecessor:	–	A	A	B	B, C	D	E, F

⇒ ∄ no predecessor for activity A. So, the starting activity

Diagram:-



Note:- B & C have same starting pt. So, they cannot end/converge in E. So, use Dummy:-



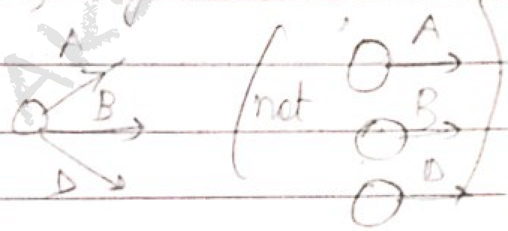
Q. Draw network diagram of project using following info:-

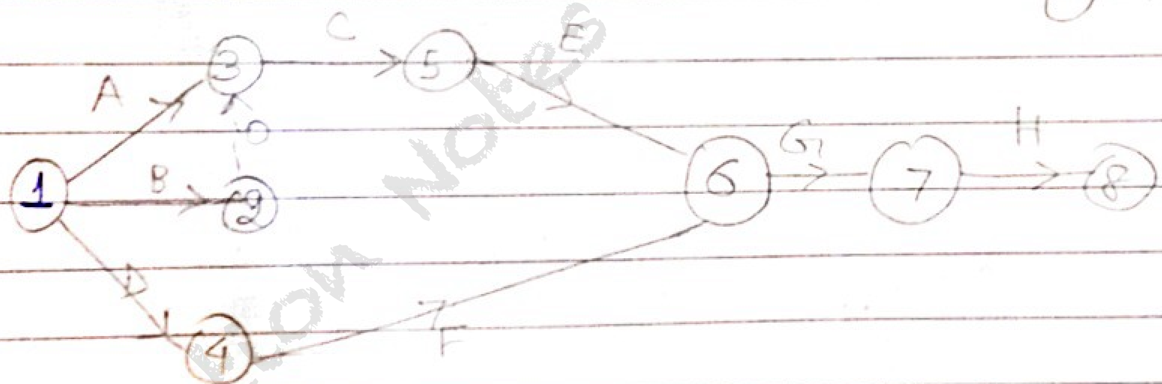
Activity : A B C D E F G H

Immediate predecessor : – – A,B – C D E,F G

They are starting activities

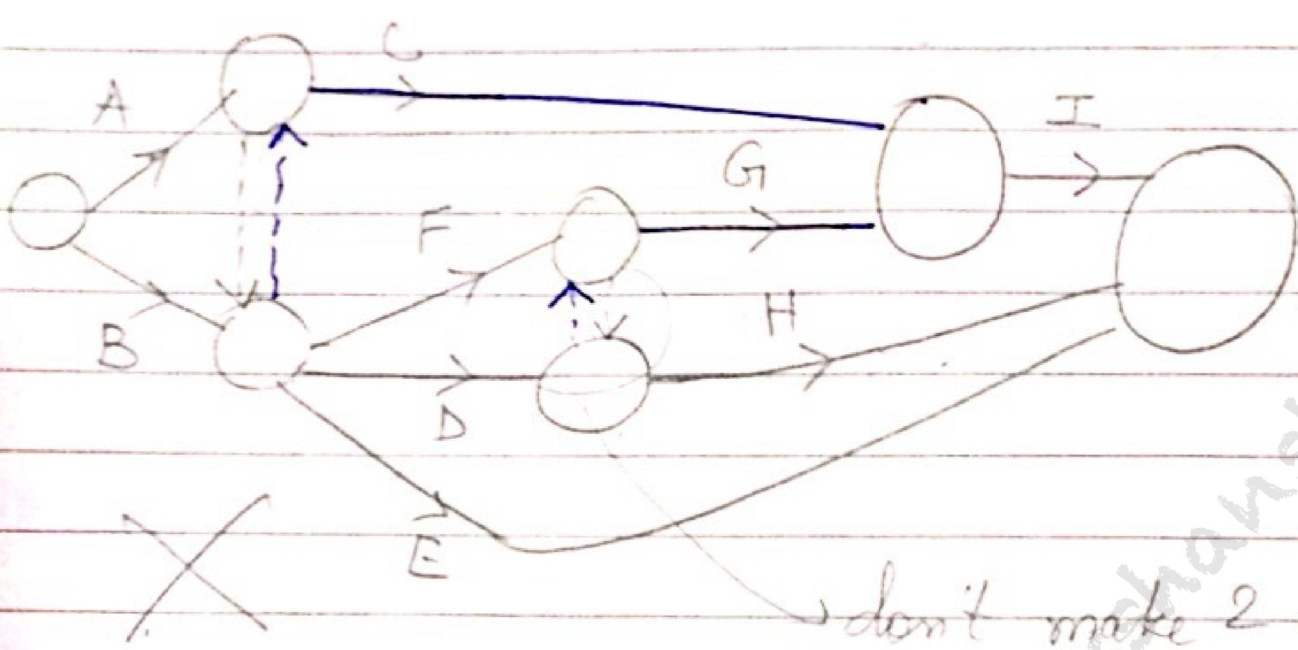
So, they can be :



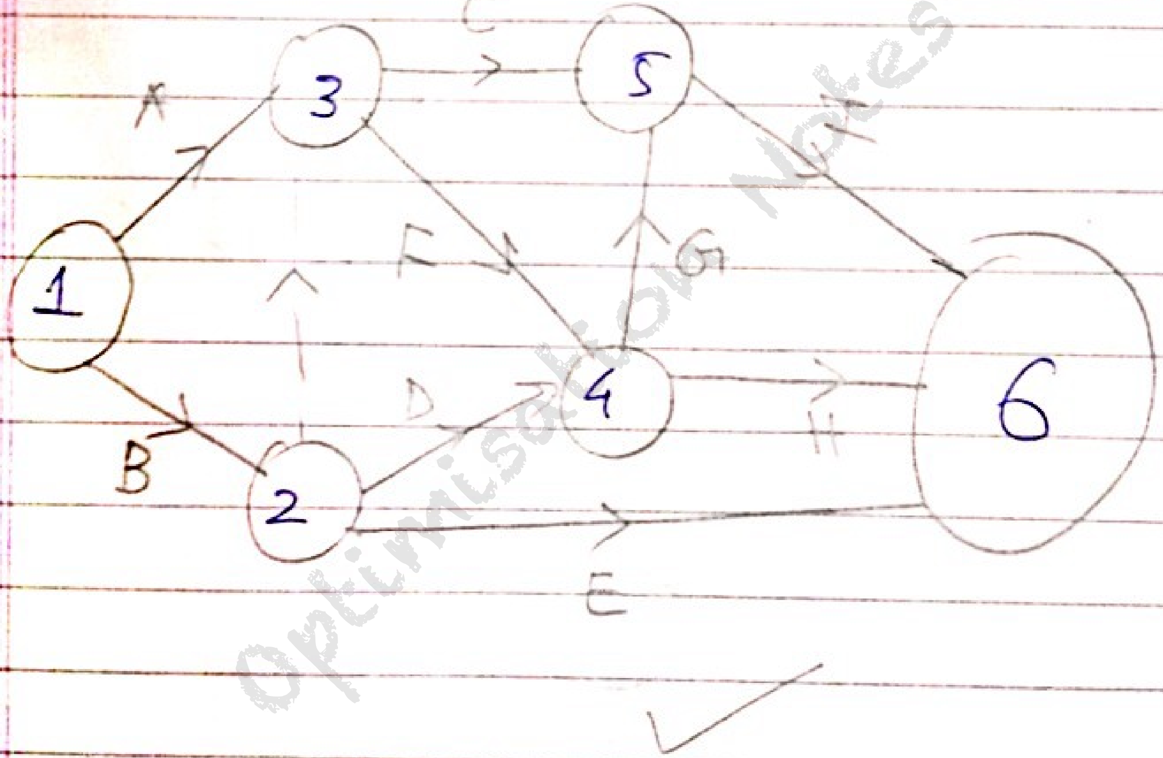


Q. Construct a network diagram where activities satisfy following requirement:-

1. A & B are starting activities of the project ;
2. A & B precede C. ;
3. B precedes D & E ;
4. A & B precede F ;
5. F & D precede G & H ;
6. C & G precede I ;
7. E, H, I are terminal activities



don't make 2 dummy
∵ Arrowhead can only be in one dirn



METHOD - 1

CRITICAL PATH METHOD (CPM)

☆ Conventions :

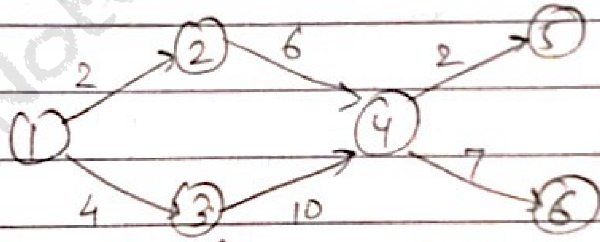
- Node : Also called EVENT.

$1 \xrightarrow{A} 2$: here, A can also be written as (1, 2).

i.e $(i, j) = i \longrightarrow j$

☆ Critical activity :

consider a project :



activity 4 → 5 or 4 → 6 cannot be started until (4+10) 14 hours are passed. So, although (2+6) = 8 hours' activity is complete, there is delay in going from 4 → 5 or 6 as we have to wait till 14 hours/days/units are passed.

So, here, activities: 1 → 3 → 4 are CRITICAL ACTIVITIES

1 → 2 → 4 are NON CRITICAL ACTIVITIES

☆ Critical path : longest possible time to complete the project.

