SIGNALS & SYSTEM GUIDE

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Signals and System Notes, First Edition

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Introduction

Laplace Transform (LT) (an integral transform)

↳ a transformⁿ from time domain to complex frequency domain.

Let $f(t)$: time varying f^n (signal)

So,

$$\boxed{F(S) = \int_0^\infty e^{-st} f(t)\,dt}$$

, S : kernel $= \sigma + j\omega$

$\sqrt{-1}$; ω → rad/s ; σ → neper/sec

We have $f(t) \xrightarrow{\mathcal{L}} F(s)$

↳ applying LT

eg: let $f(t) = e^{-at}$.

Find $\mathcal{L}(f(t))$

$$= \int_0^\infty e^{-st} \cdot e^{-at}\,dt$$

$$= \int_0^\infty e^{-(s+a)t}\,dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$\Rightarrow \mathcal{L}(e^{-at}) = \frac{0-1}{-(s+a)} = \frac{1}{s+a}.$$

↳ Find $\mathcal{L}(1)$

$$= \mathcal{L}\left(\lim_{a\to 0} e^{-at}\right)$$

$$= \lim_{a\to 0}\left(\mathcal{L}(e^{-at})\right)$$

$$= \lim_{a\to 0}\left(\frac{1}{s+a}\right) = \frac{1}{s}.$$

eg: let $f(t) = 1$

Find $\mathcal{L}(f(t))$

$$\Rightarrow F(s) = \int_0^\infty e^{-st}(1)\,dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^\infty$$

$$= \left(\frac{0-1}{-s}\right)$$

$$\Rightarrow \mathcal{L}(1) = \frac{1}{s}.$$

Same ans.

Small letter : Instantaneous variable
capital letter : Laplace variable.

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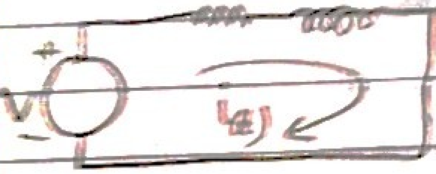
- **Applicⁿ of Laplace Transform to solⁿ of D.E:**

Consider an eq. of RL series circuit excited by a (1) voltage source with initial value of current

$i_{(t)}\big|_{t=0} = I_0$

Magnitude of DC voltage i/p = V.
(Find $i_{(t)}$ using Laplace transformⁿ)



Now, Devt. of D.E i.e writing KVL in series ckt.

$$Ri + L\frac{di}{dt} = V \quad \text{with } i_{(t)}\big|_{t=0} = I_0.$$

By Faraday's Law,

flux

$$|e| = \frac{d\phi}{dt} \text{ or } \frac{d\psi}{dt}$$

Previous knowledge

$$\& \ \phi = Li$$

generally constt (w.r.t t) except for special case

$$\text{So, } L = \frac{\partial\phi}{\partial i}$$

$$|e| = \frac{d(Li)}{dt}$$

$$\Rightarrow |e| = L\frac{di}{dt}$$

Applying LT

Here voltage is constt (DC)

So,

$$Ri_{(t)} + L\frac{di_{(t)}}{dt} = V_{(t)} = V.$$

→ V = constt

$$R\,I(s) + L\{sI(s) - I_0\} = \frac{V}{s}$$

complex frequency $(= \sigma + j\omega)$ — from ↗

σ: Neper/sec — unit comes whenever log or exponential fⁿ comes.

ω: rad/s

LT : Laplace Transform

Note

$$f(t) \xrightarrow{LT(\mathcal{L})} F(s)$$

Then $\mathcal{L}\, \frac{df(t)}{dt} = sF(s) - f(0)$

$= sF(s) - f(t)\Big|_{t=0}$

Theorem : If $f(t) \xrightarrow{\mathcal{L}} F(s)$

Then $\mathcal{L}[a\,f(t)] = a\,F(s)$

Theorem :- $\mathcal{L}(1) = \frac{1}{s} \Rightarrow \mathcal{L}(V) = \frac{V}{s}$

$$\Rightarrow I(s)\,[R + sL] = \frac{V}{s} + LI_0$$

$$\Rightarrow I(s) = \frac{V}{s(R+sL)} + \frac{LI_0}{(R+sL)}$$

Now, apply inverse LT to get $i(t)$

Note :- we have a table of LT given to us :-

$f(t)$	$F(s)$
1	$1/s$
e^{-at}	$1/s+a$
$\sin(\omega)t$	$\omega/s^2+\omega^2$
$\cos(\omega)t$	$s/s^2+\omega^2$
t^n	$\sqrt{n}/s^{n+1}$

use them to find $\mathcal{L}^{-1}$

eg :- $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$

$$i(t) = \underbrace{\mathcal{L}^{-1}\left(\frac{V}{s(R+sL)}\right)}_{\text{done using (M1) \& (M2) (PTO)}} + \underbrace{\mathcal{L}^{-1}\left(\frac{LI_0}{R+sL}\right)}_{\text{Solve separately}} \rightarrow (1)$$

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(M1) $\dfrac{V}{s(R+sL)} = \dfrac{V}{L}\left(\dfrac{1}{s(s+\alpha)}\right)$

where $\alpha = \dfrac{R}{L}$

$\dfrac{1}{s(s+\alpha)} = \dfrac{k_1}{s} + \dfrac{k_2}{s+\alpha}$

$1 = k_1(s+\alpha) + k_2(s)$

$k_1 = \dfrac{1}{s+\alpha}\Big|_{s=0} = \dfrac{1}{\alpha}$

$k_2 = \dfrac{1}{s}\Big|_{s=-\alpha} = -\dfrac{1}{\alpha}$

(M2) By inspection,

$\dfrac{1}{s(s+\alpha)} = \left[\dfrac{(s+\alpha)-s}{s(s+\alpha)}\right]\dfrac{1}{\alpha}$

$= \dfrac{\cancel{s+\alpha}}{s\cancel{(s+\alpha)}}\dfrac{1}{\alpha} - \dfrac{\cancel{s}}{\cancel{s}\alpha(s+\alpha)}$

$= \dfrac{1}{s\alpha} - \dfrac{1}{\alpha(s+\alpha)}$

$\Rightarrow \dfrac{1}{s(s+\alpha)} = \dfrac{1}{\alpha}\left(\dfrac{1}{s} - \dfrac{1}{s+\alpha}\right)$

So, $\mathcal{L}^{-1}\left(\dfrac{1}{s(s+\alpha)}\right) = \dfrac{1}{\alpha}\left[\mathcal{L}^{-1}\left(\dfrac{1}{s}\right) - \mathcal{L}^{-1}\left(\dfrac{1}{s+\alpha}\right)\right]$

$= \dfrac{1}{\alpha}\left(1-e^{-\alpha t}\right) \rightarrow (3)$

Now,

$$\mathcal{L}^{-1}\left(\frac{1}{R+SL}\right) = \mathcal{L}^{-1}\left(\frac{1}{L}\right)\left(\frac{1}{s+\alpha}\right)$$

$$= \left(\frac{1}{L}\right) e^{-\alpha t} \longrightarrow (4)$$

$\rightarrow$ from table

Combining (3) & (4) in (1).

$$\Rightarrow i_{(t)} = \frac{V}{L}\,\mathcal{L}^{-1}\left(\frac{1}{s(s+\alpha)}\right) + \frac{L I_0}{L}\,\mathcal{L}^{-1}\left(\frac{1}{s+\alpha}\right)$$

$$\Rightarrow i_{(t)} = \underbrace{\frac{V}{L}\left(\frac{1}{\alpha}\right)\left(1-e^{-\alpha t}\right)}_{\text{from (3)}} + I_0\left(e^{-\alpha t}\right)$$

$$\Rightarrow \boxed{i_{(t)} = \frac{V}{R}\left(1-e^{-Rt/L}\right) + I_0 e^{-Rt/L}}$$

This is the final response.

$\rightarrow$ Checking sol^n:-

At $t=0$, $i_{(t)} = \frac{V}{R}(1-1) + I_0 = I_0$

So, true ✓

$\rightarrow$ At $t \to \infty$,

Physical meaning

$i_{(t)}$ becomes $\frac{V}{R}$ $\Rightarrow$ L becomes short circuited at steady state

Due to sudden switching L comes into play even though voltage is DC.

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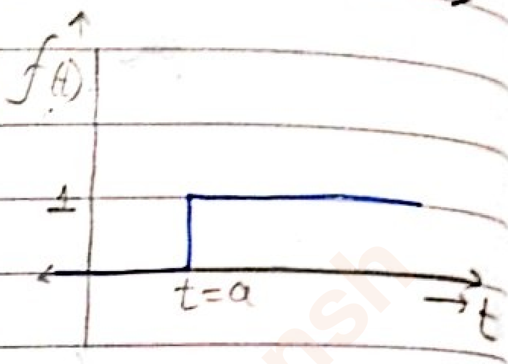
Article 1.4 Some Useful Signal Models

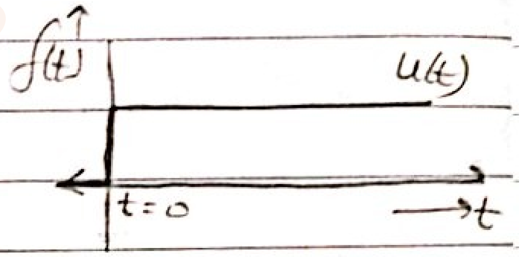
(1) Unit Step Function ($u(t)$)



Shifted unit step f^n $\left\{ u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases} \right.$

↳ at $t = a$, its jump discontinuity

eg: DC generator
Initially 0. when switch on, 220 V.



Non shifted unit step f^n $u(t) = \begin{cases} 1 & , t > 0 \\ 0 & t < 0 \end{cases}$

↳ $t = 0$, pt. of discontinuity.

- Transform^{ns} can be applied to this f^n.

eg: Laplace transform^n :-

$u(t)$; $t > 0$

$$\mathcal{L}\, u(t) = \frac{1}{s} \quad \left(\because \mathcal{L}(1) = \frac{1}{s} \right)$$

$$\int_0^\infty e^{-st} u(t)\,dt = \int_0^\infty e^{-st} (1)\,dt = \frac{1}{s}$$

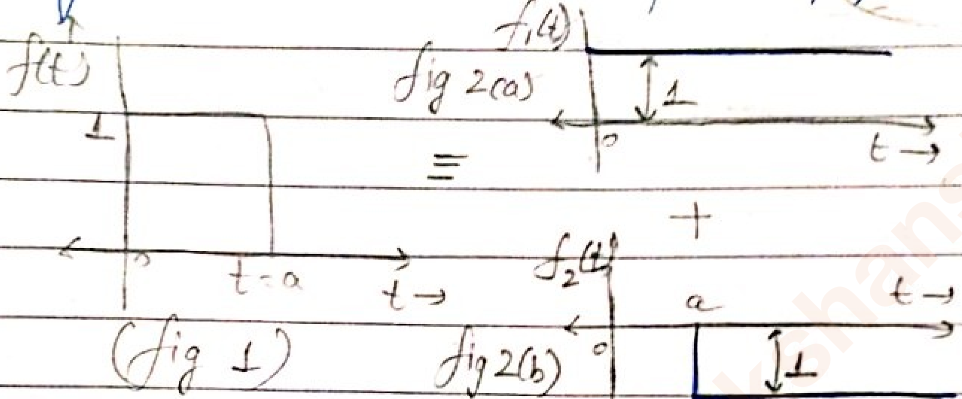
- This signal has major applic^n in communic^n.

☆ Applic^{ns}
↳ Generated voltage from DC Gen. → $220\,u(t)$
(Generator rating, say, 2.2 KW, 220 V, 1500 rpm)
So, it has a switching f^n from OFF to ON.

LT : valid for electrical sys → with wires. we know what is $t = 0$

Fourier Transform^n : for communic^n sys → no wires. We dont know $t = 0$. Start from $-\infty$.

→ In communic^n engineering GATE signal or WINDOW signal is used in the form of pulse.



(fig 1) fig 2(a) fig 2(b)

1 signal ⟶ Broken into 2 signals.

with reference to fig(1) & fig(2(a)) and 2(b), we get,

$$f(t) = \underbrace{f_1(t)}_{u(t)} + \underbrace{f_2(t)}_{(-1)\,u(t-a)}$$

Hence, $f(t) = u(t) - u(t-a)$ ∵ on −y axis.

$$\boxed{\begin{array}{l} \star\ \mathcal{L}\,u(t-a) = e^{-as}\left(\dfrac{1}{s}\right) \\ \quad \rightarrow a = 0 \\ \qquad \mathcal{L}\,u(t) = \dfrac{1}{s}. \end{array}}$$

$\mathcal{L}[G(t)] = ?$; $G(t)$: Window/Gate signal $= u(t) - u(t-a)$

$= \mathcal{L}[u(t) - u(t-a)]$

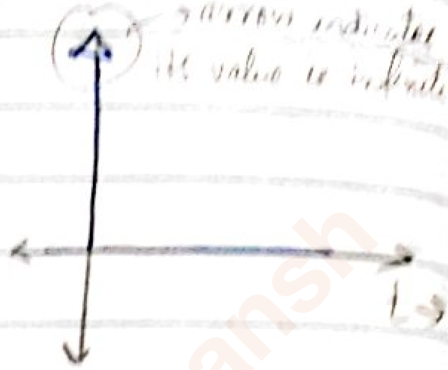
$= \mathcal{L}[u(t)] - \mathcal{L}[u(t-a)] = \dfrac{1}{s} - e^{-as}\dfrac{1}{s}$

$= \dfrac{1}{s}(1 - e^{-as})$

∵ Area = 1
→ also called DIRAC-DELTA f^n

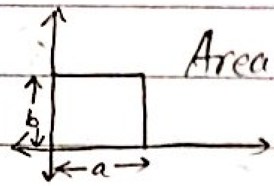
⟨2⟩ Unit Impulse function ($\delta(t)$)

Shifted unit impulse $\delta(t-a) = \begin{cases} \infty ; & t = a \\ 0 ; & t \neq a \end{cases}$



Normal unit impulse $\delta(t) = \begin{cases} \infty ; & t = 0 \\ 0 ; & t \neq 0 \end{cases}$

- Physical significance or applicⁿ of $\delta(t)$

(a) 

Area = A = 1

keeping A = 1, as constt.,
Let $a \to 0$
Then $b \to \infty$
Then, substitute $a = t$ (time)
$b = \delta(t)$

If a f^n satisfies :

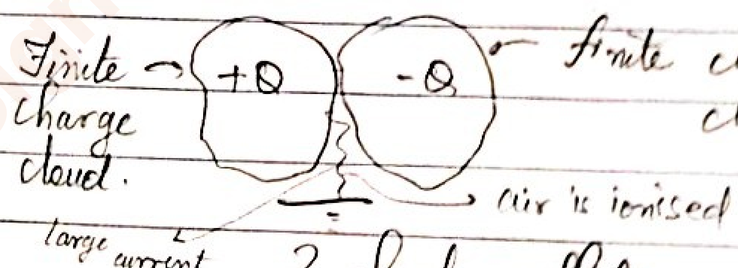
$$\lim_{t \to 0} \int_{-\infty}^{\infty} f(t)\,dt = 1$$

[$\int_{-\infty}^{\infty}$ the limits are approximate]

don't give limits ∵ $t \to 0$

Then, such a f^n is called impulse f^n.

(b) The concept of THUNDERING.



Here, finite charge
$\lim_{t \to 0} \int f(t_{\text{thundering}})\,dt = Q$

2 clouds colliding suddenly (impulse) giving rise to :- thundering current strength ~ kA
persistence ≃ μ sec.

This effect was first given by Mathematician DIRAC using f^n $\delta(t)$.

* Mechanical applicⁿ :- In SPACE DOMAIN :- eg: hammering a nail on a wall. Area of nail^2 = $dx dy$
* Thrust = Force × Area

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- Main Points (including properties)

P (i) $\delta(t)$ is not STRICTLY a f^n. It's a distrib^n.
Why not a f^n :- ∵ It doesn't obey theory of continuity.

P (ii) However, in practice, it has some applic^{ns}.
electrical applic^n ← (Impulse test of Generator / Transformer) ← in time domain
based on rating

P (iii) What do you mean by STRENGTH OF Impulse f^n?
from defin^n :-

$$\delta(t) = \begin{cases} \infty ; t = 0 \\ 0 ; t \neq 0 \end{cases}$$

⇒ Based on defin^n, ∞ cannot be strength of f^n ∵ physically unrealizable.
Still such signals are necessary in practical applic^{ns}
Then, strength is defined as :-
The pt. from where it took birth i.e the AREA magnitude is defined as Strength of Impulse f^n.
⇒ Base Area = 1 → or any value
$\int f(t)\,dt$
↳ So, $1\,\delta(t)$

P (iv) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$ (∵ we kept area = 1)

P (v) $\phi(t)\,\delta(t) = \phi(0)\,\delta(t)$

any f^n, given

(∵ $\delta(t)$ has value only at $t = 0$. So, $\phi(t)$ will contribute to product only at $t = 0$.)
(At $t \neq 0$, it becomes $\phi(t) \cdot \delta(0) = \phi(t) \cdot 0 = 0$.
So, no point)

... anything is a f^n, that means its continuous. If not continuous, then, its a distribution.

Q) Prove that $\sin(t^2 - \frac{\pi}{2})\,\delta(t) = -\delta(t)$

Sol^n :- Sol^n is valid only at $t=0$. [Property (v)]

At $t=0$

So, $\sin(0 - \frac{\pi}{2})\,\delta(t)\Big|_{t=0}$

$= (-1)\,\delta(t)\Big|_{t=0} = -\delta(t)$

H.P

Q) Prove that $\frac{\omega^2+1}{\omega^2+9}\cdot\delta(\omega-1) = \frac{1}{5}\,\delta(\omega-1)$

Put $\omega = 1$ (to make $\delta(0)$)

LHS

$\frac{1+1}{1+9}\,\delta(\omega-1)$

$= \frac{1}{5}\,\delta(\omega-1) = RHS$

H.P

ω: So, now, we have Impulse f^n in frequency domain (not time domain). So, the f^n becomes

$\delta(\omega - a) = \begin{cases} \infty & \omega = a \\ 0 & \omega \neq a \end{cases}$

---o---

$\Rightarrow \delta(t-T) = 0 \; ; t \neq T$

P(vi) $\phi(t)\,\delta(t-T) = \phi(T)\,\delta(t-T)$

Cond^n :- $\phi(t)$ is ctc. at $t = T$.

(Again same reason, $\because$ f^n exists only at $t = T$. So, $\phi(t) \to \phi(T)$)

P(vii) $\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt = \phi(0)\underbrace{\int_{-\infty}^{\infty}\delta(t)\,dt}_{\text{Area}=1} = \phi(0)$

Subject to cond^n; $\phi(t)$ is continuous at $t=0$

SAMPLING / SHIFTING PROPERTY of $\delta(t)$

P(viii) Relation b/w $\delta(t)$ & $u(t)$.