- To find critical path in a network, we use 2 steps :

    S1) Forward pass

    S2) Backward pass.

S1) Forward Pass

In forward pass, we calculate Earliest start time (ES) $\forall$ nodes & Earliest start time for node i (ES_i)

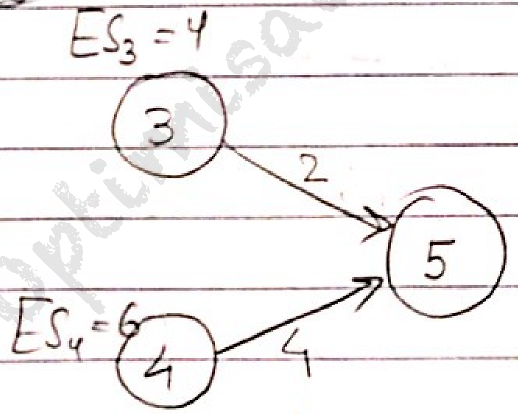
So, we find $ES_1 - ES_n$.

☆ Note : $ES_1 = 0$, always.

formula ☆ $$ES_i = \max_k [ES_k + t_{ki}]$$

ES_k → Earliest start time of node k which appears before i

eg



$$ES_5 = \text{Max} [ES_3 + t_{35}, ES_4 + t_{45}]$$

2 terms ∵ 2 activities are ending at 5.

$$= \text{Max} [4+2, 6+4]$$

$$\Rightarrow ES_5 = 10$$

IIly, keep finding $\forall$ nodes.

10 days is the min. time req^d after which any activity from node 5 can be started.

S2) Backward Pass:

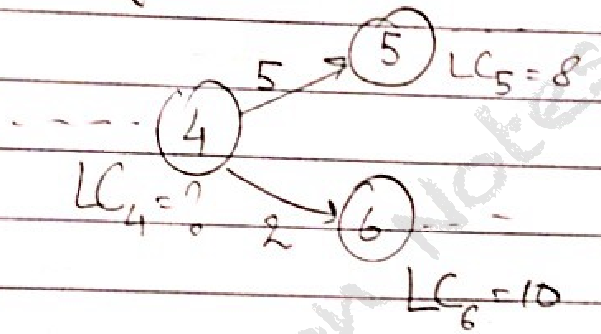
In backward pass we calculate latest completion time $\forall$ nodes (LC)

☆ LC of last node = ES of last node

i.e $LC_n = ES_n$

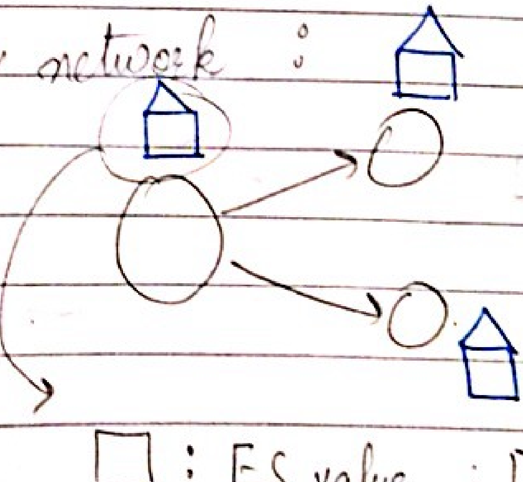
formula ☆ $$LC_k = \min_j \left[LC_j - t_{kj} \right]$$

eg) Taking a small part of a network:



$$LC_4 = \min(LC_5 - t_{54}, LC_6 - t_{64})$$
$$= \min(8-5, 10-2)$$
$$\Rightarrow LC_4 = 3$$

☆ Convention: In a network:



$\square$: ES value ; $\square_i = ES_i$

$\triangle$: LC value ; $\triangle_j = LC_j$

★ Critical path is a continuous path.

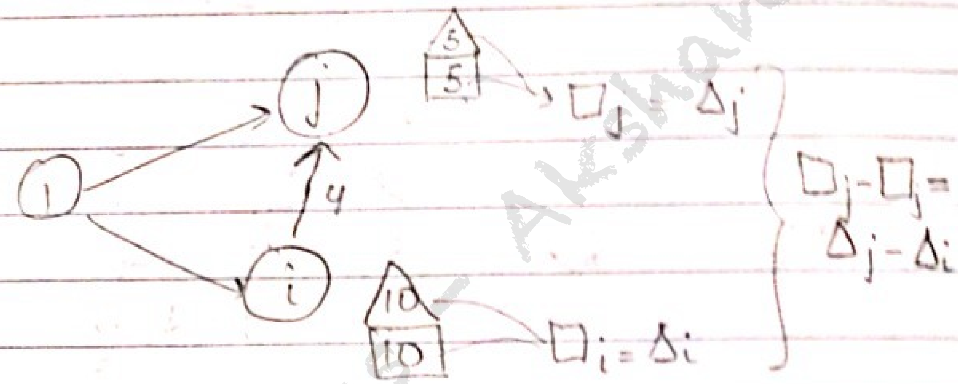
★ (i, j) will be a critical activity if

(a) $\square_i = \Delta_i$

(b) $\square_j = \Delta_j$

(c) $\square_j - \square_i = \Delta_j - \Delta_i = t_{ij}$

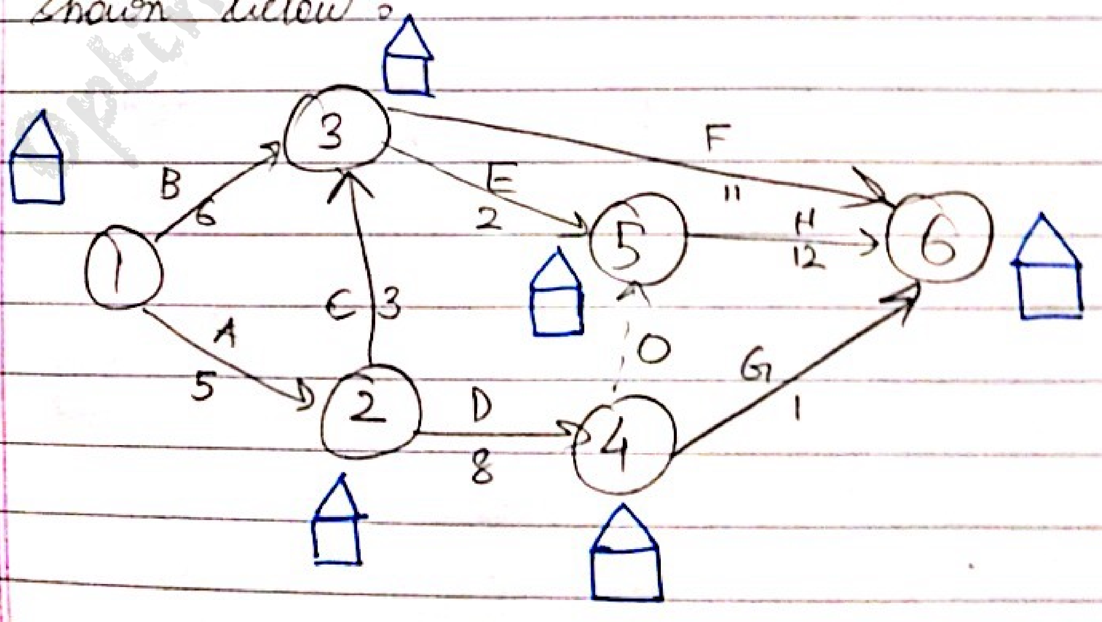
(eg)



BUT, $\underbrace{\square_j - \square_i}_{5} \neq \underbrace{t_{ij}}_{4}$

So, (i, j) is not a critical activity.

Q. Determine critical path of the project network shown below:



Idea, calculate △ & □ for each node.

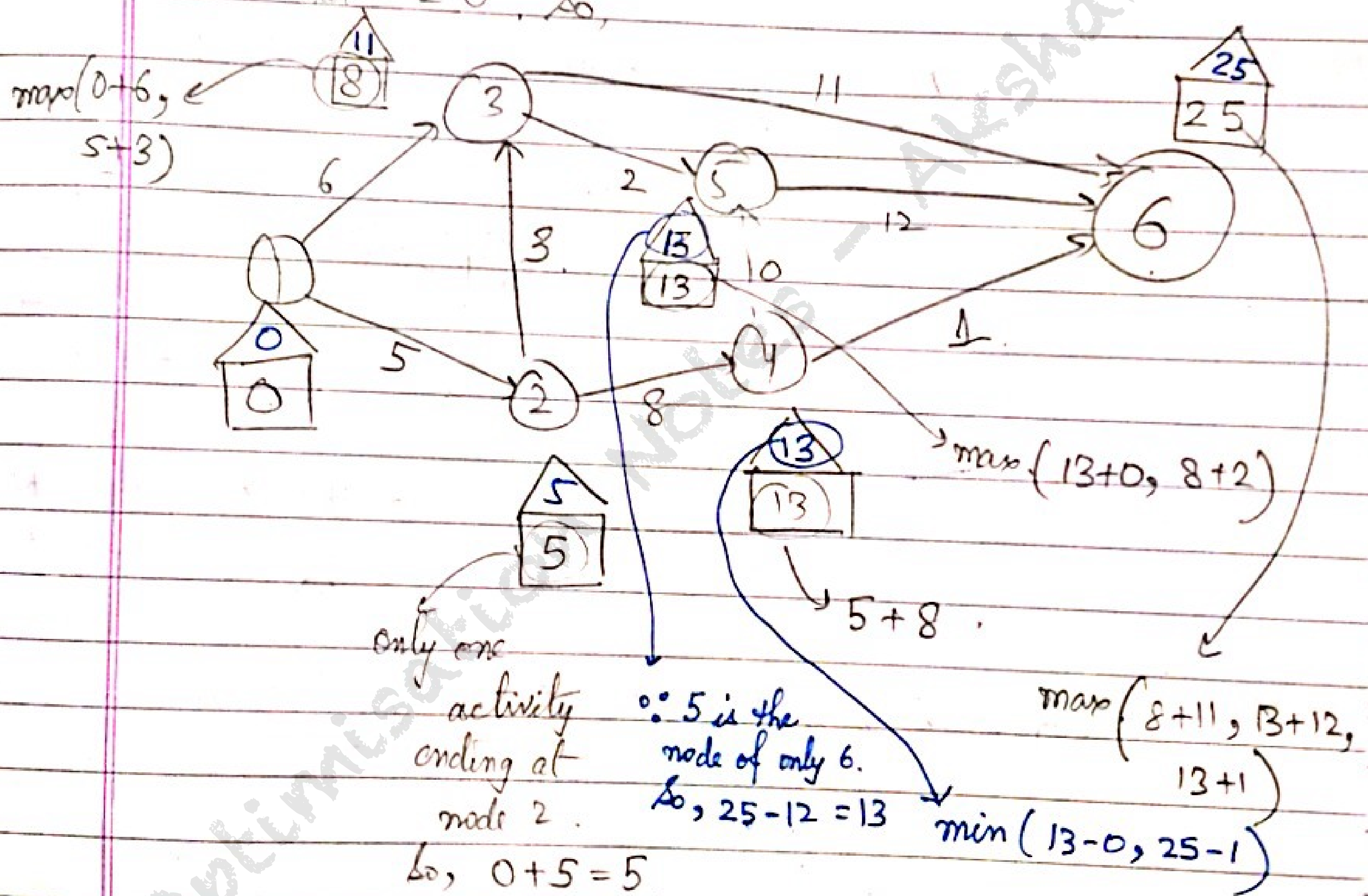
△ : LC

□ : ES.

So, make △ over □ at each node, as shown.

Forward Pass

Now, start from ES, i.e □ for node 1 that = 0. So,



Backward Pass

Next, start from LC_6. Make it same as ES_6.

So, $LC_6 = 25$

$LC_3 = \min(25-11, 13-2) = 11$

$LC_2 = \min(11-3, 13-8) = 5$

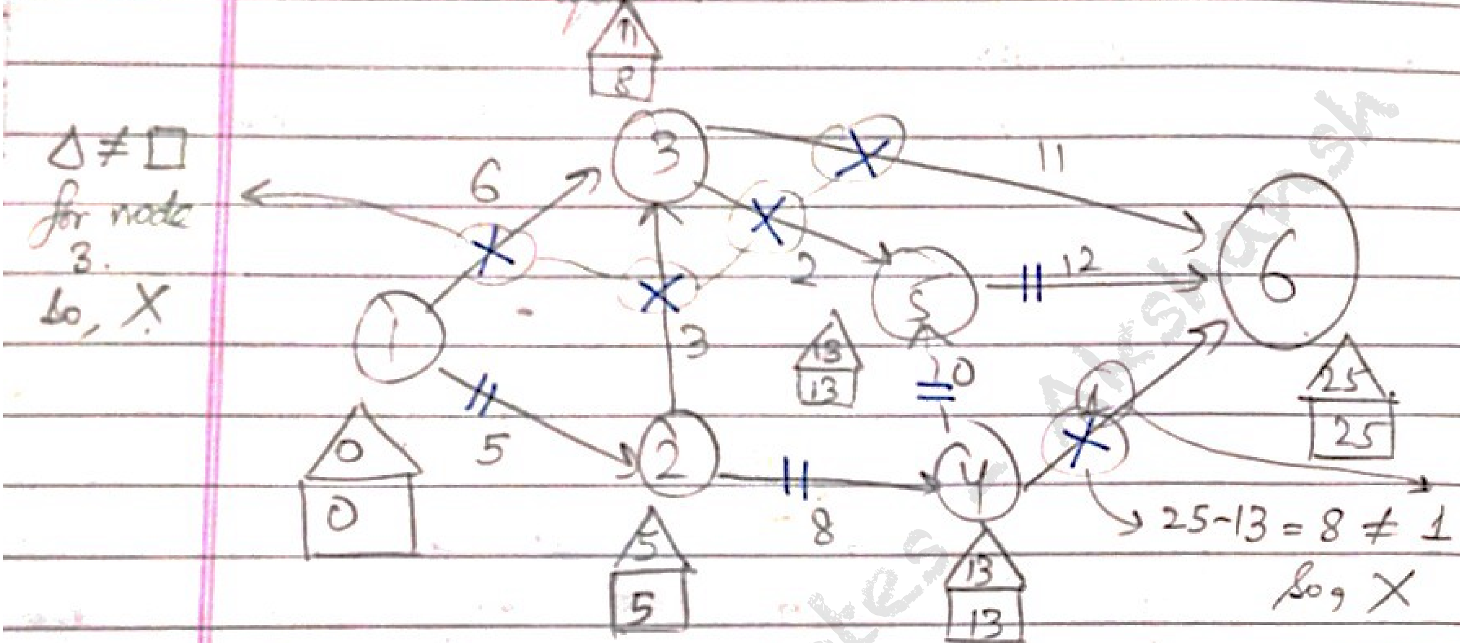
$LC_1 = \min(11-6, 5-5) = 0$

Now finding critical path :

Idea :- $\triangle = \square$ (in values).

See that this happens.

Now, start joining the activities where its equal



every time is marked by $\xrightarrow{\parallel}$ → move along that

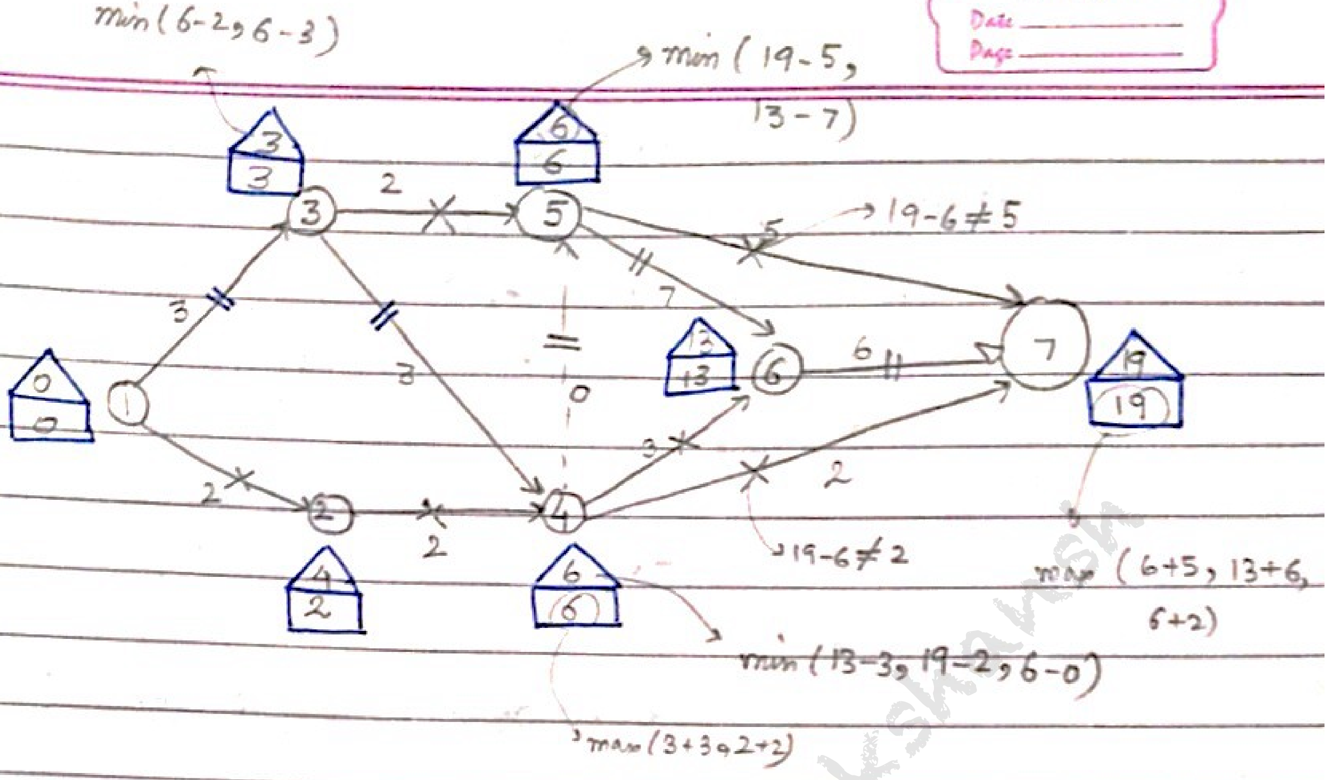
or $\xrightarrow{\times}$ → don't move.

So, path becomes (Critical path)

(1) → (2) → (4) → (5) → (6).

Max. time taken to complete activities = 25 units

Q. Determine the critical path in the following network :-
(also find duration of project)



Duration of project = 19

So, critical path = 1 → 3 → 4 → 5 → 6 → 7

Corresponding activities are called critical activities (i.e., I cannot delay any such activity without delaying the durⁿ of project)

§ FLOATS (basically, the amount of delay I can make in any activity)

Total Slack time for activities are called floats.