(M1) Approximate method / By Inspection

based on graphical observ^n

* RC circuit is a differentiator circuit.
We saw, $\delta(t) \simeq \frac{d}{dt} u(t)$. So, in theory, we can say, unit step fn. passed from a differentiator ckt, we get an impulse fn.

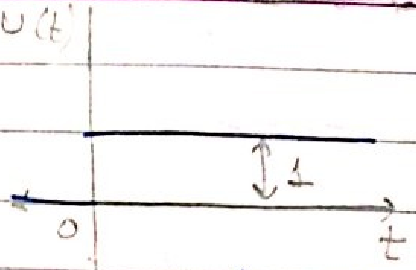

fig 1(a)

By inspection on fig 1(a) & fig 1(b)

Approxim^n :-

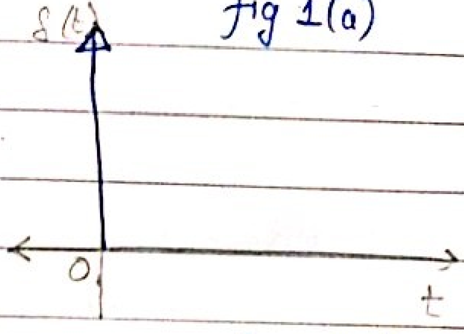

fig 1(b)

$$* \boxed{\delta(t) \simeq \frac{d}{dt} u(t)}$$

$\left(\because u(t) \text{ is constt} = 1 \right.$
So, $\frac{d}{dt}(1) = 0$
i.e $\left. \delta(t) = 0 \right)$

(M2) Rigorous Method / Analytical method

Let $I = \int_{-\infty}^{\infty} \frac{du(t)}{dt} \phi(t)\,dt$; where $u(t)$ = unit step f^n.
$\phi(t)$ = Any f^n of time

Using Integr^n by parts (ILATE)

$$\left\{ \begin{array}{c} \int_a^b fg\,dx = \left[f \int g\,dx \right]_a^b - \int_a^b \left[\left(\frac{d}{dx} f \right) \int g\,dx \right] dx \longrightarrow (1) \\ f = f(x) \ \& \ g = g(x) \end{array} \right\}$$

Let $g = \frac{du(t)}{dt}$ & $f = \phi(t)$; $a \to -\infty$; $b \to \infty$.

$$\therefore I = \left[\left\{ \phi(t) \int \frac{du(t)}{dt}\,dt \right\} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\underbrace{\frac{d\phi(t)}{dt}}_{= \dot{\phi}(t)} \int \frac{du(t)}{dt}\,dt \right) dt$$

(from (1))

$$\Rightarrow I = \phi(t)\, u(t) \Big|_{t \to -\infty}^{t \to \infty} - \int_{-\infty}^{\infty} \dot{\phi}(t)\, u(t)\,dt \longrightarrow (4)$$

$\dot{\phi}(t) \to$ another f^n of time

If $f(t) \xrightarrow{\mathcal{L}} F(s)$

Then $\mathcal{L}\frac{d}{dt} f(t) = sF(s) - f(0)$

where $f(0) = f(t)\big|_{t=0}$

Proof

As per the profile of particular f^n:-

At $t \to -\infty$, $u(t) = 0$ & for $t > 0$, $u(t) = 1$

$$\therefore \quad I = \phi(t)\, u(t)\Big|_0^\infty - \underbrace{\int_0^\infty \dot{\phi}(t)\, dt}$$

(−∞ to ∞ doesn't exist, exists only from 0 → ∞. u(t) = 1 over there)

$$= \phi(\infty) - \phi(t)\Big|_{t=0}^{t\to\infty}$$

$$= \phi(\infty) - [\phi(\infty) - \phi(0)]$$

$$\Rightarrow I = \phi(0)$$

from sampling property of $\delta(t)$

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\, dt = \phi(0) \rightarrow (P(vii))$$

Now $$I = \int_{-\infty}^{\infty} \frac{du(t)}{dt}\phi(t)\, dt = \phi(0) \rightarrow (3)$$

Comparing P(vii) & (3), we get

$$\boxed{\frac{du(t)}{dt} = \delta(t)}$$

↳ Rigorous proof.

* $\mathcal{L}[f(t)] = F(s)$
$F(s) = \mathcal{L}\, f(t)$
$= \int_{-\infty}^{\infty} e^{-st} f(t)\,dt$.

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* Additional observ^n on $\delta(t) \simeq \frac{dU(t)}{dt}$.

LHS

$\mathcal{L}\,\delta(t)$ = Laplace Transform^n of $\delta(t)$

f^n or signal which exists only at $t=0$.

$$\mathcal{L}\,\delta(t) = \int_0^{\infty} e^{-st}\,\delta(t)\,dt$$

Hence, $\mathcal{L}(\delta(t)) = \int_0^{\infty} e^{-0}\,\delta(t)\,dt$

$$= \int_0^{\infty} \delta(t)\,dt = 1.$$

Now

RHS

$$\mathcal{L}\,\frac{d}{dt} f(t) = sF(s) - f(0)$$

So Now,

$F(s) = \mathcal{L}\, f(t)$

or $\mathcal{L}(u(t))$

& $\mathcal{L}\, u(t) = \int_0^{\infty} e^{-st} \underbrace{u(t)}_{1} dt$

$$= \frac{1}{s}$$

$\Rightarrow \mathcal{L}\,\frac{d}{dt} u(t) = s\left(\frac{1}{s}\right) - 0$

$\Rightarrow \mathcal{L}\,\frac{d}{dt} u(t) = 1$ approx value at $t=0$

$=$ RHS.

Proof of $\mathcal{L}\,\frac{df(t)}{dt} = sF(s) - f(0)$

$$\mathcal{L}\, f(t) = F(s) = \int_0^{\infty} \underset{\text{II}}{e^{-st}}\, \underset{\text{I}}{f(t)}\,dt$$

$$= \left\{ f(t) \int e^{-st}\,dt \right\}_0^{\infty} - \int_0^{\infty} \left(\frac{df(t)}{dt} \int e^{-st}\,dt \right) dt$$

$$= \left[f(t) \left(\frac{e^{-st}}{-s} \right) \right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} \frac{df(t)}{dt} e^{-st}\,dt$$

$$F(s) = f(0)\left(\frac{1}{s}\right) + \frac{1}{s}\left(\mathcal{L}\left(\frac{df(t)}{dt}\right) \right)$$

$\times s$ both sides, we get

$$\Rightarrow sF(s) = f(0) + \mathcal{L}\left(\frac{df(t)}{dt}\right)$$

$$\Rightarrow \mathcal{L}\left(\frac{df(t)}{dt}\right) = sF(s) - f(0)$$

So, LHS = RHS.

H.P.

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Chapter - 1

here, $f(t)$ may be a DC signal (like 110 V, 220 V.) or $\sin \omega t$, $\cos \omega t$

Article 1.1 SIGNAL ENERGY & SIGNAL POWER

- Energy of a signal $(f(t)) = E_f = \int_{-\infty}^{\infty} f^2(t)\, dt$

  ↳ for a real signal

eg

$\rightarrow$ R

Energy $= \int i^2 R\, dt$ So, a square f^n ⇓ NON LINEAR f^{ns}.

$= \int \frac{V^2}{R} dt$

✓ eg: $V = 2i$: linear (Resistance)

$P = i^2(2)$: Non linear.

- Complex signal $(g(t)) = p(t) + j\, q(t)$

  $j = \sqrt{-1}$

  variable taken for example. f^n of time, another f^n of time.

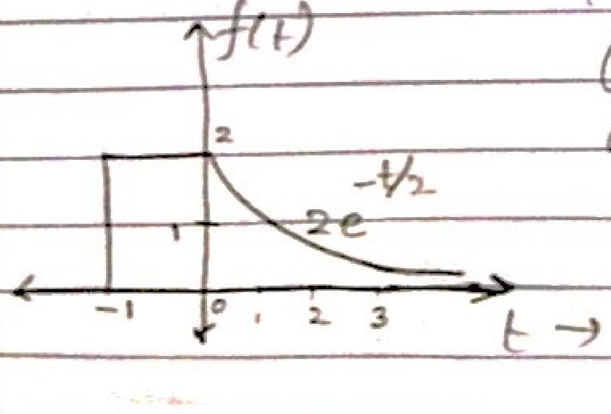
- Energy of a complex signal $(g(t))$

$$E_f = \int_{-\infty}^{\infty} \{|f(t)|\}^2 dt$$

→ any f^n $f(t)$

→ Here, $f(t) = e^{j\omega t}$: std.

eg



Given a signal. Find its energy & power.

★ Power of signal = P_f

★ Let $v(t) = V_0 \sin\omega t$
$i(t) = I_0 \sin\omega t$.

Power $= \int^{dt} v(t)\, i(t)\, dt$
$= \int V_0 I_0 \sin^2\omega t\, dt$

$= \int V_0 I_0 \left(\frac{1}{2}\right)(1-\cos 2\omega t)$ → becomes zero

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$$E_f = \int_{-\infty}^{\infty} (f(t))^2 dt.$$

$$= \int_{-1}^{0} (f_1(t))^2 dt + \int_{0}^{\infty} (f_2(t))^2 dt$$

$$= \int_{-1}^{0} (2)^2 dt + \int_{0}^{\infty} \left(2e^{-t/2}\right)^2 dt$$

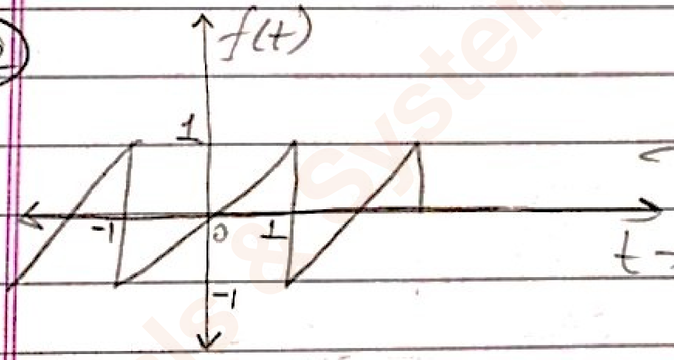
$$= 4[t]_{-1}^{0} + 4\int_{0}^{\infty} e^{-t} dt$$

$$= 4(-)(-1) + 4\left(\frac{e^{-t}}{-1}\right)_0^{\infty}$$

$$= 4 + 4(1)$$

$\Rightarrow E_f = 8$ units

eg. 2



Find the power for given signal

It is a periodic f^n

If $f(t) = f(t+T)$ then its a periodic f^n with period "T".

P_f = Power of signal $\left(\cancel{P_f}\right)$ $(f(t))$

$$= \lim_{T\to\infty} \left\{ \frac{1}{T} \underbrace{\int_0^T [f(t)]^2 dt}_{\text{energy}} \right\} \rightarrow \frac{\text{energy}}{\text{time}} = \text{Power}.$$

Imp. → Say $f(t) = t$. So, we get power a f^n of time. Physically that doesn't happen. So, we take $T \rightarrow \infty$.
→ over whole period (instantaneously its possible)

Alternatively, $P_f = \lim_{T \to \infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 \, dt \right]$ ★

Note :- We know,

$$E_f = \int_{-\infty}^{\infty} [f(t)]^2 \, dt \text{ is energy.}$$

$$P_f = \lim_{T \to \infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 \, dt \right]$$

then, why not we use the formula

$$P_f = \frac{E_f}{T} \;?$$

Reason :- E_f is a fixed value.

So, $\frac{E_f}{T}$ will be a f^n of time.

But in a cts. sys, Power should be constt.

So, we make $\int_{-T/2}^{T/2} [f(t)]^2 dt$ as a f^n of time s.t $\frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 dt$ is independent of t.

eg (2) continued

To find P_f

$$P_f = \frac{1}{T} \int_{-T/2}^{+T/2} \{f(t)\}^2 \, dt$$

Upper limit − Lower

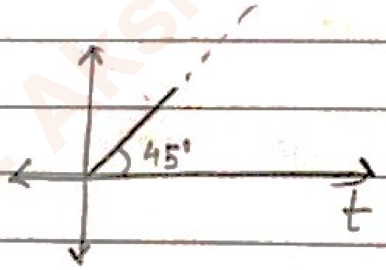
$$\Rightarrow P_f = \frac{1}{(1-(-1))}\int_{-1}^{1}(t^2)\,dt.$$

$$= \frac{1}{2}\left[\frac{t^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3}-\left(-\frac{1}{3}\right)\right] = \frac{1}{3} \quad \text{Ans}$$

UNIT Ramp Signal: It's of the form:-

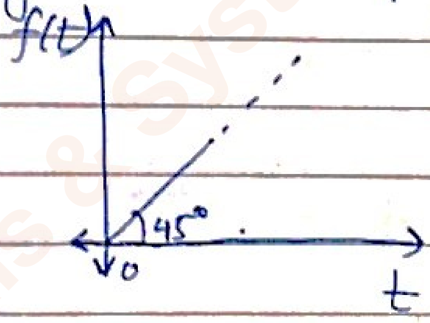
∵ tan(45°)



$$\boxed{\text{Ramp Signal},\ R(t) = t}$$

The above example (2) was a ramp signal.

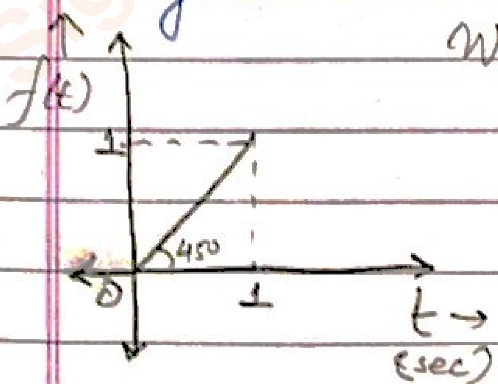
Q1) Now, given unit ramp f^n as:-



Such signals are used in communic^n, instrument^n, electronics...

To generate the signal, as follows:-

What will be expression for this f(t)?

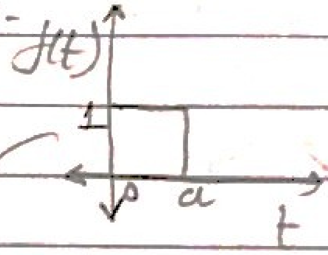


Idea: We have to stop the f^n R(t). So, in such cases (of stopping), unit step f^n u(t) is used.

Idea :- $[u(t) - u(t-a)]$ acts as a conversion factor to stop a signal

---

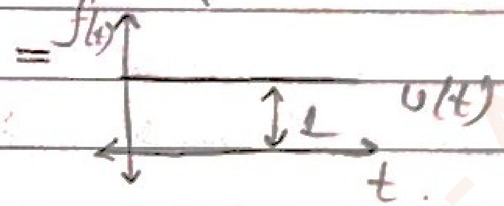
Note :- Suppose we have to generate the following pulse :-



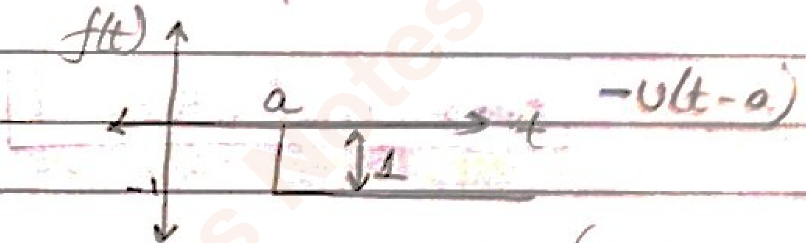
Derive the expr^n

Basically, Stopping the f^n $u(t)$ to get

=



+



So, $f(t) = u(t) + [-u(t-a)] = (u(t) - u(t-a))$

Signal strength or Signal expression

Continuing Q. 1

Using the above concept in Q.1

We get $f(t) = t\,[\,u(t) - u(t-1)\,]$

Signal expression $(f(t) = t)$

Check

for $0 < t < 1$, $u(t-1) = 0$
$u(t) = 1$

for $t > 1$, $u(t) = 1$
$u(t-1) = 1$

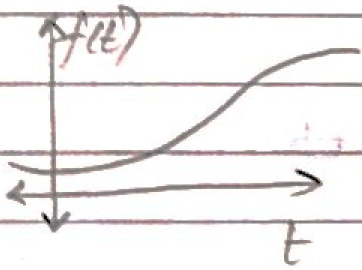
Check defin^n of unit step f^n

Date
Page

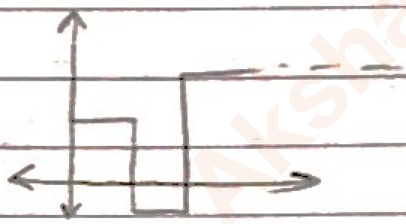
CLASSIFICⁿ OF SIGNALS

1. Continuous time & Discrete time.
2. Analog & Digital Signals.
3. Periodic & Aperiodic Signals.
4. Deterministic & Random Signals.

(1)

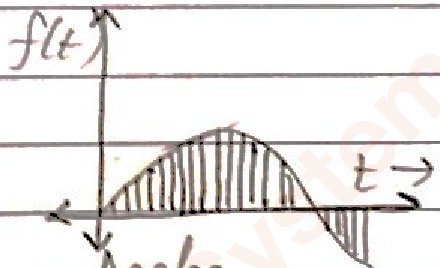


cts signal.

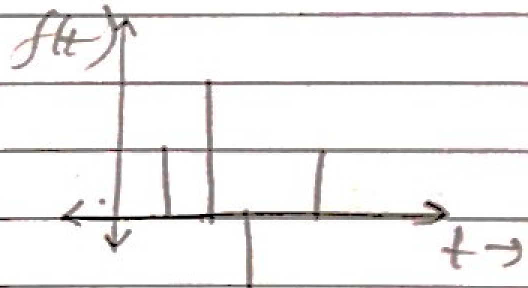


discrete time signal
(Involves jumping)
or discreteness

(2)



Analog
(sth like cts. signal)
(These lines have less gap)
(Tips of spectrum make a locus & continuity)



Digital signal.
The gap b/w spectrum lines is more.

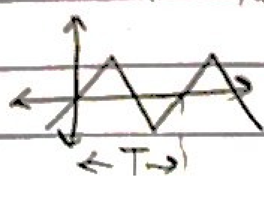
(3)

Periodic Signal:-
If a f^n repeats itself after a specific gap of time.

So, $f(t) = f(t \pm T)$
T : Period.

Aperiodic signal:-
Signal which is not periodic

eg)



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(4) Deterministic :-
Signal which is known, or whose value can be deter-mined (from physical process or using mathematical expression)

Random signal.
Any time, signal can have any value. Its value cannot be determined exactly

(8) ARTICLE 1:3
SIGNAL OPERATION

☆ UNIT IMPULSE f^n (continued).