Types

Total Float (TF)

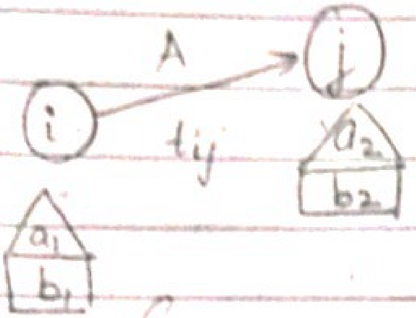
Max. time available to complete the activity − (minus) the time reqd to complete the activity. Formula for activity (i, j)

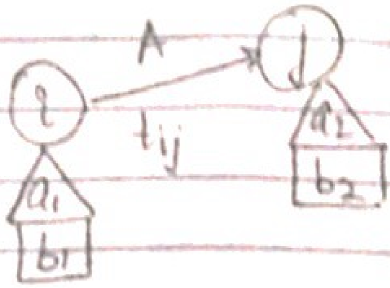
$$TF_{ij} = \overbrace{\Delta_j}^{LC_j} - \overbrace{\square_i}^{ES_i} - t_{ij}$$

Free float (FF)

$$FF_{ij} = ES_j - ES_i - t_{ij}$$

$$= \square_j - \square_i - t_{ij}$$





Here,
Slack time = $(a_2 - b_1 - t_{ij})$

starting at this time

&

working for this much time to complete

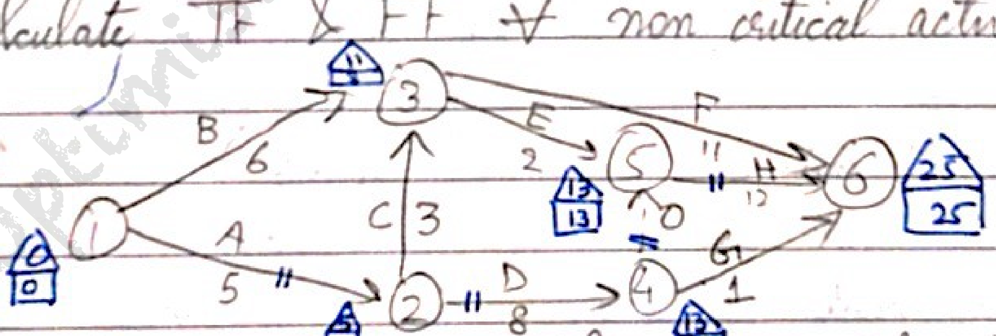
waiting for this much time after completing

Idea: we want to start activity early (ASAP)

Slack time = $b_2 - b_1 - t_{ij}$

* Note:- Floats are calculated ONLY for NON Critical Activities.
(for Critical activities, its = 0).

Q. Calculate TF & FF $\forall$ non critical activities in (done before):



As solved previously, the non critical activities were found as:
B, C, E, F, G.

Non Critical	Duration	TF	FF
B	6	11-0-6 = 5	8-0-6 = 2
C	3	11-5-3 = 3	8-5-3 = 0
E	2	13-8-2 = 3	13-8-2 = 3
F	11	25-8-11 = 6	25-8-11 = 6
G	1	25-13-1 = 11	25-13-1 = 11

Note: Meaning: TF = 5 for B.
i.e, I have slack time of 5 days. So, start late by 5 days.
FF = 2.
If I start early, I'll complete early & will have to wait for 2 days.

METHOD-2

8 PERT (Program Evaluation & Review Technique)

→ or Project.

- ✓ considers probabilistic time durⁿ.
- ✓ To estimate durⁿ of activity use

Most likely time estimate (m)
- ✓ If something goes wrong & something as per the plan. Then, time taken to complete activity.

Optimistic time estimate (a)
- ✓ how much time it'll take if nothing goes wrong. (least possible time)

Pessimistic (b) time estimate
- ✓ how much time reqd to complete activity if every thing goes wrong.

> ✓ Duration of activity : denoted by t_e

Std. normal variate (Z) : $\frac{X-\mu}{\sigma}$; Std. deviation = $\sqrt{\text{variance}}$

STEPS :

S1) Draw network of project.

S2) Find/compute the expected duration (t_e) of all activities. Also find variance (σ^2)

S3) Find critical path of the network ($\because t_e$ is now known).

$$t_e = \frac{a + 4m + b}{6}$$ (based on β - distribution)

$$\sigma^2 = \left(\frac{b-a}{6}\right)^2$$

Ex) Following are the activities of a project with optimistic (a), most likely (m) and pessimistic (b) time estimates:

(weeks)

Activity	a	m	b
1 → 2	1	1	7
1 → 3	1	4	7
1 → 4	2	2	8
2 → 5	1	1	1
3 → 5	2	5	14
4 → 6	2	5	8
5 → 6	3	6	15

(a) Draw network of project (may/may not be given in exam - network)

(b) Find expected dur^n & variances ∀ activities

(c) What is the expected project length?

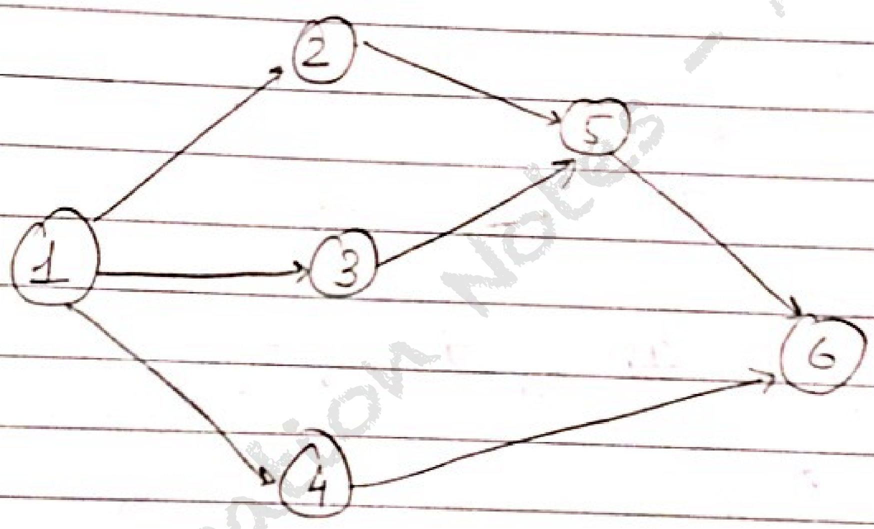
(d) Calculate variance & std. deviation of project length.

(e) What is probability that project will be completed atleast 4 weeks earlier than expected

(f) What is probability that project will be completed no more than 4 weeks later than expected time

(g) If project's due time = 19 weeks, what is the probability of meeting the deadline

Solⁿ (a) Network :



Formulas :

(b) $t_e = \frac{a + 4m + b}{6}$, $\sigma^2 = \left(\frac{b-a}{6}\right)^2 = \frac{(b-a)^2}{36}$

(c) Expected Project length, T_E = sum of t_e $\forall$ critical activities

(d) σ_E^2 = Sum of σ^2 $\forall$ critical activities

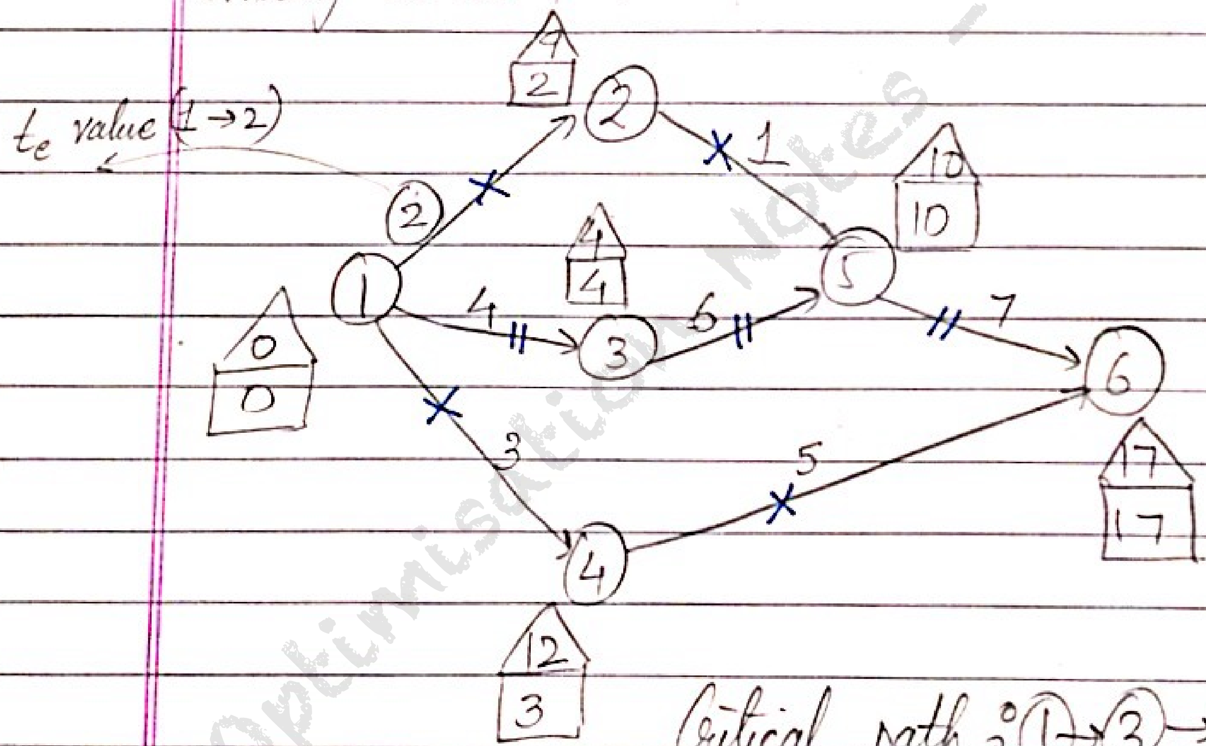
Standard deviation = $\sqrt{\sigma_E^2} = \sigma_E$

Finding values of paths using formulas

(b)

Activity	a	m	b	t_e	σ^2
$\boxed{1 \to 2}$	1	1	7	2	1
$\boxed{1 \to 3}$	1	4	7	$\boxed{4}$	$\boxed{1}$
$1 \to 4$	2	2	8	3	1
$2 \to 5$	1	1	1	1	0
$\boxed{3 \to 5}$	2	5	14	$\boxed{6}$	$\boxed{4}$
$4 \to 6$	2	5	8	5	1
$\boxed{5 \to 6}$	3	6	15	$\boxed{7}$	$\boxed{4}$

Finding critical path



Critical path : $1 \to 3 \to 5 \to 6$

Path length = 17

Now, mark the critical paths in table.

(c) Expected project length, $T_E = 4 + 6 + 7 = 17$ weeks

(d) Variance of project length $= \sigma_E^2 = 1 + 4 + 4 = 9$.

Std. deviation, $\sigma_E = 3$.

Let T_s be the time req^d to complete project. Then T_s has normal distrib^n with mean T_E & variance σ_E^2

Then,

$$\text{Std. normal variate } (Z) = \frac{T_S - T_E}{\sigma_E}$$

(e) $T_s = 17 - 4$ weeks $= 13$

Now,

finding $P(T_s \leq 13)$

$$= P\left(Z \leq \frac{13-17}{3}\right)$$

$$= P(Z \leq -1.33)$$

(after seeing from z-distrib^n table ∵ value given in exam)

⇒ Prob. $= 0.0918$

i.e around 9% chance is there that project completes in 13 weeks.

(f) Here, $T_s = 17 + 4$ weeks $= 21$

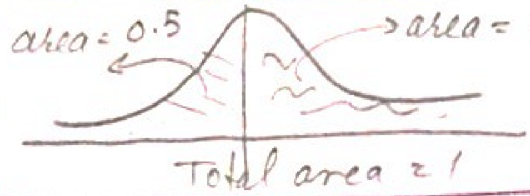
finding $P(T_s \leq 21)$

$$= P\left(Z \leq \frac{21-17}{3}\right)$$

$$= P(Z \leq 1.33)$$

$= 0.9082$ → from table. (90% chance so, high probability)

Std. normal
★ Variate
curve



★ $Z \leq 0$
$= 0.5$

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(g) Now, $T_s = 19$

$P(T_s \leq 19)$

$\Rightarrow P\left(Z \leq \frac{19-17}{3}\right)$

$\Rightarrow P(Z \leq 0.667)$

$= 0.7486$ (from table)

So, $\simeq 75\%$ chance is there that we meet deadline

extra (h) finding $P(17 \leq T_s \leq 19)$

$= P\left(Z \leq \frac{19-17}{3}\right) - P\left(Z \leq \frac{17-17}{3}\right)$

$- P(Z \leq 0)$

Idea : Z curve has total area = 1.
So, half of the area = 0.5.
Illy, if $Z \leq 2$.
So, this has value < 0.5
Illy, $Z \geq 1$
This has value > 0.5

(i) Finding $P(T_s \geq 15)$

$= 1 - P(T_s < 15)$

$= 1 - P\left(Z < \frac{T_s - T_E}{\sigma_E}\right)$ ✓

Ex-2 following are the activities in a project with a, b & m values: (hours)

Activity	Predecessors	a	m	b	t_e	σ^2
A	—	4	6	8	6	4/9
B	—	1	4.5	5	4	4/9
C	A	3	3	3	3	0
D	A	4	5	6	5	1/9
E	A	0.5	1	1.5	1	1/36
F	B, C	3	4	5	4	1/9
G	B, C	1	1.5	5	2	4/9
H	E, F	5	6	7	6	1/9
I	E, F	2	5	8	5	1
J	D, H	2.5	2.75	4.5	3	1/9
K	G, I	3	5	7	5	4/9

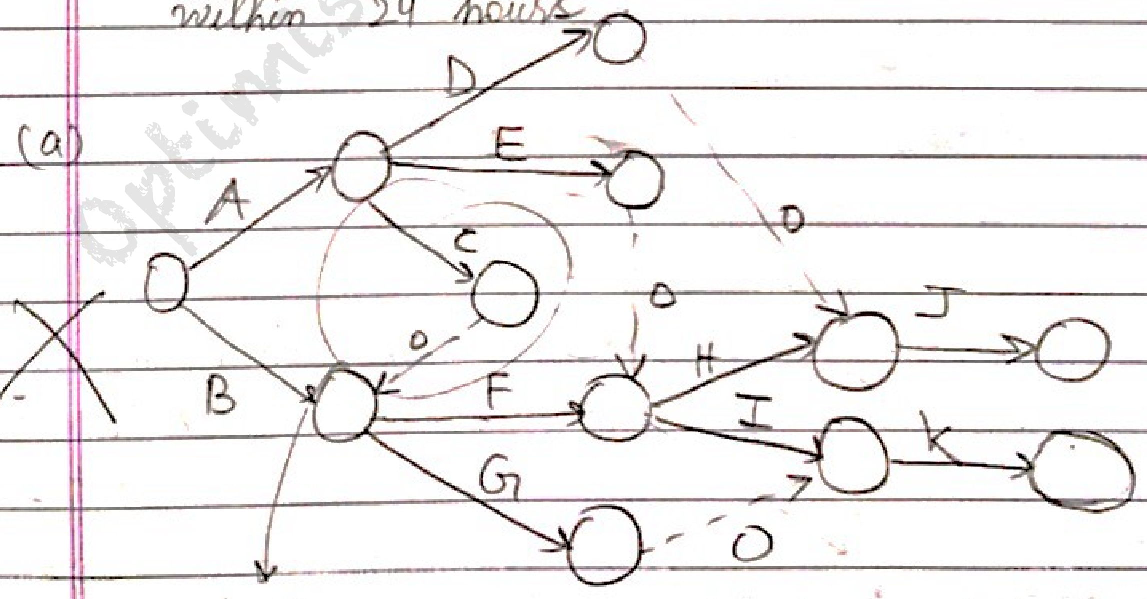
J & K are ending (last) activities → This line, even if not mentioned should be clear from Predecessor column (J, K not there anywhere)

(a) Draw network

(b) Find critical path

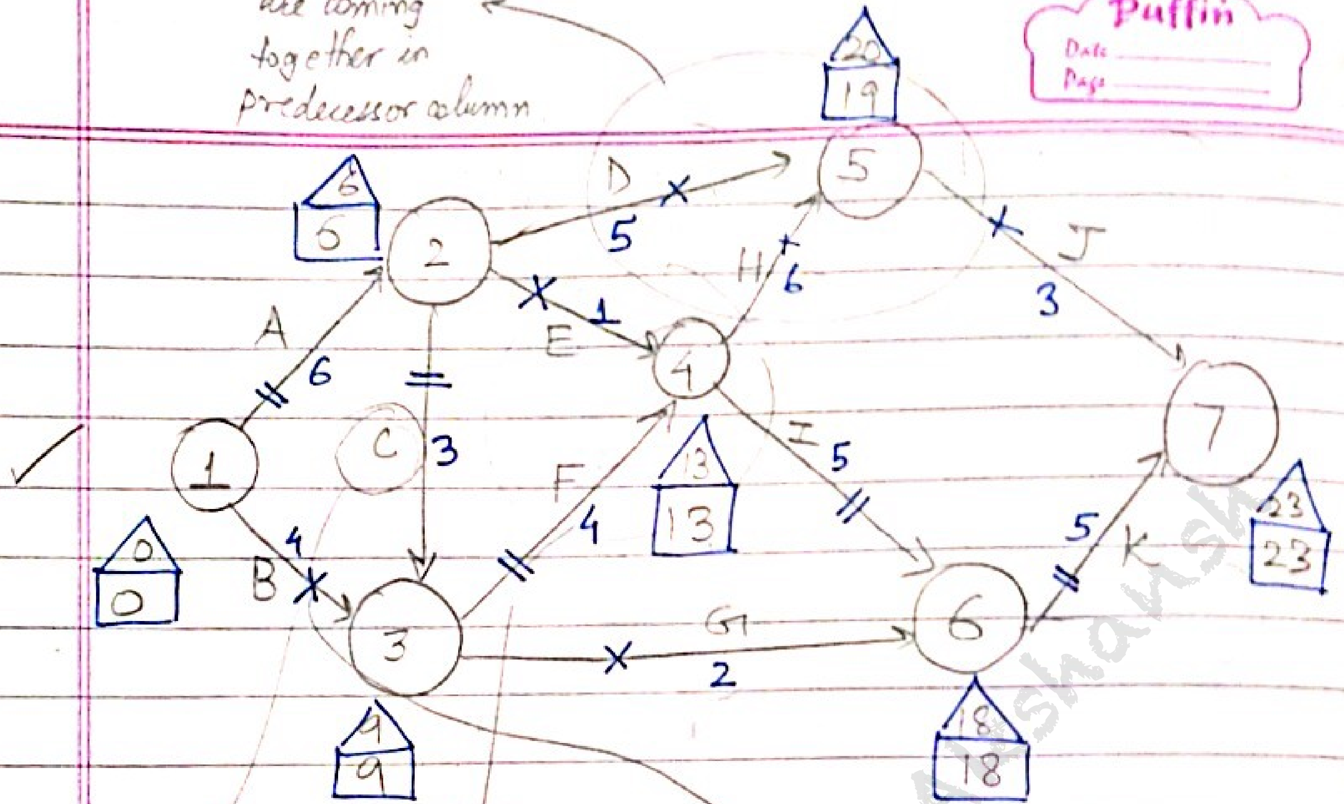
(c) Find probability that project will be completed within 24 hours

(a)



don't do like this (PTO)

∵ D & H are coming together in predecessor column



∵ we saw from table & got to know that, B & C occur together.