Problem
(1) Prove :-

$$\int_{-\infty}^{\infty} \delta(t-2)\cos\left(\frac{\pi t}{4}\right)dt = 0.$$

Sol^n:- let $f(t) = \delta(t-2)\cos\left(\frac{\pi t}{4}\right)$

From property of Dirac f^n, $\delta(t-2) = \begin{cases} \text{exist} & t=2 \\ 0 & t\neq 2 \end{cases}$

Hence, $\cos\left(\frac{\pi t}{4}\right)$ has no contribution on the integral for $t \neq 2$.

Hence, at $t=2$; $\cos\left(\frac{\pi(2)}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$.

Hence, $\int_{-\infty}^{\infty} \delta(t-2)\cos\left(\frac{\pi t}{4}\right)dt = 0$.

(Aliter : Use P(vi)
$\phi(t)\,\delta(t-T) = \phi(T)\,\delta(t-T)$
$\phi(T) \rightarrow \cos\left(\frac{\pi(2)}{4}\right)$
$\delta(t-T) \rightarrow \delta(t-2)$)

refer graph in the end
UE : Upper envelope
LE : lower envelope
★ For a varying f^n, $\frac{UE + LE}{2} = DC\ f^n$.

§ ARTICLE 1.4.
FUNCTION (Signal)

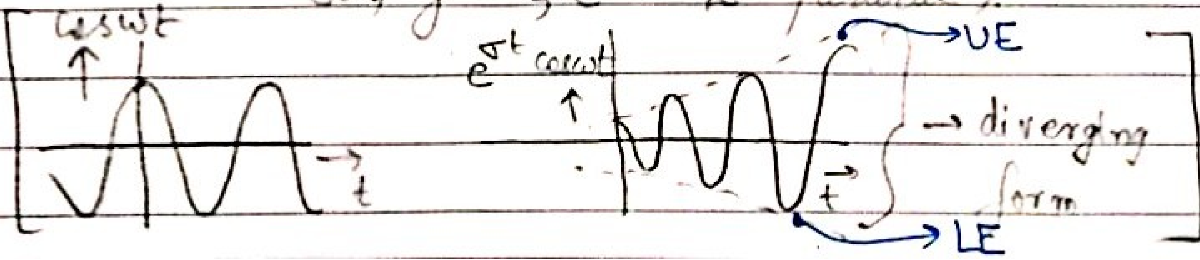
Section 3 : $\boxed{\text{The exponential } f^n\ (e^{st})}$

★ Main Points.

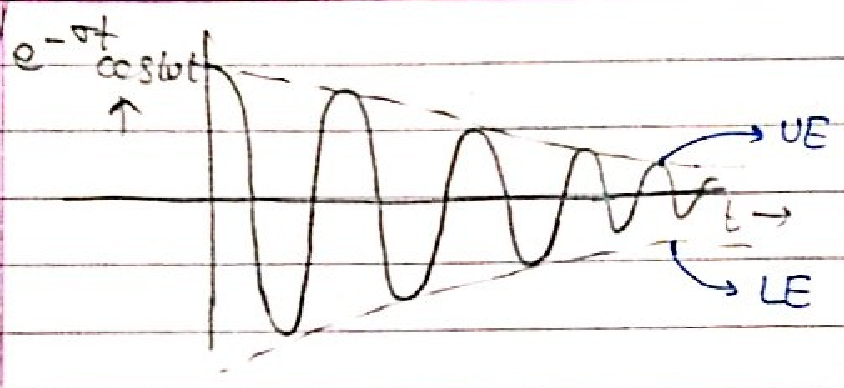
1. S : known as KERNEL
   - When used: Its Major Applicⁿ :-
     If $f(t)$ is a time varying f^n
     Then, $F(s)$ = Laplace Transform of $f(t)$
     $$= \int_0^{\infty} e^{-st} f(t)\, dt$$

2. Physical meaning of s : Its called complex frequency
   $$\boxed{s = \sigma + j\omega}$$
   $\omega \rightarrow$ rad/sec
   $\sigma \rightarrow$ Neper/sec
   $\rightarrow$ Related to phenomena of log

3. $e^{st} = e^{(\sigma + j\omega)t}$
   $= e^{\sigma t} \cdot e^{j\omega t} \equiv e^{i\theta} \rightarrow \text{cis}\,\theta$
   $= (e^{\sigma t})\, [\cos\omega t + j \sin\omega t]$
   $= e^{\sigma t} \cos\omega t + j\, e^{\sigma t} \sin\omega t$.
   ($\cos\omega t$ & $\sin\omega t$ are sustained oscillⁿˢ.
   So, f^n, e^{st} is periodic).



σ : means → damping phenomena in physical sys.
⇓
Resistance, friction, etc.



} exponentially decaying f^n

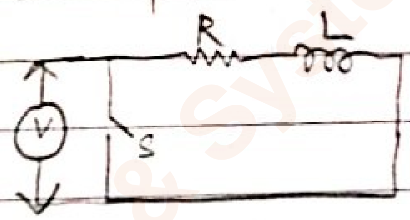
4. *Special case of e^{st}*

(a) $K e^{(0)t} = K$
→ for $s=0$

(b) Monotonic exponential ⇒ exponential f^n is monotonically increasing
→ such characteriz^n is possible for e^{st} only when $e^{st} = e^{\sigma t}$
$\Rightarrow \omega = 0$

—x—

Note :- Special case explaining meaning of σ



when s is closed after $t=0$

$Ri + L\frac{di}{dt} = 0$

an example explaining the idea of σ.
which means damping

extra

$\Rightarrow Ri = -L\frac{di}{dt}$

$\Rightarrow \log i = -\frac{Rt}{L} + \log c.$

$\Rightarrow i = i_0 e^{-Rt/L}$

→ gives damping effect

—x—

Nice : Inertial elements :- m, J, L, C . . .

Energy storing elements (Same?)

↳ Notice their forms :- $\frac{1}{2}mv^2, \frac{1}{2}J\omega^2$ → mechanical

$\frac{1}{2}LI^2, \frac{1}{2}CV^2$ → electrical

(c) $\cos\omega t$ is sinusoidal.

Now,

$$e^{st} = e^{\sigma t}(\cos\omega t + j\sin\omega t)$$

↳ If $\sigma = 0$,

$$s = j\omega$$

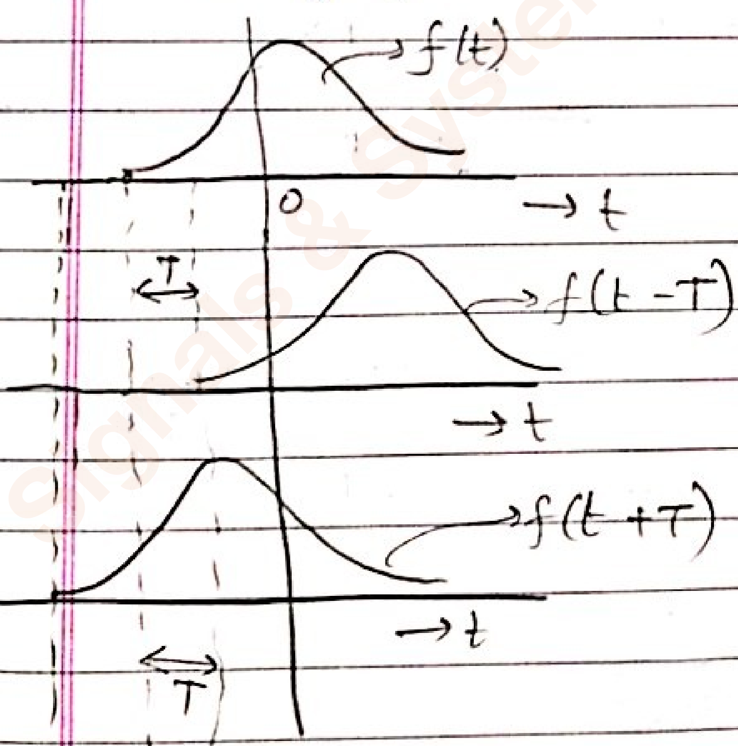
(d) e^{st} varies exponentially (shown in the converging & diverging graphs drawn before)

—— x ——

§ ✯ ARTICLE 1.3.

Some useful SIGNAL OPERATIONS.

✱ TIME shifting.



Adding or subtracting any constt. or f^n with time in $f(t)$, the function shifts.

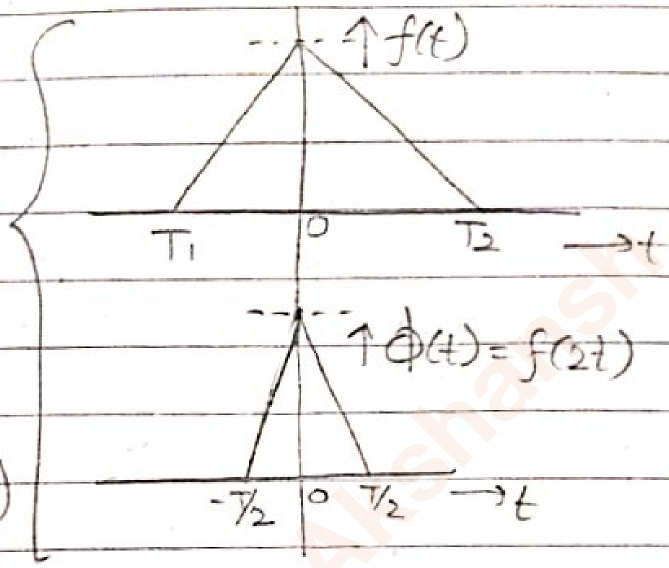
✯ TIME-Scaling

In general, scaling can be done as:-

$\phi(t) = f(at)$

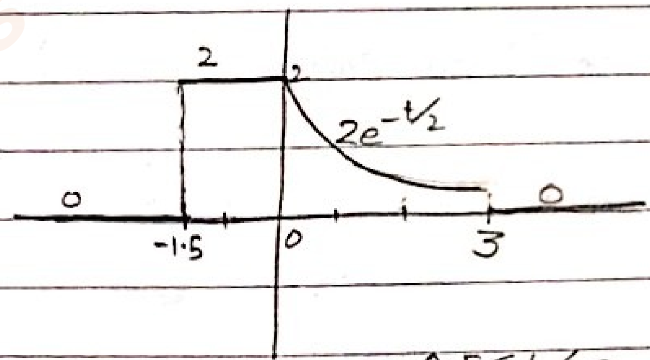
or $\phi(t) = f\left(\frac{t}{a}\right)$

example of time scaling (Functional value unchanged)



Q Given, $f(t) = \begin{cases} 2 & ; -1.5 \leq t \leq 0 \\ 2e^{-t/2} & ; 0 \leq t < 3 \\ 0 & ; \text{otherwise} \end{cases}$

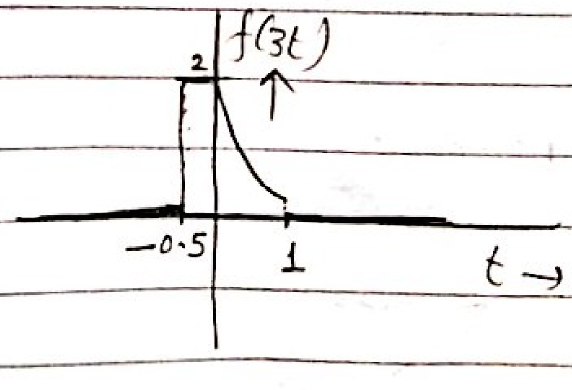
Find :- $f(3t)$ & plot it.



We multiplied t by 3. So, only time scaling changes i.e, only f^n of t changes. So,

$$f(3t) = \begin{cases} 2 & ; -1.5 \leq 3t \leq 0 \rightarrow -0.5 \leq t \leq 0 \\ 2e^{-3t/2} & ; 0 \leq 3t < 3 \rightarrow 0 \leq t < 1 \\ 0 & ; \text{otherwise} \end{cases}$$

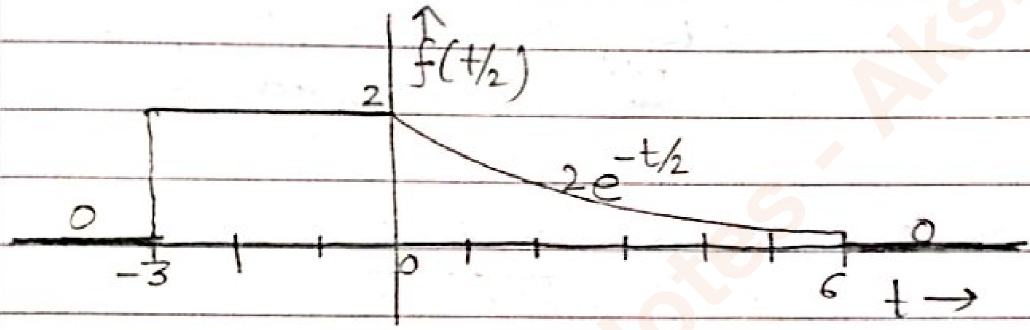
Plotting :-



Q. Again, $f(t)$, same as previous ques. (given)

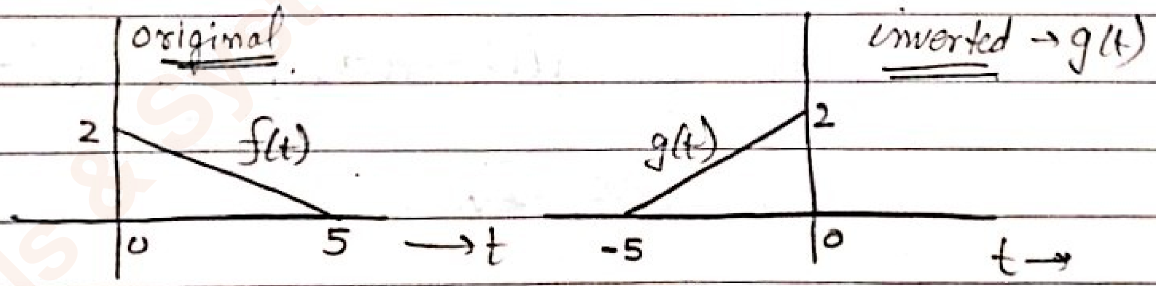
Find $f\left(\frac{t}{2}\right)$ & plot.

$$f(t/2) = \begin{cases} 2 & ; -1.5 \le t/2 \le 0 \\ 2e^{-(t/2)/2} & ; 0 \le t/2 < 3 \\ 0 & ; \text{otherwise.} \end{cases}$$



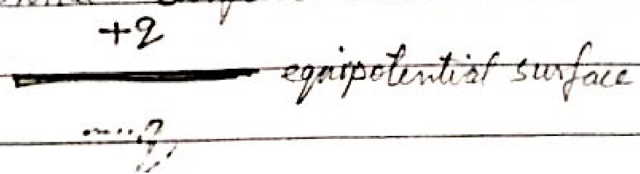
★ TIME - Inversion ≡ Mirror image : along y axis (or ordinate, or dependent var. axis)

ex



$$g(t) = f(-t)$$

★ Practical applicⁿ of mirror image :-
Consider an equipotential surface. If a charge +q is above the surface. So, it can be visualised as, a -ve charge is below the surface, to generate the equipotential surface over which net charge = 0



* Autonomous Sys : Sys. is driving response without giving i/p

* Even & Odd f^n :-

$$f(t) = f(-t) \; : \; \text{EVEN } f^n$$

$$f(t) = -f(-t) \; : \; \text{ODD } f^n$$

- Properties :-

P1) Even f^n × odd f^n = Odd f^n
P2) Odd f^n × ~~even~~ odd f^n = even f^n
P3) even f^n × even f^n = even f^n.

way to remember :- Take odd as − even as +.
So. $(+) \times (-) = (-)$
$(-) \times (-) = (+)$
$(+) \times (+) = (+)$

In general, any signal f(t) can be written :-

$$f(t) = \frac{1}{2} f(t) + \frac{1}{2} f(t)$$

$$= \frac{1}{2} f(t) + \frac{1}{2} f(-t) + \frac{1}{2} f(t) - \frac{1}{2} f(-t)$$

$$\Rightarrow f(t) = \underbrace{\frac{1}{2}\left[f(t) + f(-t) \right]}_{\text{even}} + \underbrace{\frac{1}{2}\left[f(t) - f(-t) \right]}_{\text{odd}}$$

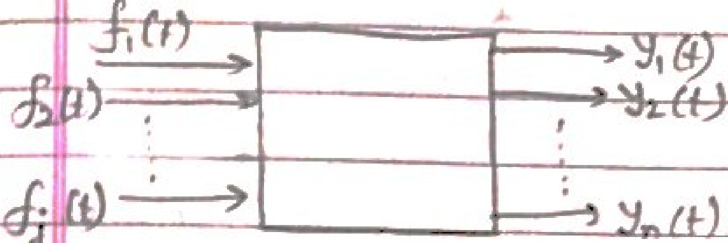
- Applic^n of even & odd f^n :-

✓ In harmonic analysis (= fourier series) of power sys. signals, interference of communic^n sys. by power sys, applic^n in power electronics (inverter & converter, using active devices like transistors, MOSFET, Thyristor)

★ General form of Fourier Series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

§ Systems & Classifications



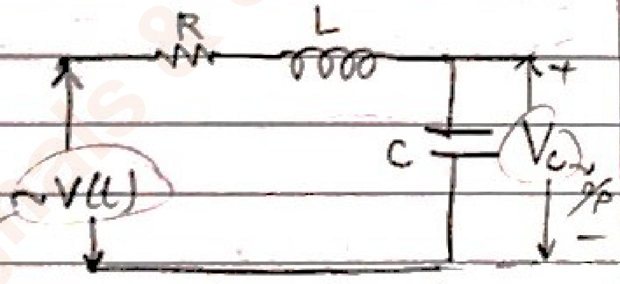
Represent^n of a system

★ In this subject, we see every sys. as a black box with i/p & o/p.

We don't concentrate on inside mechanism, rather, on terminal cond^{ns} → i/p & o/p variables

done in EMech.

Series RLC circuit eg (2):



i/p Single i/p single o/p sys.

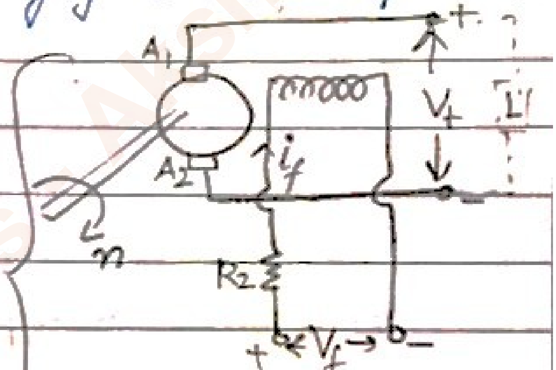
Nature: both electrical.

$$v(t) = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

$$\& \; v_c(t) = \frac{1}{C}\int i\,dt$$

Practical eg. (1) :- Simple DC separately excited generator :-

(a) Varying Prime Mover speed (n):



E_g = Generated emf $= f(n, i_f)$

(b) Field current (i_f) can be controlled (above fig.)