∵ we saw starting & ending nodes of E & F are together

$9-0 \neq t_{ij}$
$9-0 \neq t_{ij}$ $\Big\} \neq 4.$
So, ✗

(b) Finding t_e & σ^2

$t_e = \dfrac{a+4m+b}{6}$, $\sigma^2 = \dfrac{(b-a)^2}{36}$

Critical path is
$1 \to 2 \to 3 \to 4 \to 6 \to 7$

(c) Mark the critical activities seeing critical path.
(like $1 \to 2$ is A. So, mark A ∵ t_e & σ^2)

Let T_e = expected project length $= 6+3+4+5+5$
$= 23$ hours

$\sigma_e^2 = 4/9 + 0 + 1/9 + 1 + 4/9 = 2$

$\sigma_e = \sqrt{2} = 1.414$

let T_s be the project completion time.

$$P(T_s \leq 24) = P\left(Z \leq \frac{T_s - T_e}{\sigma_e}\right)$$

$$= P\left(Z \leq \frac{24-23}{1.414}\right)$$

$$= P(Z \leq 0.71)$$

$$= F(0.71)$$

→ cumulative probability for std. normal distribution.

$$= 0.7612$$ → comes from table. Will be given in exam

extra : Probability that time taken is b/w 24 & 26 hours

$$= P(24 \leq T_s \leq 26)$$

$$= F\left(\frac{26-23}{1.414}\right) - F\left(\frac{24-23}{1.414}\right)$$

✓

CLASSICAL OPTIMIZATION THEORY

Chapter-18

NON LINEAR PROGRAMMING PROBLEMS

↳ when objective f^n and/or any constraint is non linear.

Let A be a symmetric matrix of order 3×3

we wont get order > (3×3) in exam.

matrix is symmetric if $a_{ij} = a_{ji}$

let

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

→ Principle diagonal elements.

↳ $a_2 = b_1$, $a_3 = c_1$, $b_3 = c_2$.

✓ Principal minors of order 1 :

→ 1×1

→ Principal diagonal matrices of order 1×1

$$|a_1| = a_1$$
$$|b_2| = b_2$$
$$|c_3| = c_3.$$

✓ Principal minors of order 2 :-

→ 2×2

$$\underbrace{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}}_{\text{for } a_1}, \underbrace{\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}}_{\text{for } b_2}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

↳ deleting rows & column having a_1

Note ☆ Order of matrix = n
Order of minor = k.
Then, no. of minors = ${}^{n}C_{k}$.

✓ Principal minors of order (3) :- 3×3

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Notation :-

Δ_1 : 1st ord. Principal minors.
Δ_2 : 2nd " " "
Δ_3 : 3rd " " "

☆ Note :
In all previously got principal minors, top LHS element is a principal diagonal element

☆ Leading principal minors

of order 1 : $D_1 = |a_1| = a_1$

2 : $D_2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

3 : $D_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

∴ a_1 should lead, i.e., should be there in top LHS

A

★ A symmetric matrix A is called

a) +ve definite if $D_1 > 0$, $D_2 > 0$, $D_3 > 0$

b) −ve definite, if $D_1 < 0$, $D_2 > 0$, $D_3 < 0$

odd order < 0 even order > 0

c) +ve semidefinite, if $\Delta_k \geq 0 \; \forall k$

Principal minors

d) −ve semidefinite, if $\Delta_k \begin{cases} \leq 0 & \forall \text{ odd } k \\ \geq 0 & \forall \text{ even } k \end{cases}$

★ If above 4 cond^ns are not satisfied, then, matrix A is called indefinite.

★ HESSIAN MATRIX or HESSIAN (H)

↳ a matrix used to check if the stationary pt is min / max.

Let f be a f^n of 3 variables x_1, x_2, x_3.

So, consider $f(x_1, x_2, x_3)$

So, Hessian matrix is defined by 2nd order partial derivatives of x_1, x_2, x_3 as shown :-

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$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

If f is a f^n of 2 variables (x_1, x_2)

So, $H\left[f(x_1, x_2)\right] =$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

→ will come in exam

* Gradient of f, $\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x_3}\right]$

$$= \begin{bmatrix} a_1 x_1 + a_2 x_2 + a_3 x_3, \\ b_1 x_1 + b_2 x_2 + b_3 x_3, \\ c_1 x_1 + c_2 x_2 + c_3 x_3 \end{bmatrix}, \text{say}$$

So, Hessian matrix for this is :-

$$H = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

eg If $f = x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_3 + 16x_2x_3 + 4x_1x_2$.

Find ∇f & H.

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right]$$

$$\Rightarrow \nabla f = \begin{bmatrix} 2x_1 + 4x_2 + 4x_3, \\ 4x_1 + 8x_2 + 16x_3, \\ 4x_1 + 16x_2 + 8x_3 \end{bmatrix}$$

→ note: write in order: $(\ )x_1 + (\ )x_2 + (\ )x_3$

So,

$$H = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 16 \\ 4 & 16 & 8 \end{bmatrix}$$

Note: In exam:

We will be asked only to solve for cond^n of stationary pt

☆ If Hessian of f is −ve definite, then, stationary pt. gives local max. value of f^n f.

☆ If Hessian of f is +ve definite, then, stationary pt. gives local minimum value of f.

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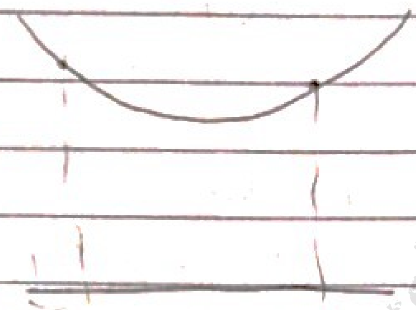
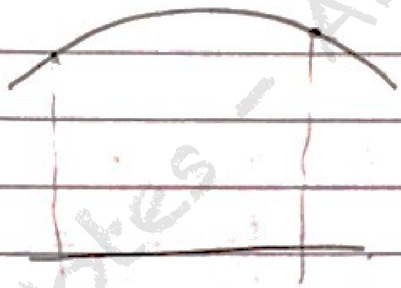
☆ If Hessian of a f^n is +ve definite, then given f^n is CONVEX

☆ If Hessian of a f^n is −ve definite, given f^n is CONCAVE.

Convex f^n :-



Concave f^n :-



☆ (NLPP)

18.2.2 Non Linear Programming Problem with inequality type constraint :

Consider Max $Z = f(x_1, x_2, \ldots x_n)$

$$g_1(x_1, x_2, \ldots x_n) \leq 0$$
$$g_2(x_1, x_2, \ldots x_n) \leq 0$$
$$\vdots$$
$$g_m(x_1, x_2, \ldots x_n) \leq 0$$

& $x_1, x_2, \ldots x_n \geq 0$.

Now, use necessary cond^n to get stationary pts.
So, do, $f'(x) = 0$.

↳ KKT's conditions*
(Karush-Kuhn-Tucker conditions)
(will be asked in exam to write KKT's cond^n)

$n+m$ eq^{ns} for $n+m$ unknowns

> Cond^n ①

$$\frac{\partial f}{\partial x_i} - \lambda_1 \frac{\partial g_1}{\partial x_i} - \lambda_2 \frac{\partial g_2}{\partial x_i} - \cdots - \lambda_m \frac{\partial g_m}{\partial x_i} = 0$$

↳ $i = 1, 2, \ldots n$.

> Cond^n ②

$$\lambda_j g_j = 0 \;;\; j = 1, 2, \ldots m$$

> Cond^n ③

$$g_j \leq 0 \;;\; j = 1, 2, \ldots m.$$

*> Cond^n ④

$$\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$$

↳ If its minimiz^n problem & $\geq$ type constraints are there, $\lambda_1, \lambda_2, \ldots \lambda_m \leq 0$. *

eg ① Write KKT's condⁿ & solve :

Maximize $f = 2x_1^2 - 7x_2^2 + 12x_1x_2$

s.t. $2x_1 + 5x_2 \leq 98$ (In exams, mainly linear constraints will come)

(g) $x_1, x_2 \geq 0$

Proof for KKT's condⁿs

LAGRAGE'S FUNCTION

We define a fⁿ, $L = f - \lambda_1 g_1 - \lambda_2 g_2 - \lambda_3 g_3 \ldots$

↳ no. of constraints = no. of λ's → Lagrange's parameters

Then,

do: $\frac{\partial L}{\partial x_1} = 0$ $\frac{\partial L}{\partial x_2} = 0$

$\frac{\partial L}{\partial \lambda_1} = 0$ $\frac{\partial L}{\partial \lambda_2} = 0$

KKT's condⁿ :- (here, $L = f - \lambda g$) → $2x_1 + 5x_2 - 98$

Condⁿ 1) $\frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0$

$\Rightarrow (4x_1 + 12x_2) - \lambda(2) = 0 \longrightarrow (1)$

$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0$

$\Rightarrow (-14x^2 + 12x_1) - \lambda(5) = 0 \longrightarrow (2)$

Condⁿ 2) $\lambda g = 0 \Rightarrow \lambda(2x_1 + 5x_2 - 98) = 0 \longrightarrow (3)$

Condⁿ 3) $g \leq 0$

$\Rightarrow 2x_1 + 5x_2 - 98 \leq 0 \longrightarrow (4)$

Condⁿ 4) $\lambda \geq 0 \longrightarrow (5)$

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Solving the KKT's cond^n:

Case ① :- $\lambda = 0$.

from ①&② $\begin{cases} eq^n ① \rightarrow 4x_1 + 12x_2 = 0. \\ eq^n ② \rightarrow -14x_2 + 12x_1 = 0. \end{cases}$ → Solving

we get

$x_1 = x_2 = 0$.