(c) $E_g = f(i_f, n)$

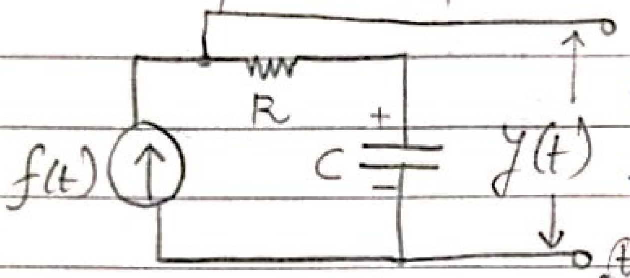
(d) So, the given fig. above shows a 2 i/p single o/p system. i_f, n → E_g

Nature: electrical, mechanical

If 'i' is not considered as o/p (only $V_c(t)$) is considered as o/p, then, present ex. indicates single i/p single o/p sys.

* Variable shouldn't exist in limits of integr^n to make it definite integral
- Definite integral :- $\int_{x=x_1}^{x=x_2} f(x)\,dx$
- Indefinite integral $\int_{x=0}^{x=x} f(x)\,dx$

eg ③ : Basic represent^n of an RC network



Also called forcing fn.

★ $f(t)$: i/p (Current source)

★ $y(t)$: o/p (Voltage source)

KVL, $y(t) = R f(t) + \frac{1}{C}\int_{-\infty}^{t} f(t)\,dt$

limit in terms of var. (Indefinite integral): means that we can find at any time 't'.

$-\infty$ → ∵ we don't know when capacitor was charged.

$$\left(v = iR + \frac{1}{C}\int i\,dt \right)$$

★ R & C : parameters (fixed values)

★ Energy storing elements (inertial elemts) : the forms

Nice

	Element	Force	Energy
Electrical	L	$L(di/dt)$	$\frac{1}{2} L i^2$
	C	$C(dV/dt)$	$\frac{1}{2} C V^2$
Non electrical (mechanical)	m	$m(dv/dt)$	$\frac{1}{2} m v^2$
	J	$J(d\omega/dt)$	$\frac{1}{2} J \omega^2$

Further expansion of y(t) leads to :-

$$y(t) = R f(t) + \underbrace{\frac{1}{C}\int_{-\infty}^{0} f(t)\,dt}_{\text{Initial cond}^n} + \frac{1}{C}\int_{0}^{t} f(t)\,dt \quad \text{— ⓐ}$$

$$\Rightarrow y(t) = R f(t) + \underbrace{\boxed{v_c(0)}}_{\text{initial voltage across capacitor}} + \frac{1}{C}\int_{0}^{t} f(t)\,dt$$

★ Symbol:- $i_L(0^+)$: current across inductor just after $t=0$.
$v_c(0^-)$: capacitor vol. just before $t=0$

★ $f(t)$: forcing f^n :- ∵ its forcing I/p to get response

∴ $y(t)$ = Zero i/p sol^n + Zero State response → meaning done later

or $\underbrace{\text{Zero i/p response}}$ = $v_c(0)$ (Initial cond^n)

Response or o/p when i/p = 0.

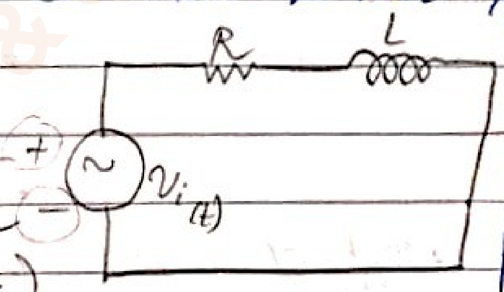
CLASSIFIC^n OF SYSTEMS :-

1. Linear & Non-linear.
2. Constt. Parameter & Time Varying Parameter.
3. Instantaneous & Dynamic Sys.
4. CAUSAL & Non-Causal Sys.
5. Lumped Parameter & Distributed parameter. → eg. transmission power line. Inductance present in the line is the parameter (distributed in space) → PTO
6. Continuous & Discrete time sys.
7. Analog & Digital Sys.

where parameter at particular place

(1) Linear & Non-Linear Systems:

eg



Instantaneous polarity (Req^d with AC)

KVL :

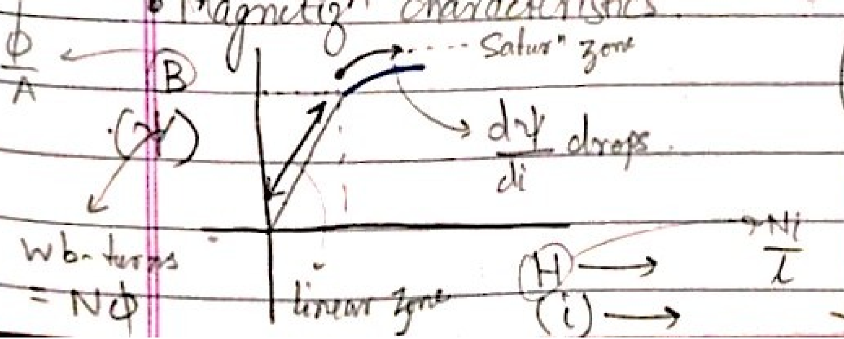
$Ri + L\frac{di}{dt} = v_{i(t)}$

1'st ord. linear D.E

R & L are parameters. Constt instantaneously

→ ψ vs i or B vs H

• Magnetiz^n characteristics



} ex. of Varying parameter

Explanation of Distributed parameter

- ~~mathematical explanation~~
- explanation for distrib^n parameter :

$$L = \frac{N\phi}{I} = \frac{N(N\phi)}{NI}$$

$$= N^2\left(\frac{\phi}{NI}\right) = N^2\left(\frac{\text{flux}}{\text{mmf}}\right)$$

$$= N^2(\text{Permeance})$$

$$\Rightarrow L = N^2\left(\frac{\mu A}{l}\right)$$

So, $L \propto \frac{1}{l}$

The long length of transmission line is divided or distributed in diff^t sections & L is found separately for them.

mathematical explanation:

Consider the D.E

$$\frac{dy(t)}{dt} + 3y(t) = f(t) \rightarrow (a)$$

: Checking its linearity

generally, physical significance is : $y(t)$ = response / o/p

$f(t)$ = forcing f^n / i/p

M1

By common sense :

General idea, if sq., cube etc terms are not there, then its linear

eg: $5\frac{d^2y}{dt^2} + 6y\frac{dy}{dt} + 7y = 23\sin\omega t$

Its non linear : $y \times \frac{dy}{dt}$: square term

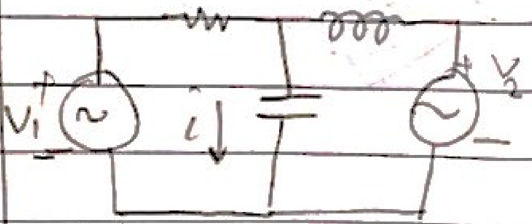
Linearity

Algebraic: y^2, y^3 — sq. term, Cube term.

Differential: $y\frac{dy}{dt}$, $\frac{dy}{dt}\cdot\frac{d^2y}{dt^2}$ — sq. term, sq. term

M2 Using Principle of Superposition (Pos)

eg : Find i



Make $V_2 = 0$ & find current due to V_1 (say i_1)

Then, make $V_1 = 0$ & find current due to V_2 (say i_2)

So, $i = i_1 + i_2$

Date
Page

(M2) Checking linearity : POS.

Consider 2 i/ps [$f_1(t)$ & $f_2(t)$] be present

& corresponding o/ps be $y_1(t)$ & $y_2(t)$ resp.
Hence, sys can be expressed as:

$$f_1(t) = \frac{dy_1(t)}{dt} + 3y_1(t) \longrightarrow (1)$$

$$f_2(t) = \frac{dy_2(t)}{dt} + 3y_2(t) \longrightarrow (2)$$

($\times k_1$) with eq^n (1) & ($\times k_2$) with eq^n (2) & adding both eq^{ns}, we get

$$k_1 f_1(t) + k_2 f_2(t) = \frac{d}{dt}(k_1 y_1(t)) + \frac{d}{dt}(k_2 y_2(t)) + 3k_1 y_1(t) + 3k_2 y_2(t)$$

$$\Rightarrow k_1 f_1(t) + k_2 f_2(t) = \frac{d}{dt}[k_1 y_1(t) + k_2 y_2(t)] + 3(k_1 y_1(t) + k_2 y_2(t)) \rightarrow (b)$$

Comparing (a) & (b), we get.

$$\left.\begin{aligned} k_1 y_1(t) + k_2 y_2(t) &= y(t) \\ k_1 f_1(t) + k_2 f_2(t) &= f(t) \end{aligned}\right\} \text{general proof}$$

So, POS is valid.

(Putting $k_1 = k_2 = 1$, linearity is checked again by POS)
↳ Particular proof.

M3 * Special concept on linearity : Geometric linearity.

eg ① Consider a st. line $y = mx$; m : constt.

x : i/p

Consider 2 systems 1 & 2 y : o/p

So, $y_1 = mx_1 \rightarrow a$

$y_2 = mx_2 \rightarrow b.$

Adding ⓐ & ⓑ

$\Rightarrow y_1 + y_2 = m(x_1 + x_2)$: Its the eq^n of a line. So, linear.

eg ② Consider $y = mx + c$

Two systems ① & ② So, $x_1 \rightarrow y_1$

$x_2 \rightarrow y_2$

So, $y_1 = mx_1 + c$

$y_2 = mx_2 + c$

Adding $y_1 + y_2 = m(x_1 + x_2) + 2c \rightarrow$ ⓒ

But, if we give i/p as $(x_1 + x_2)$, we get

$y = m(x_1 + x_2) + c \rightarrow$ ⓓ

ⓒ $\neq$ ⓓ

So, geometric linearity not always true

Now, in above eg ①, when $x = 0$, $y = 0$

eg ②, $x = 0$, $y = c$.

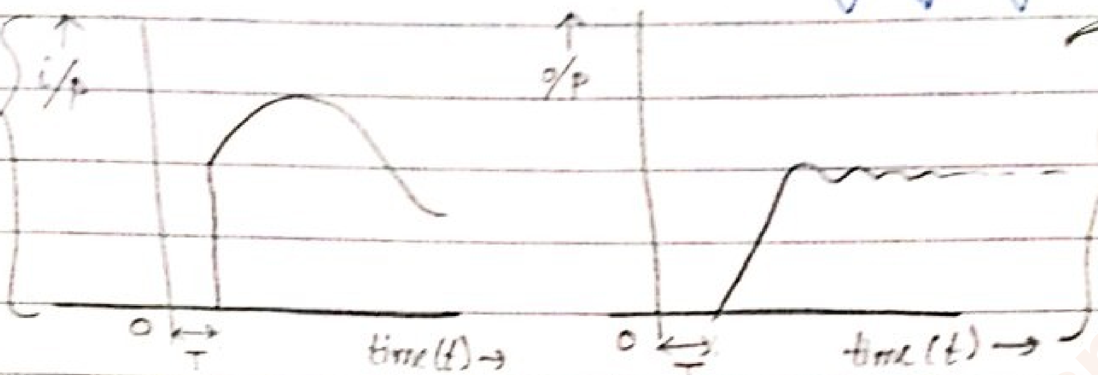
So, eg ① is memory less system

eg ② is memory full system.

So, geometric linearity is true only for memory less system.

(2) Time Invariant & Time Varying Systems

Indications: If $\exists$ delayed i/p $\Downarrow$ $\exists$ delayed o/p



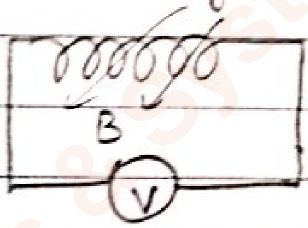
Time Invariant Sys.

- Generally, the const coeff. of D.E. describing an engineering (or any practical) system are known as PARAMETERS

eg. (1) In an R-L sys., $Ri + L\frac{di}{dt} = v(t)$, R & L are parameters

→ Time Invariant sys: Parameters not varying exactly
Time Varying Sys: Parameters vary with time

eg. (2) Consider a magnetic sys



$emf = e = \frac{d\psi}{dt}$; $\psi = Li$ (Parameter)

So, if • Parameter not varying
↳ $\frac{d\psi}{dt} = L\frac{di}{dt}$

• Parameter varying
↳ $\frac{d\psi}{dt} = L\frac{di}{dt} + i\frac{dL}{dt}$

$= L\frac{di}{dt} + i\frac{dL}{d\theta}\frac{d\theta}{dt}$ (ω)

or $L\frac{di}{dt} + i\frac{dL}{dx}\frac{dx}{dt}$ (v)

So. emf changing with speed change or angular speed change.

- eg (3): In Control sys., having eqns:-

$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(x)$

→ M, B, K : parameters

$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = f(\theta)$ (Torque)

→ J, B, K : parameters

meaning : in eg ② (previous pg.).

$$\frac{d\psi}{dt} = L\frac{di}{dt} + i\frac{dL}{dt} \quad : \text{directly}$$

$$\& \quad = L\frac{di}{dt} + i\frac{dL}{d\theta}\frac{d\theta}{dt} \quad : \text{Indirectly}$$

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- When parameters are f^{ns} of time (directly or indirectly), then, the concerned sys. is called time - ~~in~~ variant sys.

③ Instantaneous & Dynamic Sys.

- Instantaneous sys : memory less
- Dynamic Sys : finite memory or Memory full

→ Meaning eg①: Consider a resistance; forcing f^n is given as i/p. Immediately response comes out ⇒ * ∃ no time delay b/w i/p & o/p

eg②: $v = Ri$

eg③: $F = Kx$

→ Meaning eg①: Consider an inductor. Current (i), as a forcing f^n is given as i/p. So, o/p is given as $e = L\frac{di}{dt}$

↳ If L = constt.,
So, ∃ delay in o/p.
due to time gap (dt) involved

eg②: $v = Ri + L\frac{di}{dt}$

* ∃ delay ⇒ memory is stored ⇒ it's finite memory sys.

eg③: $F = f\frac{dx}{dt} + Kx$

✪ Extra :- Physical significance of constt. of integr^n.

$$E = \int\left(L\frac{di}{dt}\, i\right)dt$$

$$= \frac{1}{2}Li^2 + C_1$$

If $i = 0$, $E = 0$

$$\therefore C_1 = 0$$

$$\Rightarrow E = \frac{1}{2}Li^2.$$

So, constt. of integr^n helps to see INITIAL cond^n.

* In causal sys. o/p can't start before giving i/p
* In non causal sys. o/p comes before giving i/p.

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(4) Causal & Non-Causal Sys :-

- CAUSAL Sys. (Physical or non-anticipative sys)
  - → o/p = $y(t)$ ≡ Response
  - i/p = $f(t)$ ≡ forcing fn.
  - → defin^n :- $y(t)$ for any instant (t_0) depends only on the value of $f(t)$ for $t \leq t_0$
  - → i.e., seeing past cond^n (not future)
- Non-causal sys (Anticipative sys)
  - ↳ sys. disobeying defin^n of causal sys.

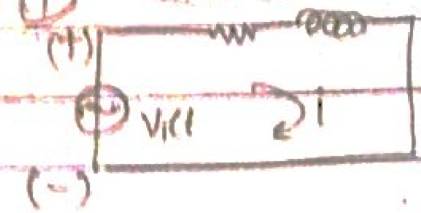
response (o/p) of a sys. when forcing f^n (i/p) is withdrawn

☆ Time domain analysis of Cts. time Systems.

Total response = Zero i/p response + Zero state response.

Part A

eg ①



$V_i(t) = Ri + L\frac{di}{dt}$

If $V_i(t) = 0$,

$Ri + L\frac{di}{dt} = 0$ → Zero i/p

→ an autonomous sys.

→ It'll give zero i/p response

→ complementary f^n

$v(t)$: i/p

$i(t)$: o/p (response)

- let a circuit has eq^n

$$Ri + L\frac{di}{dt} = V_m \sin\omega t$$

→ Part (A) : $Ri + L\frac{di}{dt} = 0$ → Sol^n is Zero i/p response

→ sol^n : → $i_{zi}(t) = c_1 e^{-Rt/L}$

zi → Zero i/p; c_1 → Integr^n const.

→ Part (B) : Choosing a sol^n from DE

Choice : $i = A\sin\omega t + B\cos\omega t$ → ②

→ A, B : unknowns.

Substituting eq^n ② in part (A), find A & B using variable separable method.

Now, Particular integral, i_{zs} can be known

zs → Zero state

So, $i_{Total}(t) = i_T(t) = i_{zi}(t) + i_{zs}(t)$

$= i_{CF} + i_{PI}$

PI → particular integral

Naming in maths : complementary f^n

Naming in electronics : Zero i/p response

Zero state response

* In a sys, whenever ∃ a gap/delay b/w i/p & response (∃ dl term)
so, the element due to which delay came, that element will form D.E in that sys.

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So, $I_T(t) = A\sin(\omega t) + B\cos(\omega t) + C_1 e^{-Rt/L}$

Putting initial condn,
$I = i_0$, at $t = 0$
So, C_1 will be known
So, $I_T(t) = A\sin(\omega t) + B\cos(\omega t) + C_1 e^{-Rt/L}$
→ A, B, C_1 : all known now.
Hence, the Solution

eg ② Part-I : Z-I response
Let, $Ri + L\frac{di}{dt} = 0$ → ①

Now, let the time domain operator, $\frac{d}{dt} = (D)$, say

* Explanation of writing $\frac{d}{dt} = D$
→ we know
$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ → Take care of Initl condn

Now, using $\frac{d}{dt} = D$,
we have to take care of only 'I condn'
+, it reduces an algebraic eqn

So, $(R + LD)i = 0$

Aliter. ↓ frequency domain operator

Applying LT on eqn ①
⇒ $R\,I(s) + L[s\,I(s) - i(0)] = 0$
⇒ $(R + sL)I(s) = L\,i(0)$: algebraic eqn
Assuming $i(0) = 0$, then,
$(R + sL)I(s) = 0$, basically same nature eqn got.

Concept * In a sys, the order of D.E $\propto$ no. of energy storing element.

eg:- In R-L circuit → 1 Energy storing ⇒ order = 1 | In RLC circuit → 2 Energy storing ⇒ Order = 2

* In general :-

Using concept of operator $\left(D \equiv \frac{d}{dt}\right)$,
we can write, in general:

$$Q(D)\, y_0(t) = 0 \quad \text{---(1)} \quad ; \; y_0(t) \text{ Zero i/p response}$$

$$\Rightarrow \left[D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_0\right] y_0(t) = 0 \longrightarrow (2)$$

eg: for RLC circuit,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

Put $\frac{d}{dt} = D \Rightarrow \left(LD^2 + RD + \frac{1}{C}\right)q = 0$

$$\Rightarrow \left(D^2 + \frac{R}{L}(D) + \frac{1}{LC}\right) = 0$$

Comparing with (2)

$$\Rightarrow a_{n-1} = a_1 = R/L, \quad a_0 = 1/LC$$

So, general solⁿ, $y_0(t) = ce^{\lambda t} \longrightarrow (3)$; c, λ : constts.

↳ eg: $Ri + L\frac{di}{dt} = v(t)$

For zero i/p, $R + LD = 0$

$$Ri = -L\frac{di}{dt} \Rightarrow \frac{di}{i} = -\frac{R}{L}dt$$

$$\Rightarrow \log i - \log c = -\frac{R}{L}dt$$

$$\Rightarrow i = ce^{-Rt/L} \quad (\equiv ce^{\lambda t})$$

From (3)

$$\left.\begin{aligned} D\,y_0(t) &= \lambda c e^{\lambda t} \\ D^2 y_0(t) &= \lambda^2 c e^{\lambda t} \;\dots\dots \\ D^n y_0(t) &= \frac{d^n y_0(t)}{dt^n} = \lambda^n c e^{\lambda t} \end{aligned}\right\} \longrightarrow 3(A)$$

Substituting 3(A) results in q^n (2), we get

$$\lambda^n c e^{\lambda t} + a_{n-1}\lambda^{n-1} c e^{\lambda t} + \dots + a_1 \lambda c e^{\lambda t} + a_0 c e^{\lambda t} = 0$$

$$\Rightarrow \left(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0\right) c e^{\lambda t} = 0$$

$$\Rightarrow \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \longrightarrow (4)$$

→ Characteristic eq^n (Auxiliary eq^n in Maths 3)
(can also be got using LT)

→ $Q(\lambda) = 0$.

→ eg (1): R-L series circuit

$$Ri + L\frac{di}{dt} = v(t)$$

For zero i/p $cond^n$,

$$Ri + L\frac{di}{dt} = 0$$

Applying LT, → $i(t)\big|_{t=0^-}$ → just before t=0

$$RI(s) + L(sI(s) - i(0)) = 0$$

For $i(0) = 0$

$$\Rightarrow R\,I(s) + sLI(s) = 0$$

$$\Rightarrow R + sL = 0$$

$$\Rightarrow s + \frac{R}{L} = 0.$$

$(\equiv a_1\lambda + a_0 = 0)$

→ here, a form of characteristic eq^n

→ eg (2): RLC series circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$; Applying LT,

$$Ls^2Q(s) + RsQ(s) + \frac{1}{C}Q(s) = 0$$

$$\Rightarrow s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad (\equiv \lambda^2 + a_1\lambda + a_0 = 0)$$

→ form of 2nd ord. characteristic eq^n.

★ LTI : Linearly Time Invariant

So, $Q(\lambda) = 0$ is called as characteristic eq^n
↳ $Q(\lambda)$ is called characteristic polynomial

★ Solution of eq^n ④
(Just like general sol^n of n^{th} order D.E solved (by auxiliary eq^n method))

→ Case I :
Real & distinct roots (i.e, roots are $\lambda_1, \lambda_2, \ldots$)
So, $y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots\ldots$

→ Case II :
Real & equal roots (i.e. $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda$)
$y_0(t) = (c_1 + c_2 t + c_3 t^2 + \ldots) e^{\lambda t}$

→ Case III :
Complex roots (pairs) (i.e, $\lambda \pm i\beta$)
$y_0(t) = ce^{\lambda t} \cos(\beta t + \theta)$
↳ phase shift ; comes with initial value.

Q Numerical Examples

Q.1) Find zero i/p component of the response for the following LTI system
$(D^2 + 3D + 2) y(t) = D f(t)$
↳ given $y_0(0) = 0$ & $\dot{y}_0(0) = -5$.

Sol^n :- $D^2 + 3D + 2 = 0$.
Characteristic eq^n : $s^2 + 3s + 2 = 0$: Laplace transform
or $\lambda^2 + 3\lambda + 2 = 0$: Classical analysis

Q 1) M2 Using LT :

$$(s^2+3s+2)\,Y(s) = s\,F(s)$$

$$\Rightarrow TF = \frac{Y(s)}{F(s)}$$

$$\Rightarrow TF = \frac{s}{s^2+3s+2}$$

$$\Rightarrow TF = \frac{s}{(s+2)(s+1)}$$

$$= \frac{A}{s+2} + \frac{B}{s+1}$$

$$\Rightarrow B = s \quad (s=-1)$$

$$A = -s \quad (s=-2)$$

$$= \frac{s}{s+1} - \frac{s}{s+2}$$

$$= \frac{s+1-1}{s+1} - \frac{s+2-2}{s+2}$$

$$= 1 - \frac{1}{s+1} - 1 + \frac{2}{s+2}$$

$$\Rightarrow TF = \frac{2}{s+2} - \frac{1}{s+1}$$

Now, $h(t) = \mathcal{L}^{-1}(TF)$

$$= \mathcal{L}^{-1}\left(\frac{2}{s+2} - \frac{1}{s+1}\right)$$

$$= 2\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 1\cdot\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow h(t) = 2e^{-2t} - e^{-t}$$

$$\text{or } h(t) = \underbrace{(2e^{-2t} - e^{-t})\,u(t)}_{\text{same as got before}}$$

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$\Rightarrow (\lambda+1)(\lambda+2) = 0$

$\Rightarrow \lambda = -1, -2 \quad (=s)$

Real & distinct roots

So, sol^n: $y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$\Rightarrow y_0(t) = c_1 e^{-t} + c_2 e^{-2t}$ —— (1)

Putting initial $cond^{ns}$ on eq^n (1)

$\Rightarrow 0 = c_1 e^0 + c_2 e^0 \Rightarrow \boxed{c_1 = -c_2}$ $\quad (\because y_0(0) = 0)$

& $\dot{y}_0(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$

$\Rightarrow -5 = -c_1 - 2c_2$ $\quad (\because \dot{y}_0(0) = -5)$

$\Rightarrow c_1 + 2c_2 = +5$

$\Rightarrow -c_2 + 2c_2 = +5$

$\Rightarrow \boxed{c_2 = +5}$

& $\boxed{c_1 = -5}$ } Zero i/p component

Hence, particular sol^n of eq^n (1),

$y_0(t) = -5e^{-t} + 5e^{-2t}$ Ans

So, in eq^n (1) :- c_1, c_2 : got by initial $cond^n$

$e^{-t} + e^{-2t}$: got by the nature of eq^n.

Note:- ~~for~~ Idea (Reason) for sol^n of Imaginary pair roots.

PROOF : $y_0(t) = C e^{\alpha t} (\cos(\beta t + \theta))$ only for phase diff

Let roots are $\alpha \pm j\beta$ } Distinct

So, $y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, let $c_1 = c_2 = \frac{1}{2}$, say

$= \frac{1}{2} e^{(\alpha + j\beta)t} + \frac{1}{2} e^{(\alpha - j\beta)t}$

$= e^{\alpha t} \left[\frac{e^{j\beta t} + e^{-j\beta t}}{2} \right]$

$= e^{\alpha t} \left[\frac{e^{j\theta} + e^{-j\theta}}{2} \right] \quad ; \theta = \beta t$

Now, we know,

★ $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$ & $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$\Rightarrow y_0(t) = e^{\alpha t} \cos\theta$

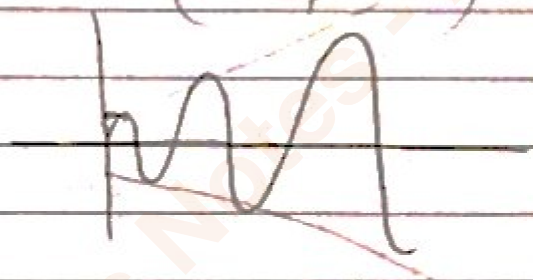
$\Rightarrow y_0(t) = e^{\alpha t} \cos\beta t$

→ Case I : $a > 0$.

($\because e^{\alpha t}$ term)

Solution will be a DIVERGING Sinusoid

($\because \cos\beta t$ term)

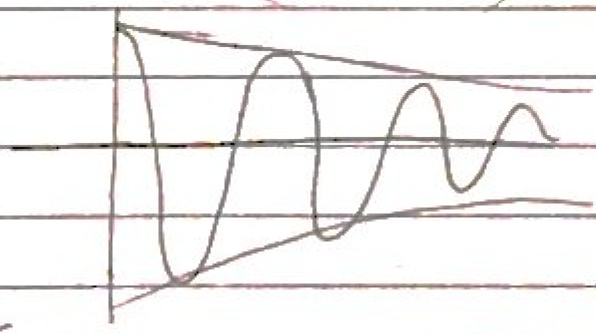


→ Case II : $a < 0$.

($\because$ of $e^{\alpha t}$ term)

Solution will be DECAYING Sinusoid

($\because$ of $\cos\beta t$)



Q.2) $(D^2 + 6D + 9)y(t) = (3D+5) f(t)$

↳ Initial cond^{ns} :- $y(0) = 3$ & $\dot{y}(0) = -7$

Find: Zero-i/p component of response.

Solⁿ : Finding z-i solⁿ

↳ ⇒ $f(t) = 0$ for z-i

⇒ RHS has to be zero.

So, solving :- $D^2 + 6D + 9 = 0$

$\Rightarrow (D+3)^2 = 0$

$\Rightarrow D = -3, -3$

So, from auxiliary eqⁿ :- $\lambda_1 = \lambda_2 = -3 = \lambda$, say

Real & equal roots

$\Rightarrow y(t) = (C_1 + C_2 t)e^{\lambda t}$

$\Rightarrow y(t) = (C_1 + C_2 t)e^{-3t} \longrightarrow$ eqⁿ (1)

Putting initial condⁿ on eqⁿ (1)

$y(0) = 3$

$\Rightarrow 3 = C_1 + C_2(0)$

$\Rightarrow \boxed{3 = C_1}$

$\dot{y}(0) = -7$

Now, $\dot{y}(t) = -3C_1 e^{-3t} \underbrace{- 3C_2 e^{-3t} + C_2 e^{-3t}}_{\text{chain rule}}$

$\Rightarrow \dot{y}(0) = -7 = -3C_1(1) - 3C_2(1) + C_2(1)$

$\Rightarrow -7 = -3(3) - 3C_2 + C_2$

$\Rightarrow 2 = -2C_2$

$\Rightarrow \boxed{C_2 = -1}$

So, zero i/p components are :- $C_1 = 3$ & $C_2 = -1$

Particular solⁿ :-

$y(t) = (3 - t)e^{-3t}$ Ans

Q.3) $D^2 + 4D + 40 = (D+2) f(t)$: Given

↳ Initial condⁿ :- $y(0) = 2$ & $\dot{y}(0) = 16.78$

Find zero-i/p component

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Solⁿ :- for z-i response, RHS = 0.

Solving for characteristic eqⁿ :-

$$D^2 + 4D + 40 = 0$$

Writing auxiliary eqⁿ

$$\Rightarrow \lambda^2 + 4\lambda + 40 = 0$$

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 160}}{2}$$

$$\Rightarrow \lambda = \frac{-4 \pm 12i}{2} = -2 \pm 6i$$

$$\Rightarrow \underbrace{\lambda_1 = -2+6j, \ \lambda_2 = -2-6j}_{\text{imaginary root pair}}$$

M1 So, solⁿ :-

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$= c_1 e^{(-2+6j)t} + c_2 e^{(-2-6j)t}$$

$$\Rightarrow y(t) = e^{-2t}\left[c_1 e^{j6t} + c_2 e^{-j6t}\right] \longrightarrow (1)$$

Putting initial condⁿ

$$y(0) = 2$$

$$\Rightarrow 2 = e^0\left[c_1 e^0 + c_2 e^0\right]$$

$$\Rightarrow \boxed{c_1 + c_2 = 2} \longrightarrow (2)$$

Now,

$$\dot{y}(t) = e^{-2t}\left[j6c_1 e^{j6t} - j6c_2 e^{-j6t}\right]$$

$$- 2e^{-2t}\left[c_1 e^{j6t} + c_2 e^{-j6t}\right]$$

$$\dot{y}(0) = 16.78$$

$$\Rightarrow 16.78 = e^0\left[j6c_1 e^0 - j6c_2 e^0\right] - 2e^0\left[c_1 e^0 + c_2 e^0\right]$$

$$\Rightarrow 16.78 = j6c_1 - j6c_2 - 2(c_1 + c_2)$$

$$\Rightarrow 16.78 = 6j\left[c_1 - c_2\right] - 2(2) \quad \text{from (2)}$$

$$\Rightarrow 20.78 = 6j(c_1 - c_2)$$

$$\Rightarrow C_1 - C_2 = \frac{20.78}{6j}$$

$$= \left(\frac{20.78}{6}\right)\left(\frac{j}{j^2}\right)$$

$$= \left(\frac{20.78}{6}\right)(-j)$$

$$\Rightarrow C_1 - C_2 = -3.463j \rightarrow (3)$$

From (2) & (3),

$C_1 = 1 - 1.73167j$
$C_2 = 1 + 1.73167j$ } Zero i/p components.

Hence, solⁿ is

from eqⁿ (1)

$$y(t) = e^{-2t}\left[(1-1.73167j)e^{j6t} + (1+1.73167j)e^{-j6t}\right]$$

(M2) General solⁿ given as

$$y_0(t) = ce^{\alpha t}\cos(\beta t + \theta)$$

here, $\alpha = -2$, $\beta = 6$.

$$\Rightarrow y_0(t) = ce^{-2t}\cos(6t + \theta) \rightarrow (4)$$

Now, with initial condⁿˢ given, we can find value of c & θ

Given :- $y(0) = 2$

& $\dot{y}(0) = 16.78 = \left.\frac{dy}{dt}\right|_{t=0}$

From (4) $y(0) = 2$

$$\Rightarrow 2 = ce^0\cos(\theta)$$

$$\Rightarrow 2 = c\cos\theta \rightarrow (5).$$

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Now,

$\dot{y}_0(t) = -2ce^{-2t}\cos(6t+\theta) - ce^{-2t}\sin(6t+\theta)(6)$

Now,

$\dot{y}(0) = 16.78$

$\Rightarrow 16.78 = -2ce^{0}\cos\theta - 6ce^{0}\sin\theta$.

$\Rightarrow 16.78 = -c[2\cos\theta + 6\sin\theta]$

$\Rightarrow 16.78 = -\frac{2}{\cos\theta}[2\cos\theta + 6\sin\theta]$

$\Rightarrow 16.78 = -4 - 12\tan\theta$

$\Rightarrow 20.78 = -12\tan\theta$

$\Rightarrow \tan\theta = -\frac{20.78}{12}$ $\left(= -\underbrace{1.732}_{\sqrt{3}}\right)$

$\Rightarrow \theta = -59.99° \simeq -60° = -\frac{\pi}{3}$

Using this in (5)

$\Rightarrow c = 3.99 \simeq 4.$

So, z-i components are :- $c = 4$

$\theta = -60°$

& $y(x) = y_g(x) = 4e^{-2t}\cos(6t - 60°)$ Ans

8 UNIT IMPULSE RESPONSE $h(t)$

↳ o/p due to $\delta(t)$ that is applied at i/p

∵ $\delta(t)$ exists only at $t=0$. So, $\forall\, t \neq 0$, $\delta(t)$ doesnt exist

✲ ARTICLE 2.3 : RULES

Rule → (1) Consider LTIC Sys.

↳ (Linear Time Invariant) Sys

Rule → (2) Describing eq^n :-

$$Q(D)\,y(t) = P(D)\,f(t)$$

→ $y(t)$: o/p or response

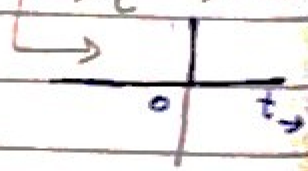
→ $f(t)$: i/p or forcing f^n

→ $Q(D) = 1.D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_0$

↳ $P(D) = b_n D^n + b_{n-1} D^{n-1} + \dots + b_1 D + b_0$

✲ Note : Additional pts. on $h(t)$

(A) $h(t)$ is the response in time domain when i/p is $\delta(t)$ → unit impulse f^n $\begin{cases} \infty ; & t=0 \\ 0 ; & t \neq 0 \end{cases}$



$\delta(t) \rightarrow$ System $\rightarrow h(t)$

i/p (forcing f^n) o/p (response)

(B) Transfer f^n = $T(s)$

$$\Rightarrow T(s) = \frac{\text{LT of o/p}}{\text{LT of i/p}}$$

↳ s.t initial cond^{ns} are relaxed

i.e $f(t)\big|_{t=0} = 0$

$$\text{So, } T(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[f(t)]} = \frac{Y(s)}{F(s)} \longrightarrow (1)$$

; also, we know, $\mathcal{L}(\delta(t)) = 1$ } PROOF

* $\frac{d}{dt}$ is seen as a limiting oper^n : $\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$

* $D \left(= \frac{d}{dt}\right)$ is not having any limits. Seen as multiplic^n oper^n = $D\,f(t)$ Butthe

PROOF : $\mathcal{L}(\delta(t)) = 1$

$$\hookrightarrow \int_0^{\infty} e^{-st}\,\delta(t)\,dt \quad \text{(By defin}^n\text{ of LT)}$$

$$= \int_0^{\infty} e^{0}\,\delta(t)\,dt \longrightarrow \because \delta(t) \text{ exists only at } t=0$$

$$= \int_0^{\infty} \delta(t)\,dt$$

$$= 1 \quad (\because \text{area of delta f}^n = 1)$$

From (1), $y(t)$ becomes $h(t)$ when $f(t) = \delta(t)$

$$\Rightarrow Y(s) = \mathcal{L}\,h(t)$$

when $T(s) = \dfrac{\mathcal{L}[h(t)]}{\mathcal{L}[\delta(t)]}$

$$\Rightarrow T(s) = \mathcal{L}[h(t)]$$

$$\Rightarrow \boxed{h(t) = \mathcal{L}^{-1}(T(s))}$$

(c) $\mathcal{L}\,f(t) = F(s)$

or $\mathcal{L}^{-1}(F(s)) = f(t)$

ex - Let $F(s) = \dfrac{1}{(s+1)(s+2)}$

So $\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\dfrac{1}{(s+1)(s+2)}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] \quad \text{Partial fraction}$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$$

[illegible]

★ Diff b/w D & $\frac{d}{dt}$: $\left(D \equiv \frac{d}{dt}\right)$ $\left(\frac{d}{dt} f(t) = D f(t)\right)$

d/dt : operator; has meaning only when applied to $f(t)$

D : an algebraic operator; can be used as a multiplier

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = e^{-1t} - e^{-2t}$$

$$\Rightarrow \mathcal{L}^{-1}[F(s)] = e^{-t} - e^{-2t}$$

Rule → (3) $h(t) = b_n \delta(t) + \left([P(D)\, y_n(t)]\, u(t)\right)$

→ $\delta(t)$: unit impulse f^n

→ $u(t)$: unit step f^n.