Checking in eq^{ns} ①, ②, ③, ④, ⑤.

$4(0) + 12(0) - 2(0) = 0$ ✓

$-14(0) + 12(0) - 5(0) = 0$ ✓

$0(2(0) + 5(0) - 98) = 0$ ✓

$2(0) + 5(0) - 98 \leq 0$ ✓

$0 \geq 0$ ✓

So, its satisfied. But, its trivial sol^n, ∵ we are basically not producing anything. So, discarded.

Case ② :- $\lambda \neq 0$

from ③, $12x_1 + 5x_2 - 98 = 0 \longrightarrow$ ⑥.

Solving eq^{ns} ①, ②, ⑥, we get.

① × 5 − ② × 2, we get

$20x_1 + 60x_2 - 10\lambda = 0$

$24x_1 - 28x_2 - 10\lambda = 0$

$\Rightarrow -4x_1 + 88x_2 = 0$

$\Rightarrow -x_1 + 22x_2 = 0 \longrightarrow$ ⑦

Solving ⑥ & ⑦

$\Rightarrow 44x_2 + 5x_2 = 98$

$\Rightarrow 49x_2 = 98$

$\Rightarrow x_2 = 2$ & $x_1 = 44$.

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Using x_1 & x_2 in (1)

$\Rightarrow 4(44) + 12(2) - 2\lambda = 0$

$\Rightarrow \lambda = 100 \geq 0$ ✓

So, $x_1 = 44$, $x_2 = 2$, $\underbrace{f = 4900}_{\text{Putting in objective } f^n}$.

Now, Sufficient cond^n: Finding Hessian matrix,

$$\frac{\partial f}{\partial x_1} = 4x_1 + 12x_2$$

$$\frac{\partial f}{\partial x_2} = -14x_2 + 12x_1$$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 12 \\ 12 & -14 \end{bmatrix}$$

$D_1 = |4| = 4 > 0$ ✓

$D_2 = \begin{vmatrix} 4 & 12 \\ 12 & -14 \end{vmatrix} < 0$.

From previous defin^{ns}, it is neither +ve definite nor negative definite. So, nothing can be said whether its max. or min.

Q Solve using KKT's cond^ns :-

$$\text{Max :- } \overset{=f}{Z} = 10x_1 - x_1^2 + 10x_2 - x_2^2$$

$$\text{s.t} \quad x_1 + x_2 \leq 14$$
$$-x_1 + x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

here,
$$g_1 = x_1 + x_2 - 14$$
$$g_2 = -x_1 + x_2 - 6$$

KKT's cond^ns are :

Cond^n (1)
$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \lambda_2 \frac{\partial g_2}{\partial x_1} = 0$$
$$(10 - 2x_1) - \lambda_1(1) - \lambda_2(-1) = 0 \longrightarrow (1)$$
$$\& \quad \frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial g_2}{\partial x_2} - \lambda_2 \frac{\partial g_2}{\partial x_2} = 0$$
$$\Rightarrow (10 - 2x_2) - \lambda_1(1) - \lambda_2(1) = 0 \longrightarrow (2)$$

Cond^n (2)
$$\lambda_1 g_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 14) = 0 \longrightarrow (3)$$
$$\& \quad \lambda_2 g_2 = 0 \Rightarrow \lambda_2(-x_1 + x_2 - 6) = 0 \longrightarrow (4)$$

Cond^n (3)
$$g_1 \leq 0 \Rightarrow x_1 + x_2 - 14 \leq 0 \longrightarrow (5)$$
$$g_2 \leq 0 \Rightarrow -x_1 + x_2 - 6 \leq 0 \longrightarrow (6)$$

Cond^n (4)
$$\lambda_1, \lambda_2 \geq 0 \longrightarrow (7)$$

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Solving KKT's condⁿ :

Case ① $\lambda_1 = 0$, $\lambda_2 = 0$

from ① & ② $10 - 2x_1 = 0 \Rightarrow x_1 = 5$
$10 - 2x_2 = 0 \Rightarrow x_2 = 5$.

Using in ③ & ④.

$0(5+5-14) = 0$ ✓

$0(-5+5-6) = 0$ ✓

$5+5-14 \leq 0$ ✓

$-5+5-6 \leq 0$ ✓

$\lambda_1, \lambda_2 \geq 0$ ✓

Case ② : $\lambda_1 \neq 0$, $\lambda_2 \neq 0$.

from ③ & ④,

as $\lambda_1, \lambda_2 \neq 0$, so, $x_1 + x_2 - 14 = 0$
$-x_1 + x_2 - 6 = 0$

$\Rightarrow x_2 = 10, \; x_1 = 4$

Using these values in eqⁿˢ ①, ②

$10 - 2(4) - \lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_1 - \lambda_2 = 2$

$10 - 2(10) - \lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 + \lambda_2 = -10$.

$\Rightarrow \lambda_1 = -6 \leq 0$ ✗
$\lambda_2 = -4 \leq 0$ ✗

eqⁿ ⑦ not satisfied

$4 + 10 - 14 \leq 0$ ✓

$-4 + 10 - 6 \leq 0$ ✓

So, this solⁿ is discarded ∵ eqⁿ ⑦ not satisfied.

Case (3) :- $\lambda_1 \neq 0$, $\lambda_2 = 0$

from (3), $x_1 + x_2 - 14 = 0$. from (4) is satisfied ✓

Using $\lambda_2 = 0$ in eq^n (1) & (2).

$\Rightarrow \quad 10 - 2x_1 - \lambda_1 = 0$.

$\quad 10 - 2x_2 - \lambda_1 = 0$.

$\Rightarrow \quad -x_1 + x_2 = 0$.

$\Rightarrow \quad x_1 = x_2$.

$\Rightarrow \quad x_1 + x_1 - 14 = 0$

$\Rightarrow \quad x_1 = 7 = x_2$

Using in eq^n (5) & (6).

$\Rightarrow \quad 7 + 7 - 14 \leq 0$ ✓

$\quad -7 + 7 - 6 \leq 0$ ✓.

Now, from (1)

$10 - 2(7) - \lambda_1 + 0 = 0$

$\Rightarrow \quad -4 - \lambda_1 = 0 \Rightarrow \lambda_1 = -4 \leq 0$. X

$\lambda_2 = 0 \geq 0$ ✓

$\therefore$ eq^n (7) is not satisfied, so, sol^n is discarded.

Case (4) :- $\lambda_1 = 0$, $\lambda_2 \neq 0$.

eq^n (3) satisfied

from eq^n (4)

$\Rightarrow -x_1 + x_2 - 6 = 0$

Using cond^n in eq^{ns} (1) & (2)

$\Rightarrow \quad 10 - 2x_1 + \lambda_2 = 0$

$\quad 10 - 2x_2 - \lambda_2 = 0$

$\Rightarrow \quad 20 - 2x_1 - 2x_2 = 0$

$\Rightarrow \quad x_1 + x_2 - 10 = 0$

$\quad -x_1 + x_2 - 6 = 0 \quad \Rightarrow x_2 = 8$

$x_1 = 2$.

Solving again 1st eq^{ns}, we find it's discarded.

So, only sol^n is $x_1 = 5,\ x_2 = 5$.

Now finding maxima & minima

$$H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

$$D_1 = |-2| = -2 < 0$$

$$D_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0.$$

$\Rightarrow$ f is -ve definite $(D_1 < 0,\ D_2 > 0)$

So, we'll get

optimum sol^n : $x_1 = 5,\ x_2 = 5$

$f_{max} = 50$.

QUADRATIC FORM, Quadratic Programming Problems and KKT's cond^{ns} for Quadratic Programming Problems :-

eg

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

& $D = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \rightarrow$ a symmetric matrix.

Then,

$$X^T D X = (x_1 \ \ x_2)\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow X^T D X = \begin{pmatrix} x_1 + 2x_2 & 2x_1 + 5x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x_1^2 + 2x_1x_2 + 2x_1x_2 + 5x_2^2$$

$$\Rightarrow X^T D X = x_1^2 + 4x_1x_2 + 5x_2^2$$

$$= f(x_1, x_2)$$

Quadratic form :

☆ A f^n, $f(x_1, x_2)$ is said to be in Quadratic form, if a matrix D (symmetric) can be defined such that, for matrix X

$$f(x_1, x_2) = X^T D X.$$

eg express $2x_1^2 - 2x_1x_2 + 3x_2^2$ in the form $X^T D X$

Idea : Check no. of variables & define X & D

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

coeff. of x_1^2

$\dfrac{\text{coeff. of } x_1x_2}{2}$

$$D = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}_{2\times 2}$$

order of matrix = no. of variables

coeff. of x_2^2

Same as a_{12}
(symmetric matrix)

So, we express it as :-

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = X^T D X.$$

Puffin

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Quadratic Programming:

* An optimiz^n problem which can be expressed in the following form:

$$\text{OPTIMIZE} \quad Z = CX + X^T D X$$

maximise / minimise

$$\text{s.t.} \quad AX \leq b$$

$$X \geq 0$$

is called Quadratic Program

→ C : Row matrix :

$(c_1 \ c_2 \ \ldots \ c_n)$

Elements $c_1, c_2, \ldots c_n$ are coeff. of Objective f^n's variables $x_1 \ x_2 \ \ldots \ x_n$.