Rule → (4) for
$n = 1 \rightarrow y_n(0) = 1$
$n = 2 \rightarrow y_n(0) = 0;\ \dot{y}_n(0) = 1$
$n = 3 \rightarrow y_n(0) = 0;\ \dot{y}_n(0) = 0;\ \ddot{y}_n(0) = 1$
⋮

Problem (Q.)

$(D^2 + 3D + 2)\, y(t) = D f(t)$

Find : unit impulse response, $h(t)$

(M1) Here, $n = 2$ (its 2nd order)

Writing Auxiliary eq^n :-

[M2 : done after (Q2)]

$(\lambda^2 + 3\lambda + 2) = 0$ (Taking RHS = 0)

$\Rightarrow (\lambda + 2)(\lambda + 1) = 0$

$\Rightarrow \lambda = -2, -1$

So, general sol^n. $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$\Rightarrow y(t) = c_1 e^{-2t} + c_2 e^{-t}$

$\Rightarrow y_n(t) = c_1 e^{-t} + c_2 e^{-2t}$ — ①

Using general formula

for $n = 2$, $y_n(0) = 0$, $\dot{y}_n(0) = 1$

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$\Rightarrow y_n(0) = c_1 e^0 + c_2 e^0$
$\Rightarrow y_n(0) = 0 = c_1 + c_2 \Rightarrow \boxed{c_1 = -c_2}$

& $\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$ (from ①)
$\Rightarrow \dot{y}_n(0) = -c_1 e^0 - 2c_2 e^0$
$\Rightarrow 1 = -c_1 - 2c_2$
$\Rightarrow c_1 + 2c_2 = -1$
$\Rightarrow -c_2 + 2c_2 = -1$
$\Rightarrow \boxed{c_2 = -1}$
& $\boxed{c_1 = 1}$

So, eq^n ① becomes
$y_n(t) = e^{-t} - e^{-2t}$

Now, general form :- $Q(D)\,y(t) = P(D)\,f(t)$
(Rule 2)

Ily, in our problem $(D^2 + 3D + 2)\,y(t) = D f(t)$
where, $P(D) = D$
; $b_n = b_2 = 0$ (i.e., coeff. of D^2 in $P(D)$)

Now, we know
$h(t) = b_n\,\delta(t) + [P(D)\,y_n(t)]\,u(t)$

$\Rightarrow h(t) = 0.\,\delta(t) + [D\cdot(e^{-t} - e^{-2t})]\,u(t)$

$\Rightarrow h(t) = D\,(e^{-t} - e^{-2t})\,u(t)$
$\Rightarrow h(t) = (-e^{-t} + 2e^{-2t})\,u(t)$
$+ \cancel{\frac{d}{dt}u(t)\,(e^{-t} - e^{-2t})}$

$\Rightarrow h(t) = (-e^{-t} + 2e^{-2t})\,u(t)$

Ans

This part not there as per property u(t) is written later, after differentiation.

V.V. Imp — Note:- $h(t)$ is value of o/p due to $\delta(t)$ applied at i/p & $\delta(t)$ exists only at $t=0$. So, $\forall\, t \neq 0$, $\exists$ no i/p ($\because \exists$ no $\delta(t)$). So, $\forall\, t \neq 0$, sys behaves as zero i/p sys. So, for finding $y(t)$ in sys, $Q(D)=0$ always.

Q 2 Determine the unit impulse response ($h(t)$) for LTIC, described by:

(1.1) $(D+2)\,y(t) = (3D+5)\,f(t)$

Solⁿ It's of the general form

$$Q(D)\,y(t) = P(D)\,f(t)$$

Here, $Q(D) = D+2$

$P(D) = 3D+5$

$b_n = b_1 = 3$

Now, $P(D) = D$

$\Rightarrow D+2 = 0$

$\Rightarrow \lambda + 2 = 0$ (Auxiliary)

$\Rightarrow \lambda = -2$

$\Rightarrow y(t) = c_1 e^{\lambda t} = c_1 e^{-2t}$

Putting given condⁿ:- $y_n(0) = 1$

$\Rightarrow y(0) = c_1 e^0 = 1$

$\Rightarrow c_1 = 1$

$\Rightarrow y(t) = e^{-2t} \longrightarrow (1)$

Now,

$$h(t) = b_n \delta(t) + [P(D)\,y_n(t)]\,u(t)$$

$$= 3\delta(t) + [(3D+5)e^{-2t}]\,u(t)$$

$$= 3\delta(t) + (5e^{-2t} - 6e^{-2t})\,u(t)$$

$3D(e^{-2t})$

$= \frac{d}{dt} e^{-2t}$

$\Rightarrow h(t) = 3\delta(t) - e^{-2t}\,u(t)$

* $\mathcal{L}^{-1}(1) = \delta(t)$; $\mathcal{L}^{-1}(kF(s)) = k\mathcal{L}^{-1}(F(s))$
* $\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

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M2) Using $\underline{LT}$

Idea: for solving using $\mathcal{L}T$,
replace D by s $\rightarrow \sigma + j\omega$

we know that if $\mathcal{L} f(t) = F(s)$
then $\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$

here, its z·i condⁿ $\Rightarrow f(0) = 0$
$\Rightarrow \mathcal{L}[Df(t)] = sF(s)$
$\Rightarrow D\,F(s) = sF(s)$
$\Rightarrow \boxed{D \Rightarrow s}$

So, in our problem,
$(s+2)Y(s) = (3s+5)F(s)$
& $TF = \frac{Y(s)}{F(s)} = \frac{3s+5}{s+2}$

$= 3\left(\frac{s}{s+2}\right) + \frac{5}{s+2}$

$= 3\left(\frac{s+2-2}{s+2}\right) + \frac{5}{s+2}$

$= 3(1) - \frac{6}{s+2} + \frac{5}{s+2}$

$\Rightarrow TF = 3 - \frac{1}{s+2}$

Now, we know
$h(t) = \mathcal{L}^{-1}(TF)$
$= \mathcal{L}^{-1}\left(3(1) - \frac{1}{s+2}\right)$

can be put later, as its '1' anyways

$\Rightarrow h(t) = 3\,\delta(t) - e^{-2t}u(t)$ (using L^{-1} properties)

Ans

Q 1) M2 Using LT

$$(s^2+3s+2)Y(s) = s F(s)$$

$$\Rightarrow TF = \frac{Y(s)}{F(s)}$$

$$\Rightarrow TF = \frac{s}{s^2+3s+2}$$

$$\Rightarrow TF = \frac{s}{(s+2)(s+1)}$$

$$= \frac{A}{s+2} + \frac{B}{s+1}$$

$$\Rightarrow B = s \quad (s=-1)$$

$$A = -s \quad (s=-2)$$

$$= \frac{s}{s+1} - \frac{s}{s+2}$$

$$= \frac{s+1-1}{s+1} - \frac{s+2-2}{s+2}$$

$$= 1 - \frac{1}{s+1} - 1 + \frac{2}{s+2}$$

$$\Rightarrow TF = \frac{2}{s+2} - \frac{1}{s+1}$$

Now, $h(t) = \mathcal{L}^{-1}(TF)$

$$= \mathcal{L}^{-1}\left(\frac{2}{s+2} - \frac{1}{s+1}\right)$$

$$= 2\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 1\cdot\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow h(t) = 2e^{-2t} - e^{-t}$$

or $h(t) = \underbrace{(2e^{-2t} - e^{-t})u(t)}_{\text{same as got before}}$

$f(t) * h(t)$

2) $\Rightarrow y(t) = \int_0^t f(t-\tau)\, h(\tau)\, d\tau$ *

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§ CONVOLUTION INTEGRAL

* MAIN POINTS :-

1) (Forcing f^n) Input, $f(t)$ → System → $y(t)$ = output (response)

2) with reference to 1), generally we can write

$$y(t) = f(t) * h(t) \; ; \; * \; (\not\equiv \text{multiplic}^n)$$

$h(t)$: unit impulse response

*: indicates convolution integral

3) If any 2 f^ns (time varying), $f_1(t)$ & $f_2(t)$ are considered & also, if LT of $f_1(t) = F_1(s)$

i.e. $\mathcal{L}(f_1(t)) = F_1(s)$

& $\mathcal{L}(f_2(t)) = F_2(s)$

So, $\mathcal{L}^{-1}(F_1(s)) = f_1(t)$

& $\mathcal{L}^{-1}(F_2(s)) = f_2(t)$

then,

$$\boxed{\mathcal{L}^{-1}(F_1(s) \cdot F_2(s)) = f_1(t) * f_2(t)}$$

4) General expression of convol^n integral:-

$$f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau$$
$$= \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau$$

expression to be used, depends on f^n.

↳ τ : dummy variable.

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5) $\boxed{y(t) = f(t) * h(t)}$

Proof

By control sys (CS)

$r(t) = i/p = f(t)$, here

$\underbrace{c(t)}_{CS} = o/p = \underbrace{y(t)}_{SS}$, here

So, $TF = T(s) = \left.\dfrac{\mathcal{L}(o/p)}{\mathcal{L}(i/p)}\right|_{\text{initial cond}^n \text{ relaxed}} = \dfrac{\mathcal{L}(c(t))}{\mathcal{L}(r(t))} = \dfrac{C(s)}{R(s)}$

$\Rightarrow T(s) = \dfrac{Y(s)}{F(s)}$; $F(s) = \mathcal{L}(f(t))$

$Y(s) = \mathcal{L}(y(t))$

Case (1) : Particular case,

Now, if $r(t) = f(t) = \delta(t)$.

Then, let the o/p $= h(t)$

$\hookrightarrow$ $\because$ by defin^n of $h(t)$, it is the o/p when i/p $= \delta(t)$

Now, as the sys. doesnt change.

$\Rightarrow T(s)$ remains same.

$\Rightarrow T(s) = \dfrac{\mathcal{L}(h(t))}{\mathcal{L}(\delta(t))}$

Now, $\left.\begin{array}{l}\mathcal{L}(h(t)) = H(s) \\ \mathcal{L}(\delta(t)) = 1 \text{ result}\end{array}\right\}$ known.

hence, $T(s) = \dfrac{H(s)}{1} = H(s)$.

Case (2) : General case,

for any i/p $r(t) = f(t)$

$T(s) = \dfrac{Y(s)}{F(s)} = H(s)$ from case (1).

$\Rightarrow Y(s) = F(s) \cdot H(s)$

Taking $\mathcal{L}^{-1}$

$\Rightarrow y(t) = \mathcal{L}^{-1}[F(s)\,H(s)]$

$\Rightarrow y(t) = f(t) * h(t)$

H.P ★

Q. Find out $\mathcal{L}^{-1}\left(\frac{1}{(s+1)(s+2)}\right)$

(M1) $= \mathcal{L}^{-1}\left(\frac{(s+2)-(s+1)}{(s+1)(s+2)}\right)$

$= \mathcal{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$

$= \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$

$= e^{-1t} - e^{-2t} \quad \left(\because \mathcal{L}(e^{-at}) = \frac{1}{s+a}\right)$

(M2) Using convolⁿ

let $F_1(s) = \frac{1}{s+1}$ & $F_2(s) = \frac{1}{s+2}$

So, $f_1(t) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

$f_2(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$

So, $\mathcal{L}^{-1}(F_1(s) \cdot F_2(s))$

$= \mathcal{L}^{-1}\left(\frac{1}{(s+1)} \cdot \frac{1}{(s+2)}\right) = f_1(t) * f_2(t)$

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$$= e^{-t} * e^{-2t}$$

$$= \int_0^t e^{-\tau} e^{-2(t-\tau)} d\tau$$

ILATE

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+1)}\cdot\frac{1}{(s+2)}\right) = e^{-t} - e^{-2t}$$

Ans

✯ IMPORTANT properties of CONVOLUTION Integral:

① COMMUTATIVE property :-

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$

② DISTRIBUTIVE property :-

$$f_1(t) * [f_2(t) + f_3(t)] = [f_1(t) * f_2(t)] + [f_1(t) * f_3(t)]$$

③ ASSOCIATIVE property :-

$$f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t)$$

④ SHIFT property :-

If $f_1(t) * f_2(t) = c(t)$

Then, $f_1(t) * f_2(t-T) = c(t-T)$

delay in result even if delay is given in one of the f^ns

OR

$f_1(t-T) * f_2(t) = c(t-T)$

$* \; f_1(t-T_1) * f_2(t-T_2) = c(t-(T_1+T_2))$.

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Ques. A sys. has
$$i/p = f(t) = e^{-t}\, u(t)$$
Unit impulse response $= h(t) = e^{-2t}\, u(t)$

Find o/p.

Now, o/p, $y(t) = f(t) * h(t)$

$$= \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t e^{-\tau}\, e^{-2(t-\tau)}\, d\tau$$

$$= \int_0^t e^{-\tau} \cdot e^{-2t} \cdot e^{2\tau}\, d\tau$$

$$= e^{-2t} \int_0^t e^{\tau}\, d\tau \quad \left(\because t \text{ is independent of } \tau \Rightarrow e^{-2t} = \text{constt} \right)$$

$$= e^{-2t}(e^{t} - 1)$$

$$\Rightarrow y(t) = e^{-2t}(e^{t} - 1)$$

$$\Rightarrow y(t) = e^{-t} - e^{-2t}$$ Ans

Q2) Given:-
$$\left.\begin{aligned} f(t) &= (1 - e^{-bt}) \\ h(t) &= e^{-at} \end{aligned}\right\} a = 0.5,\ b = 0.1$$

Find $y(t)$

Now, $y(t) = f(t) * h(t)$

$$= \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t (1 - e^{-b\tau})(e^{-a(t-\tau)})\, d\tau$$

$$= \int_0^t (1-e^{-b\tau})(e^{-at}\cdot e^{+a\tau})\,d\tau$$

$$= \int_0^t e^{-at}\cdot e^{a\tau}\,d\tau - \int_0^t e^{(a-b)\tau}\cdot e^{-at}\,d\tau$$

$$= e^{-at}\left[\int_0^t e^{a\tau}\,d\tau - \int_0^t e^{(a-b)\tau}\,d\tau\right]$$

$$= e^{-at}\left\{\left[\frac{e^{a\tau}}{a}\right]_0^t - \left(\frac{e^{(a-b)\tau}}{a-b}\right)_0^t\right\}$$

$$= e^{-at}\left\{\left(\frac{e^{at}}{a}-\frac{1}{a}\right) - \left(\frac{e^{(a-b)t}}{(a-b)} - \frac{1}{a-b}\right)\right\}$$

$$= e^{-at}\left[\frac{e^{at}}{a} - \frac{e^{at}}{a-b} - \frac{e^{-bt}}{a-b} + \frac{1}{a-b} - \frac{1}{a}\right]$$

$$= \frac{1}{a} - \frac{1}{a-b} - \frac{e^{-(a+b)t}}{a-b} + \frac{e^{-at}}{a-b} - \frac{e^{-at}}{a}$$

Now, $a = \frac{1}{2}$, $b = \frac{1}{10}$

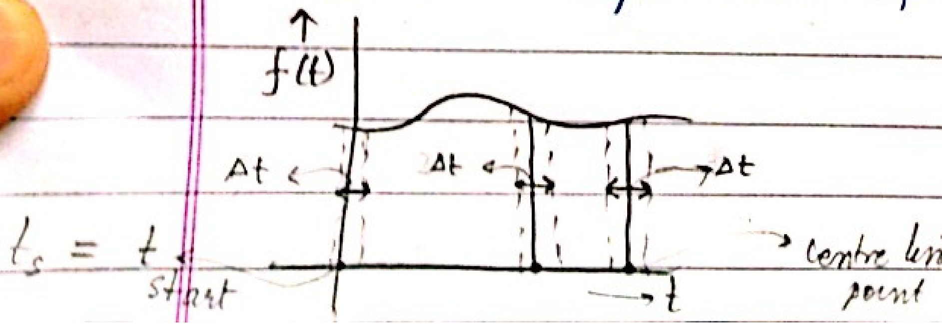
$$= 2 - \frac{5}{2} - \frac{e^{-(3t)/5}}{2/5} + \frac{e^{-t/2}}{2/5} - \frac{e^{-t/2}}{1/2}$$

$$\Rightarrow y(t) = -\frac{1}{2} - \frac{5}{2}\left[e^{-\frac{3t}{5}} - e^{-t/2}\right] - 2e^{-t/2}$$

Ans

☆ Mathematical explanation of CONVOLUTION Integral:

The signal given, can be practically considered as voltage.



Voltage Impulse = value × time

Area = f^n × abscissa

$f(t_s + \Delta t) \cdot \Delta t$

$f(t_s + 2\Delta t) \cdot \Delta t$

$f(t_s + 3\Delta t) \cdot \Delta t$

⋮

$f(t_s + n\Delta t)\, \Delta t$.

idea behind finding integral.

By changing Δt, we are progressing in front.

Now, o/p (at $t = t_0$)

$$y(t_0) = [f(t_s)\Delta t]\, h(t_0) + [f(t_s + \Delta t)\Delta t]\, h(t_0 - \Delta t) + [f(t_s + 2\Delta t)\Delta t]\, h(t_0 - 2\Delta t) + \cdots + [f(t_s + n\Delta t)\Delta t]\, h(t_0 - n\Delta t)$$

$h(t)\big|_{t=t_0}$

discrete summation

$\Delta t \simeq 0 \Rightarrow n \to \infty$

So, finally,

Discrete form
$$y(t)\Big|_{t=t_0} = y(t_0) = \sum_{i=0}^{n} [f(t_s + i\Delta t)\,\Delta t]\, h(t_0 - i\Delta t)$$

O/p

Cts form
$$y(t)\Big|_{t=t_0} = y(t_0) = \int_0^{t=t_0} f(\tau)\, h(t-\tau)\, d\tau$$

Hence, convolution formula got.

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ARTICLE 3.4

☆ TRIGONOMETRIC FOURIER SERIES
(HARMONIC ANALYSIS)

⇒ oscillⁿ.

↳ Basic Aspects / Applications / Physical significance:

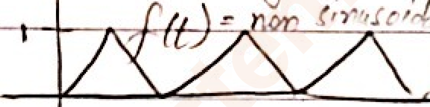
① A periodic f^n, $f(t) = f(t \pm T)$
or $= f(t \pm T_0)$ is considered.
where $T = T_0$ is the time period.
A f^n which repeats itself after a specific gap of time is known as periodic signal.

② "Fourier series" is a mathematical tool.

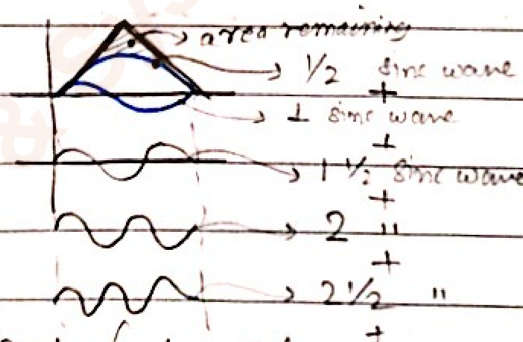
Basically, Fourier series is involved with MIXED FREQUENCY SIGNAL

Applicⁿ: full wave AC rectifier

Consider a f^n, $f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$



$= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$

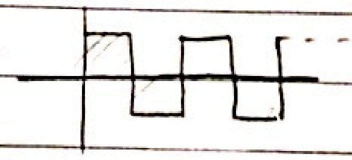


Idea: a non sinusoidal wave has to be written in terms of sine & cos. That is an infinite addⁿ

n = 1 : fundamental
n = 2 : 2nd harmonic
n = 3 : 3rd harmonic
⋮

☆ non sinusoidal signal used:
∵ easily available
: AC signal. Needs to be converted to sine & cos to use it (say, Indⁿ motor)

eg:



n = 2, 4, 6, . . . : even harmonic
n = 3, 5, 7 . . . : odd harmonic

* AC $\xrightarrow{\text{Rectifier}}$ DC   DC $\xrightarrow{\text{Inverter}}$ AC
* If $\exists$ mixed f_c in a signal, eddy current & hysteresis loss is more & sys. gets heated up.

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Fourier series is a transformⁿ from time domain to time domain.

(3) The formulⁿ w.r.t pt. no. (2) :

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

↳ $f(t)$: any time varying periodic signal with a period = $T = T_0$.

(4) $f(t)$ must consist of periodicity, several maxima & minima & discontinuity points
such f^{ns} are available to Fourier series applicⁿ.

(5) Standard Formulae: (for finding Fourier coeff.)

Fourier coefficient

$$\boxed{\begin{aligned} a_0 &= \frac{1}{T}\int_0^T f(t)\,dt \\ a_n &= \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt \\ b_n &= \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt. \end{aligned}}$$

→ frequency multiplier

*

—— x ——

• ORTHOGONAL f^{ns} : sine & cosine f^{ns}.
✓ At 90° to each other
✓ Integral over full period = 0 : Property of orthogonality.
$\left(\int_0^{2\pi} \sin\theta = \int_0^{2\pi} \cos\theta = 0\right)$

for $f = 50\,Hz$, $\omega = 314$ rad/s. $\}$ $n=1$, fundamental
$\omega = 2\times 314 \longrightarrow n=2$: Second harmonic

☆ Proof for expressions of a_n & b_n :

We know,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \longrightarrow (1)$$

ω: use ω or ω_0

$f(t) \rightarrow$ Period, $T = T_0$.

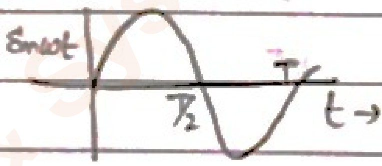
$\times$ both sides with $\sin n\omega t$ & integrating over full period, we get

$$\int_0^T f(t)\sin n\omega t\,dt = \int_0^T a_0 \sin n\omega t\,dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos n\omega t \sin n\omega t\,dt + \sum_{n=1}^{\infty} b_n \int_0^T \sin n\omega t \sin n\omega t\,dt \longrightarrow (2)$$

2 variables, n & t

Σ : varying with n (here)
$\int$: varying with t

Note :-



$\int_0^T \sin \omega t\,dt$ = Area for full time period

$\Rightarrow \int_0^T \sin\omega t\,dt = 0$.

or

$$\int_0^T \sin\omega t\,dt = \left(-\frac{\cos\omega t}{\omega}\right)_0^T = -\frac{\cos(\omega T)}{\omega} + \frac{\cos 0}{\omega}$$

$$= \frac{-1}{\omega}\left(\cos(2\pi) - \cos 0\right)$$

$2\pi \rightarrow = \omega t$

$$= \frac{-1}{\omega}(1-1) = 0.$$

☆ $n\pi = \omega t$

So, $\int_0^T \sin n\omega t\,dt = 0 \longrightarrow (3)$
(Illy)

$$\Rightarrow \int_0^T f(t)\sin n\omega t\,dt = 0 + \sum_{n=1}^{\infty} a_n \int_0^T \cos n\omega t \sin n\omega t\,dt$$

$0 \rightarrow$ from (3).

* The eq^n ① (α) has a lot of sine & cosine f^{ns}. So, if put in a transformer, there will be lot of hysteresis & eddy current losses

$$+\sum_{n=1}^{\infty} b_n \int_0^T \sin^2 n\omega t \, dt$$

$\rightarrow 0$ (same logic as in eq^n ①)

$$\Rightarrow \int_0^T f(t) \sin n\omega t = \frac{1}{2}\sum_{n=1}^{\infty} a_n \int_0^T \sin(2n\omega t)\, dt$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} b_n \int_0^T [1-\cos(2n\omega t)]\, dt$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} b_n \left\{ [T] - (\sin n\omega t)_0^T \right\}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} (b_n)(T) \qquad \rightarrow 0$$

$$\Rightarrow \int_0^T f(t) \sin n\omega t \, dt = \frac{T}{2}\sum_{n=1}^{\infty} b_n .$$

$$= \frac{T}{2} b_n$$ (Σ omitted $\because$ oper^n is done in time domain & is valid $\forall$ n)

$$= \left(\frac{2n\pi}{\omega}\right) b_n = \left(\frac{n\pi}{\omega}\right) b_n$$

$$\Rightarrow b_n = \frac{2}{T}\int_0^T f(t) \sin n\omega t \, dt$$

Extra : we see orthogonal f^{ns} in the above proof.

* Proof for finding a_n:
$\times$ $\cos n\omega t$, both sides & solve.

Aliter ☆ Finding b_n:

$$\int_0^T \sin^2 n\omega t \, dt = I_3 \text{ (say)}$$

$$\Rightarrow I_3 = \int_0^T \sin^2\theta \, dt \, , \quad \theta = n\omega t$$
$$\Rightarrow d\theta = n\omega \, dt$$

$$\Rightarrow I_3 = \frac{1}{n\omega}\int_0^{n\omega T} \sin^2\theta \, d\theta$$

$$\Rightarrow I_3 = \left(\frac{1}{n\omega}\right)\left(\frac{1}{2}\right)\int_0^{n\omega T}(1-\cos 2\theta)\,d\theta$$

$$= \left(\frac{1}{n\omega}\right)\left[\left(\frac{1}{2}\right)\left[\theta\right]_0^{n\omega T} - \underbrace{\left[\frac{\sin 2\theta}{2}\right]_0^{n\omega T}}_{\to 0}\right]$$

$$= \frac{1}{2}\left(\frac{n\omega T}{n\omega}\right)$$

$$\Rightarrow I_3 = \frac{T}{2} \rightarrow (4)$$

In the previous expression,

$$\int_0^T f(t)\sin(n\omega T)\,dt = \underbrace{\int_0^T a_0 \sin n\omega t \, dt}_{\to 0}$$
$$+ \underbrace{\int_0^T a_n \cos n\omega t \sin n\omega t}_{\to 0}$$
$$+ \int_0^T b_n \sin^2 n\omega t \, dt$$

Hence, $\int_0^T f(t)\sin(n\omega T)\,dt = b_n\left(\frac{T}{2}\right)$ (from (4))

$$\Rightarrow b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega T)\,dt$$

★ Proof of $\boxed{a_0}$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

Integrating both sides

$$\Rightarrow \int_0^T f(t)\,dt = a_0 \int_0^T dt + 0 + 0$$

(Integral over whole period)

$$\Rightarrow a_0 = \frac{1}{T}\int_0^T f(t)\,dt \,.$$

★ CONCEPT OF EXPONENTIAL FOURIER SERIES (F.S)

* Main Points : (1) What is it?

We know, $f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$

↳ trigonometric F.S.

then, $f(t) = \sum_{n\to\infty}^{+\infty} C_n e^{-jn\omega t} = \sum_{n\to-\infty}^{+\infty} D_n e^{-jn\omega t}$

↳ exponential F.S

exponential fourier coeff

(2) How did this series came into existence? → ①

✓ based on the background : $e^{j\theta} = \cos\theta + j\sin\theta$.

⇒ Implies that trigometric can be related to exponential term.

Based on eq^n ①,

$f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{-jn\omega t}$; can be easily solved.

(3) Advantage / Applicⁿ.

→ Advantage:

Exponential Fourier series, i.e., the BACKGROUND of TRANSFORMATION, called as Fourier Transform

→ Proof of

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{-(jn\omega t)} = \sum_{n=-\infty}^{+\infty} D_n e^{-jn\omega t}$$

Also,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t \, dt + \sum_{n=1}^{\infty} b_n \sin n\omega t \, dt$$

We know,

$$\cos n\omega t = \frac{e^{jn\omega t} + e^{-jn\omega t}}{2}, \quad \sin n\omega t = \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j}.$$

$$\Rightarrow f(t) = a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\omega t} + e^{-jn\omega t}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{jn\omega t} - e^{-jn\omega t}}{2j}\right]$$

$$\Rightarrow f(t) = a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n - j b_n}{2}\right] e^{jn\omega t} + \sum_{n=1}^{\infty} \left(\frac{a_n + j b_n}{2}\right) e^{-(jn\omega t)} \rightarrow ①$$

Now,

$$\sum_{n=1}^{\infty} \left(\frac{a_n + j b_n}{2}\right) e^{-jn\omega t}$$ find → equivalent to series

$$= \sum_{n \to -\infty}^{-1} \left(\frac{a_n - j b_n}{2}\right) e^{jn\omega t}$$

Imp. ✱ Trigonometric form (FS) $n \in (1, \infty)$
✱ Exponential form (FS) $n \in (-\infty, \infty)$

Finally, the series becomes,

$$f(t) = \sum_{n \to -\infty}^{\infty} C_n e^{+jn\omega t} \left(\text{or} \sum_{n \to -\infty}^{\infty} D_n e^{+jn\omega t} \right)$$

So, $\boxed{D_n = \frac{a_n - j b_n}{2}}$

So, $D_n = \frac{a_n - j b_n}{2} \left[\left(\frac{1}{2}\right)\left(\frac{1}{T}\right) \left\{ \int_0^T f(t) \cos n\omega t \, dt - \int_0^T f(t) \sin n\omega t \, dt \right\} \right]$

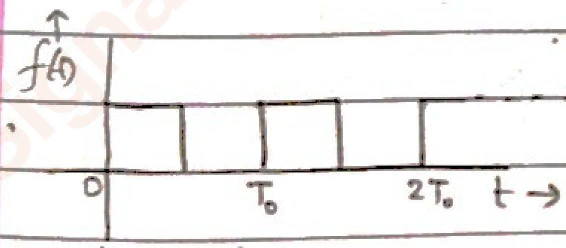
So,

$$D_n = \frac{a_n - j b_n}{2} = \frac{1}{2}\left(\frac{2}{T}\right) \left\{ \int_0^T f(t) \cos n\omega t \, dt - j \int_0^T f(t) \sin n\omega t \, dt \right\}$$

$$= \frac{1}{T} \int_0^T f(t) (\cos n\omega t - j \sin n\omega t) \, dt$$

$$\Rightarrow \boxed{D_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega t} \, dt} ; n \in (-\infty, \infty)$$

Problem.



$$f(t) = \begin{cases} 1 , & 0 < t < T_0/2 \\ 0 ; & T_0/2 < t < T_0 \end{cases}$$

Find the Trigonometric Fourier coeff.s or the series.

$$a_0 = \frac{1}{T_0} \int_0^{T_0} f(t) \, dt = \frac{1}{T_0} \int_0^{T_0/2} (1) \, dt + 0.$$

$$= \frac{1}{T_0} \left(\frac{T_0}{2} \right) = \frac{1}{2}.$$

* If a f^n is (not) symmetric about abscissa, its average value shouldn't vanish. ($a_0 \neq 0$)
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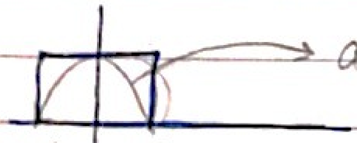
Prediction :- We know, b_n is associated with sine f^n.
So, for a f^n like

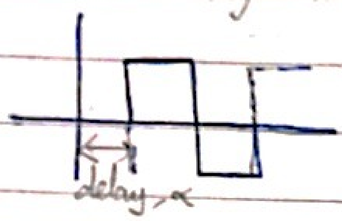
sine f^n coming inside it. So, $b_n \exists$.
& $a_n = 0$

a_n : associated with cosine f^n. So, for a f^n:
So.



$a_n \exists$. & $b_n = 0$

for a f^n with delay as :-



, both a_n & $b_n \exists$

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt = \frac{2}{T_0}\left[\int_0^{T_0/2} \cos n\omega_0 t\,dt + 0\right]$$

$$0 \rightarrow \int_{T_0/2}^{T_0}$$

$$= \frac{2}{n\omega_0 T_0}\left\{\sin n\omega_0 t\right\}_0^{T_0/2}$$

$$= \frac{2}{n\omega_0 T_0}\left(\sin \frac{n\omega_0 T_0}{2} - 0\right)$$

Now, $\omega_0 T_0 = 2\pi$

$$\Rightarrow \sin\left(\frac{n\omega_0 T_0}{2}\right) = \sin(n\pi) = 0.$$

$\Rightarrow a_n = 0$, verified with prediction.

$$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\left[\int_0^{T_0/2}(1)\sin n\omega_0 t\,dt + 0\right] \quad \int_{T_0/2}^{T_0} = 0$$

$$= \frac{2}{T_0}\left[\frac{-\cos n\omega_0 t}{n\omega_0}\right]_0^{T_0/2}$$

$$= \frac{-2}{T_0}\left[\frac{\cos\left(\frac{n\omega_0 T_0}{2}\right)}{n\omega_0} - \frac{1}{n\omega_0}\right]$$

$$\Rightarrow b_n = \frac{2}{n\omega_0}\left[1 - \cos\left(\frac{n\omega_0 T_0}{2}\right)\right] \longrightarrow (1)$$

now, $\omega_0 T_0 = 2\pi$

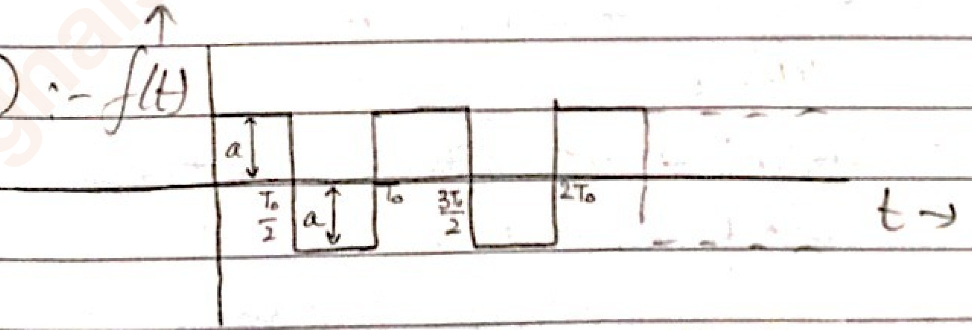
$\Rightarrow \frac{\omega_0 T_0}{2} = \pi$

$\Rightarrow \cos\left(\frac{n\omega_0 T_0}{2}\right) = \cos(n\pi) = (-1)^n$

$\Rightarrow$ eq^n (1) becomes

$$b_n = \frac{2}{2n\pi}\left[1 - (-1)^n\right] = \frac{1}{n\pi}\left[1 - (-1)^n\right]$$

Problem (2) :- $f(t)$



Find trigonometric Fourier coeff. or series.

Prediction :- (1) Graph symm. about abscissa $\Rightarrow a_0 = 0$.
(2) Sine fn. So, b_n should exist
(3) Cosine f^n not possible. So, a_n should vanish

Sol^n $a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt$

$$= \frac{1}{T_0}\left\{\int_0^{T_0/2} a\,dt + \int_{T_0/2}^{T_0} (-a)\,dt\right\}$$

$$= \frac{1}{T_0}\left(a\left(\frac{T_0}{2}\right) - a\left(\frac{T_0}{2}\right)\right)$$

$\Rightarrow a_0 = 0$ (Same as the prediction)

from fig, $f(t) = \begin{cases} a & ; 0 < t < T_0/2 \\ -a & ; T_0/2 < t < T_0 \end{cases}$

$$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\left[\int_0^{T_0/2} a\sin n\omega_0 t\,dt + \int_{T_0/2}^{T_0}(-a)\sin n\omega_0 t\,dt\right]$$

$$= \frac{2}{T_0}\left[(a)\left[\frac{-\cos(n\omega_0 t)}{n\omega_0}\right]_0^{T_0/2} - a\left(\frac{-\cos n\omega_0 t}{n\omega_0}\right)_{T_0/2}^{T_0}\right]$$

$$= \frac{2a}{n\omega_0 T_0}\left[-\cos(n\omega_0 t)\Big|_0^{T_0/2} + \cos(n\omega_0 t)\Big|_{T_0/2}^{T_0}\right]$$

$$= \frac{2a}{n\omega_0 T_0}\left[-\left[\cos\left(\frac{n\omega_0 T_0}{2}\right) - 1\right] + \left[\cos(n\omega_0 T_0) - \cos\left(\frac{n\omega_0 T_0}{2}\right)\right]\right]$$

$$\Rightarrow b_n = \frac{2a}{n\omega_0 T_0}\left[1 - \cos\left(\frac{n\omega_0 T_0}{2}\right) - \cos\left(\frac{n\omega_0 T_0}{2}\right) + \cos(n\omega_0 T_0)\right]$$

Now, $\omega_0 T_0 = 2\pi$

$\Rightarrow n\omega_0 T_0 = 2n\pi$

$$= \frac{2a}{2n\pi}\left[1 - 2\cos(n\pi) + \cos(2n\pi)\right]$$

$$\cos(n\pi) = (-1)^n = \cos(-n\pi)$$
$$\cos(2n\pi) = 1 = \cos(-2n\pi)$$

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$$\Rightarrow b_n = \frac{a}{n\pi}\left[1 + 1 - 2\cos n\pi\right]$$
$$= \frac{2a}{n\pi}\left[1 - \cos n\pi\right]$$
$$\Rightarrow b_n = \frac{2a}{n\pi}\left[1 - (-1)^n\right].$$