→ $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ i.e $X = \underbrace{(x_1 \ x_2 \ \ldots \ x_n)^T}$

we write like this to save space (in exam)

→ $b = (b_1 \ b_2 \ b_3 \ \ldots \ b_m)^T$

for m constraints.

→ A = Coeff. matrix of constraints.

$$= \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & \ldots & & a_{mn} \end{pmatrix}_{m \times n}$$

$$\hookrightarrow D = \begin{pmatrix} d_{11} & d_{12} & --- & d_{1n} \\ d_{21} & d_{22} & --- & d_{2n} \\ \vdots & & & \\ d_{n1} & d_{n2} & --- & d_{nn} \end{pmatrix}_{n\times n}$$

$\hookrightarrow D$: Symmetric matrix ☆

Defining :

- Lagrange's multiplier matrix

$$\lambda = (\lambda_1 \ \lambda_2 \ --- \ \lambda_m)^T$$

Slack variable matrix .

$$s = (s_1 \ s_2 \ --- \ s_m)^T \rightarrow m \text{ constraints.}$$

Non negativity restriction matrix .

$$X \geq 0. \qquad \left(\equiv (x_1 \ x_2 \ --- \ x_n)^T \geq 0\right)$$

$$\text{or } -X \leq 0$$

Defining slack variable matrix again for n variables as $(\mu_1 \ \mu_2 \ --- \ \mu_n)^T$

$$\text{So, } -X + [\mu] \leq 0.$$

☆ KKT's condⁿ for Quadratic Programming Problem.

main condⁿ :

$$\left[\begin{array}{c|c|c|c} -2D & A^T & -I & O \\ A & O & O & I \end{array}\right] \begin{bmatrix} X \\ \hline \lambda \\ \hline \mu \\ \hline s \end{bmatrix} = \begin{bmatrix} C^T \\ \hline b \end{bmatrix}$$

Null matrix

Identity matrix

Other condⁿˢ : $\mu_j x_j = 0 \quad \forall j$

$\lambda_i s_i = 0 \quad \forall i$

eg Write down KKT's condⁿˢ for following Quadratic Programming problem :

$$\text{Max. } Z = 6x_1 + 3x_2 - 2x_1^2 - 3x_2^2 - 4x_1x_2$$

$$\text{s.t.} \quad 1x_1 + 1x_2 \leq 1$$

$$2x_1 + 3x_2 \leq 4$$

$$x_1, x_2 \geq 0.$$

Idea : See no. of variables – 2.

no. of constraints – 2

Now, define all matrices & use formula.

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$D = \begin{pmatrix} -2 & -2 \\ -2 & -3 \end{pmatrix}$

coeff. of x_1^2 coeff of $\frac{x_1x_2}{2}$.   coeff. of x_2^2

$C = (6 \ \ 3)$

$b = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$

KKT's condn are :

$$\begin{array}{c} \\ \end{array}\left[\begin{array}{cc|cc|cc|cc} 4 & 4 & 1 & 2 & -1 & 0 & 0 & 0 \\ 4 & 6 & 1 & 3 & 0 & -1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \hline \lambda_1 \\ \lambda_2 \\ \hline \mu_1 \\ \mu_2 \\ \hline s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ \hline 1 \\ 4 \end{bmatrix}$$

$-2D$, A^T, $-I$, 0, C^T

A, 0, 0, I, b.

4 eqns $\begin{cases} \mu_1 x_1 = 0 \\ \mu_2 x_2 = 0 \end{cases}$, $\lambda_1 s_1 = 0$, $\lambda_2 s_2 = 0$

$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2, s_1, s_2 \geq 0$.

4 eqns

$\to$ 8 eqns, 8 variables. Ans

(Won't be asked to solve).

* # Revised Simplex Method

Puffin
Date: 19.12.13
Day:

↳ we reduce calcul^n & no. of columns here as compared to simplex method

ex. Solve the following LPP by revised simplex method :

$$\text{Max. } Z = 6x_1 - 2x_2 + 3x_3$$
$$\text{s.t. } \quad 2x_1 - x_2 + 2x_3 \le 2$$
$$x_1 + 4x_3 \le 4$$
$$x_1, x_2 \ge 0.$$

⇒ seeing std. relations again : $C_1 = 6,\ C_2 = -2,\ C_3 = 3$

Inverse matrix = B^{-1}

RHS column = $b = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$$\text{Constraint} : \begin{array}{ccccc} x_1 & x_2 & x_3 & s_1 & s_2 \end{array} \\ \begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \end{pmatrix}$$

Intelligently make this because all are ≤ type.

Constraint to eq^n :-

$$\begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

B^{-1}

$P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5$

coff. column of x_1

coff column of s_1

Starting table :

C_B^T	Basic	B^{-1}	Constraint(b)	Entering var.	Pivot elemt.	Ratio
0	s_1	1 0	2			
0	s_2	0 1	4			

C_B: row matrix for coeff. of slack variables

Now, find :- Simplex Multiplies, $M = C_B \cdot B^{-1}$

$$= (0\ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (0\ 0)$$

Now, find $c_j - z_j$ for NON Basic variables
(So, finding only for x_1, x_2, x_3 $\because$ for s_1 & s_2, $c_j - z_j = 0$, so, don't waste time)

$$c_1 - z_1 = c_1 - MP_1 = 6 - (0\ 0)\begin{pmatrix}2\\1\end{pmatrix} = 6$$

$$c_2 - z_2 = c_2 - MP_2 = -2 - (0\ 0)\begin{pmatrix}-1\\0\end{pmatrix} = -2$$

$$c_3 - z_3 = c_3 - MP_3 = 3 - (0\ 0)\begin{pmatrix}2\\4\end{pmatrix} = 3$$

Note: If all $c_j - z_j \leq 0$, then sol^n is optimum. Otherwise, find revised sol^n.

↳ find entering var: Var. corresponding to max. +ve $c_j - z_j$

So, in Entering var., write x_1.

C_B^T	Basic	B^{-1}	b	Ent var.	Pivot col.	Ratio
0	s_1	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	2	x_1		
0	s_2		4			

Now,

$$\text{Pivot element} = B^{-1}P_1 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

Write pivot column & min. ratio

C_B^T	Basic	B^{-1}	b	Ent. Var.	Pivot Col.	Ratio (min)
0	s_1	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	2	x_1	2	$2/2 = 1$
0	s_2		4		1	$4/1 = 4$

write outgoing vector

Now, start revising sol^n (1st Iteration)

C_B^T	Basic	B^{-1}	b	Ent. Var.	Pivot Col.	Ratio
6	x_1	$1/2$ 0	1			
0	s_2	$-1/2$ 0	3			

Idea: Make Pivot element = 1, other elements in that column = 0.

Now, do that for B^{-1} & b columns, not for Pivot.

B^{-1}		b	Pivot	
1	0	2	2	$\longrightarrow R_1/2$
0	1	4	1	$\longrightarrow R_2 \to R_2 - R_1$

Now,

$$M = C_B B^{-1} = (6\ 0)\begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} = (3\ 0)$$

Now, x_1 & S_2 are basic var. So, dont find $C_j - Z_j$ for them. Find,

$$C_2 - Z_2 = C_2 - MP_2 = -2 - (3\ 0)\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \underline{1}$$

$$C_3 - Z_3 = C_3 - MP_3 = 3 - (3\ 0)\begin{pmatrix} 2 \\ 4 \end{pmatrix} = -3$$

$$C_4 - Z_4 = C_4 - MP_4 = -3.$$

Check if $C_j - Z_j \leq 0$. Here $C_2 - Z_2 > 0$. So, that is entering var.

So write x_2 in Ent. var. column.

C_B^T	Basic	B^{-1}		b	Ent Var	Pivot Colm.	Ratio.
6	x_1	$+1/2$	0	1	x_2	$-1/2$	—
$\leftarrow$ 0	S_2	$-1/2$	0	3		$1/2$	$3/\tfrac{1}{2} = 6$

$$\text{Pivot} = B^{-1} P_2 = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$$

(2nd Iteration)

C_B^T	Basic	B^{-1}	b	Ent. Var.	Pivot Col.	Ratio.
6	x_1	$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 6 \end{pmatrix}$			
-2	x_2					

B^{-1}		b	Pivot col.	
$+\frac{1}{2}$	0	1	$-\frac{1}{2}$	$R_1 \to R_1 + (R_2)_{old}$
$-\frac{1}{2}$	0	3	$\frac{1}{2}$	$R_2 \to 2R_2$

this gives $B^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

Now,

$$M = C_B B^{-1} = (6 \;\; -2)\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = (2 \;\; 2)$$

Now, $C_j - Z_j \rightarrow$ $\left.\begin{matrix} x_1 & \times \\ x_2 & \times \end{matrix}\right\}$ Basic vars.

$$C_3 - Z_3 = C_3 - MP_3 = 3 - (2 \;\; 2)\begin{pmatrix} 2 \\ 4 \end{pmatrix} = -9$$

$$C_4 - Z_4 = C_4 - MP_4 = 0 - (2 \;\; 2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2$$

$$C_5 - Z_5 = C_5 - MP_5 = 0 - (2 \;\; 2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -2$$

All $C_j - Z_j \leq 0$.

So, soln under test is optimum.

Value of $x_1 = 4$

$x_2 = 6$.

& $x_3 = 0$ (non basic var.)

$Z_{max} = 12 \quad (24 - 12 + 0)$

end of course.