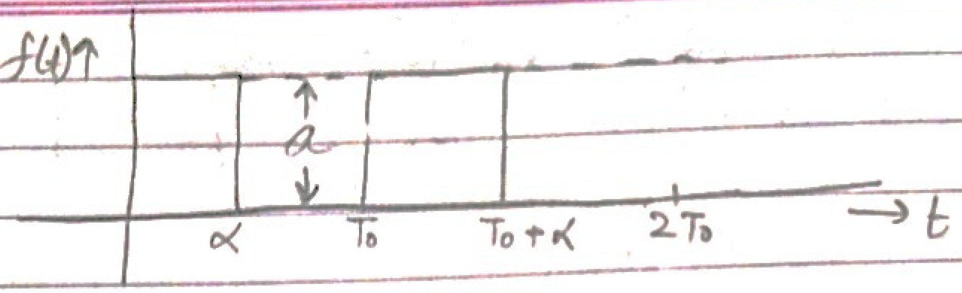
Now,

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)\,dt$$
$$= \frac{2}{T_0}\left[\int_0^{T_0/2} (a)\cos(n\omega_0 t)\,dt + \int_{T_0/2}^{T_0} (-a)\cos(n\omega_0 t)\,dt\right]$$
$$= \frac{2}{T_0}\left[\frac{a\sin(n\omega_0 t)}{n\omega_0}\Big|_0^{T_0/2} - \frac{a\sin(n\omega_0 t)}{n\omega_0}\Big|_{T_0/2}^{T_0}\right]$$
$$= \frac{2a}{n\omega_0 T_0}\left[\sin(n\omega_0 t)\Big|_0^{T_0/2} - \sin(n\omega_0 t)\Big|_{T_0/2}^{T_0}\right]$$
$$\Rightarrow a_n = \frac{2a}{n\omega_0 T_0}\left[\left(\sin\left(\frac{n\omega_0 T_0}{2}\right) - 0\right) - \left[\sin(n\omega_0 T_0) - \sin\left(\frac{n\omega_0 T_0}{2}\right)\right]\right]$$

Now $n\omega_0 T_0 = 2n\pi$

$$\Rightarrow a_n = \frac{2a}{2n\pi}\left[\sin\left(\frac{2n\pi}{2}\right) - \sin(2n\pi) + \sin\left(\frac{2n\pi}{2}\right)\right]$$
$$= \frac{2a}{2n\pi}\left[0 - 0 + 0\right]$$
$$\Rightarrow a_n = 0 \quad \text{(Just like what was predicted).}$$

Q. (Exponential form of Fourier Series)



Find the coeff (C_n or D_n) OR the exponential form of Fourier Series

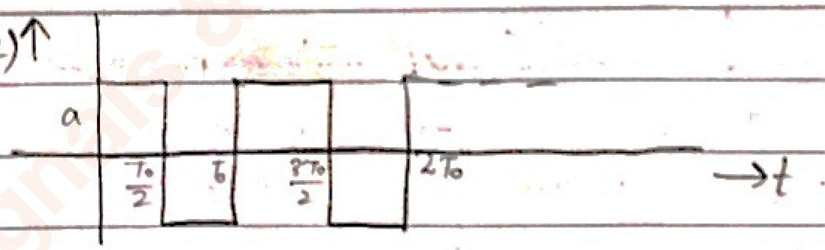
$$C_n = D_n = \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\left[\int_0^{\alpha} (a)e^{-jn\omega_0 t}\,dt + 0\right]$$

$$= \frac{a}{T_0(-jn\omega_0)}\left[e^{-jn\omega_0 t}\right]_0^{\alpha}$$

$$\Rightarrow D_n = \frac{a}{-jn(\underbrace{2\pi}_{\omega_0 T_0})}\left[e^{-jn\omega_0\alpha} - 1\right]$$

Q.



from the previous solⁿ done before — in Trigonometric F.S.

$$a_0 = 0,\ a_n = 0,\ b_n = \frac{2a}{n\pi}\left[1-(-1)^n\right]$$

range for trigonometric series $\Rightarrow n \in (1, \infty)$

Now, by formula

$$D_n = C_n = a_n - jb_n$$

$$\Rightarrow D_n = 0 - j\frac{2a}{n\pi}\left[1-(-1)^n\right] = -j\left(\frac{2a}{n\pi}\right)(1-(-1)^n)$$

Now, solve using exponential series & verifying for the same as of D_n got by Trigonometric form.

$$D_n = \frac{1}{T_0}\int_0^{T_0} f(t)\, e^{-jn\omega_0 t}\, dt \quad ; \; n \in (-\infty, \infty)$$

$$= \frac{1}{T_0}\left\{\left[\int_0^{T_0/2}(a)\, e^{-jn\omega_0 t}\, dt\right] + \left[\int_{T_0/2}^{T_0}(-a)\, e^{-jn\omega_0 t}\, dt\right]\right\}$$

$$= \frac{1}{T_0}\left\{\frac{a}{-jn\omega_0}\left(e^{-jn\omega_0 \frac{T_0}{2}} - 1\right) + \frac{(-a)}{(-jn\omega_0)}\left(e^{-jn\omega_0 T_0} - e^{-jn\omega_0 \frac{T_0}{2}}\right)\right\}$$

$$= \frac{a}{jn(2\pi)}\left\{1 - e^{-jn\omega_0 \frac{T_0}{2}} + e^{-jn\omega_0 T_0} - e^{-jn\omega_0 \frac{T_0}{2}}\right\}$$

$$= \frac{j}{j^2}\frac{a}{2n\pi}\left\{1 + e^{-j(2n\pi)} - 2e^{-j(n\pi)}\right\} \quad \left(\omega_0 T_0 = n\pi\right)$$

$$= -j\frac{a}{2n\pi}\left[1 + e^{j(-2n\pi)} - 2e^{j(-n\pi)}\right]$$

$$= -\frac{ja}{2n\pi}\left[1 + [\cos(-2n\pi) + j\sin(-2n\pi)] - 2[\cos(-n\pi) + j\sin(-n\pi)]\right]$$

$$= -\frac{ja}{2n\pi}\left[1 + [1 + j(0)] - 2[(-1)^n + j(0)]\right]$$

$$= -\frac{ja}{2n\pi}\left[2 - 2(-1)^n\right] \qquad \cos(-n\pi) = (-1)^n.$$

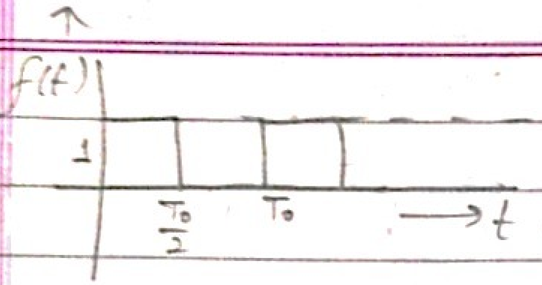
$$\Rightarrow D_n = -\frac{ja}{n\pi}\left[1 - (-1)^n\right] \quad ; \; n \in (-\infty, \infty)$$

$$\Rightarrow D_n = 2\left[-\frac{ja}{n\pi}\left(1 - (-1)^n\right)\right] \quad ; \; n \in (1, \infty)$$

range for trigonometric series

So, $D_n = -\frac{j2a}{n\pi}\left(1 - (-1)^n\right)$, same as prev. ans.

Q



As solved by Trigonometric method before,

$a_0 = 1/2$, $a_n = 0$

$b_n = \frac{1}{n\pi}\left[1-(-1)^n\right]$

$\Rightarrow D_n = a_n - j b_n$

$= 0 - j\left(\frac{1}{n\pi}\left[1-(-1)^n\right]\right)$

$\Rightarrow D_n = -\frac{j}{n\pi}\left[1-(-1)^n\right]$

$\rightarrow n \in (1, \infty)$

Solving using Exponential form of FS

$$D_n = \frac{1}{T_0}\int_0^{T_0} f(t)\, e^{-jn\omega_0 t}\, dt \quad ; \; n \in (-\infty, \infty)$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/2} (1)\, e^{-jn\omega_0 t}\, dt + \cancel{\int_{T_0/2}^{T_0} (0)\, e^{-jn\omega_0 t}\, dt}\right]$$

0

$$= \frac{1}{T_0}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_0^{T_0/2}$$

$$= \frac{-1}{jn(\omega_0 T_0)}\left[e^{-\frac{jn\omega_0 T_0}{2}} - 1\right]$$

$$= \frac{j}{j^2} \times \frac{1}{n(2\pi)}\left[1 - e^{-\frac{j(2n\pi)}{2}}\right]$$

$$= \frac{-j}{2n\pi}\left[1 - \left(\cos n\pi - j(\sin n\pi)\right)\right]$$

$$\Rightarrow D_n = \frac{-j}{2n\pi}\left[1-(-1)^n\right] \quad ; \; n \in (-\infty, \infty)$$

$$\Rightarrow D_n = 2\left[\frac{-j}{2n\pi}\left[1-(-1)^n\right]\right] \quad ; \; n \in (1, \infty)$$

$$\Rightarrow D_n = \frac{-j}{n\pi}\left[1-(-1)^n\right] \quad ; \; n \in (1, \infty)$$

→ same as prev. ans.

* Why always operation is seen in frequency domain:
Studying/developing a signal in time domain is difficult.

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Chapter - 4.

FOURIER TRANSFORM
(Continuous Fourier Transform)

§ INTRODUCTION/BASIC IDEA/APPLICATIONS/& DEFINITION

* MAIN POINTS :

1. It is a transformation (integral transformation) from time domain to PURE frequency (or frequency) domain.
2. Meaning of Pure frequency : → Laplace Transform
Pure, ∵ side by side, LT exists, where $s = \sigma + j\omega$ represents a complex frequency. So, specification of it, being pure is required.
3. With reference to pt. no. 1,

Transform^n from time domain to frequency domain

$$\boxed{F(j\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt}$$ **

→ Fourier transform
or
Direct fourier transform
or
mathematical Continuous Fourier Transform

Transform^n from frequency domain to time domain

4. Any transform must accompany an Inverse transform.
hence, Inverse F.T can be expressed as :

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega}$$

DIRICHILET's Condⁿ.

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5. If $\boxed{\left|\int_{-\infty}^{\infty} f(t)\, dt\right| < \infty}$ (i.e, area is finite)

then only, Fourier transform exists.

↳ valid, also for Fourier transform

6. Possible Application of Fourier Transform:

(a) Communication engineering

↳ modulation & demodulation.

(b) Parameter estimation using ~~3 Φ~~ F.T.

↳ for 3 Φ synchronous machine.

Q.1 If $f(t) = e^{-at} u(t)$, find $F(j\omega)$?

↳ for $a > 0$

$$F(j\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{-at} (0) e^{-j\omega t}\, dt + \int_{0}^{\infty} (1) e^{-t(a+j\omega)}\, dt$$

$$= 0 + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$\left(\because u(t) = \begin{cases} 1, & t \in (0, \infty) \\ 0, & t \in (-\infty, 0) \end{cases}\right)$$

$$\Rightarrow F(j\omega) = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty}$$

$$\Rightarrow F(\omega) = 0 - \left(\frac{1}{-(a+j\omega)}\right) = \frac{1}{a+j\omega}. \text{ Ans}$$

Note: We know,
$F(\omega) = \mathcal{F}(j\omega)$ is a complex no. { argument $= \tan^{-1}\left(\frac{y}{x}\right)$
$\Rightarrow$ we can find $\underbrace{|F(j\omega)| \ \& \ \angle F(j\omega)}_{\text{They are associated}}$

☆ Plot of $|F(j\omega)|$ vs ω : Amplitude Response
$\angle F(j\omega)$ vs ω : Phase response

eg: for $f(t) = e^{-at}u(t)$, (previous ques),

$$|F(j\omega)| = |F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\angle F(j\omega) = \arg[F(j\omega)]$$
$$= \arg\left[\frac{1}{a+j\omega}\right]$$
$$= \arg(1) - \arg(a+j\omega)$$

$1+0j$ $\quad$ $\left[\because \arg\left(\frac{a}{b}\right) = \arg(a) - \arg(b)\right]$

$$= \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{\omega}{a}\right)$$
$$= 0 - \tan^{-1}\left(\frac{\omega}{a}\right)$$
$$\Rightarrow \angle F(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

* CRITICAL OBSERVATIONS ON PREVIOUS Ques (Q.1)

1. What is the necessity for making $a>0$?
We had the integral $\int^{\infty} e^{-(a+j\omega)t}\,dt$
Clearly, integral f^n should converge as $t\to\infty$.
Now, if $a<0$, $\int_0^{\infty} e^{(a-j\omega)t}\,dt$ will come. So, we cannot talk about its convergence. It might rise up.
Convergence should be assured while doing FT.

* By Dirichlet's condⁿ, the FT of any fⁿ, $f(t)$ should converge. So, for FFT to be applicable, convergence of any fⁿ should be sure. so that a finite answer can be obtained.

* Table of STANDARD Fⁿˢ : FOURIER TRANSFORM

f^n	$F(\omega) = F(j\omega)$
• $e^{-at}(u(t))$	$\frac{1}{a+j\omega}$, $a>0$
• $e^{-at}u(-t)$	$\frac{1}{a-j\omega}$, $a>0$
• $e^{-a\|t\|}$	$\frac{2a}{s^2+\omega^2}$, $a>0$
• $te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$, $a>0$
• $t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$, $a>0$
• $\delta(t)$	1
• 1	$2\pi\,\delta(\omega)$

Q. $f(t) = \delta(t)$. Find its FT, $F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt.$$

$$\delta(t) = \begin{cases} \text{exists} & t=0 \\ 0 & t\neq 0 \end{cases}$$

$$= \int_{-\infty}^{\infty} \delta(t)\,e^{-j\omega(0)}\,dt = \int_{-\infty}^{\infty}\delta(t)\,dt$$

$\Rightarrow F(j\omega) = 1$ ($\because$ area of dirac-delta fⁿ $= 1$, from its definⁿ)

* $F\left[\frac{d^2 f(t)}{dt^2}\right] = (j\omega)^2 F(\omega)$

Properties of Fourier Transform.

P1. Symmetry Property

If $f(t) \longleftrightarrow F(\omega)$

(i.e., FT & inverse FT exists)

then, $F(t) \longrightarrow 2\pi f(-\omega)$

P2. Scaling Property

If $f(t) \longleftrightarrow F(\omega)$

then, for any real constt $= a$,

$$f(at) = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

P3. Time Shifting

If $f(t) \longleftrightarrow F(\omega)$

then, $f(t-t_0) \underset{F^{-1}}{\overset{F}{\longleftrightarrow}} e^{-j\omega t_0} F(\omega)$

→ Proof :-

$$F(f(t-t_0)) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t-t_0)\,dt$$

Let $t - t_0 = y$, then $dy = dy$ ($\because t_0 \text{ is const}$)

→ $t \to -\infty,\ y \to -\infty$

$t \to \infty,\ y \to \infty$

$$\Rightarrow F[f(t-t_0)] = \int_{-\infty}^{\infty} e^{-j\omega t_0} e^{-j\omega y} f(y)\,dy$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} e^{-j\omega y} f(y)\,dy$$

$$\left[\begin{aligned} \because e^{-j\omega t} &= e^{-j\omega(t-t_0+t_0)} \\ &= e^{-j\omega t_0} \cdot e^{-j\omega(t-t_0)} \\ &= e^{-j\omega t_0} \cdot e^{-j\omega y} \end{aligned}\right]$$

$$\Rightarrow F[f(t-t_0)] = e^{-j\omega t_0} F[f(y)]$$

$$\Rightarrow F[f(t-t_0)] = e^{-j\omega t_0} F[f(t-t_0)]$$

P5. → IIly in LT :-
If $f(t) \leftrightarrow sF(s)$
then $\frac{df(t)}{dt} = sF(s) - f(0)$.

P4. If $f(t) \longleftrightarrow F(\omega)$,

Frequency Shifting

then, $f(t)e^{j\omega_0 t} \longleftrightarrow F(\omega - \omega_0)$

↳ Proof :-

$$F\left(f(t)e^{j\omega_0 t}\right) = \int_{-\infty}^{\infty} [f(t)e^{j\omega_0 t}]e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}\, dt \rightarrow (1).$$

[By defin^n of f^n,
if $\int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt = F(\omega)$,
then $\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} f(t)dt = F(\omega-\omega_0)$]

$$\Rightarrow F\left(f(t)e^{j\omega_0 t}\right) = F(\omega - \omega_0)$$

P5. If $f(t) \longleftrightarrow F(\omega)$

Time Differentiation:

then, $\frac{df(t)}{dt} \longleftrightarrow j\omega F(\omega)$

↳ Proof :-

$$F\left(\frac{df(t)}{dt}\right) = ?$$

We know the formula for inverse FT

i.e, $F^{-1}(F\omega) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} F(\omega)d\omega. \rightarrow (A)$

↳ $f(t)$ is a f^n of ω & after solving only, we'll get $f(t)$ as a f^n of time.

Now, doing $\frac{df(t)}{dt}$ will be called as

Differentiation under sign of Integr^n

Differentiation w.r.t one variable (t) & integr^n w.r.t another variable (ω)

Partial differentiation
($\because$ 2 variables $\to t, \omega$)

So,

$$\frac{df(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(e^{j\omega t}\right) F(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)\, e^{j\omega t}\, F(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, [j\omega F(\omega)]\, d\omega$$

from (A)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, [F(\omega)]\, d\omega = f(t)$$

$$\Rightarrow \frac{d}{dt} f(t) = F^{-1}\left(j\omega F(\omega)\right)$$

P6. Time Integr^n.

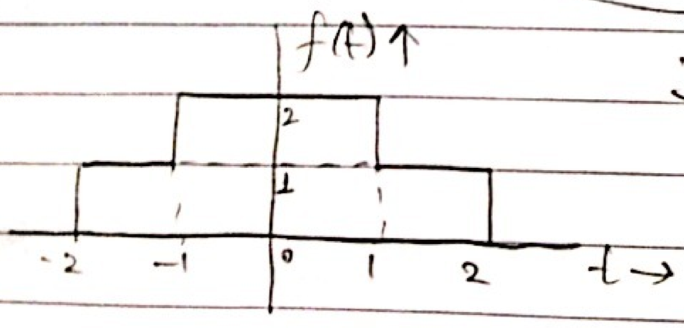
If $f(t) \longleftrightarrow F(\omega)$

then, $\int_{-\infty}^{t} f(\tau)\, d\tau = \frac{F(\omega)}{j\omega} + \pi F(0)\, \delta(\omega)$

$\to$ We know, $\delta(\omega) = \begin{cases} \infty, & \omega = 0 \\ 0, & \omega \neq 0 \end{cases}$

& $F(0) = F(\omega)\Big|_{\omega = 0}$

Problems on FT



Find :- Fourier Transform of given $f(t)$

Sol^n :- $F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$.

$$= \int_{-2}^{2} f(t)\, e^{-j\omega t}\, dt + (0) + (0)$$

$$\int_{-\infty}^{-2} (\;)\, dt \quad , \quad \int_{2}^{\infty} (\;)\, dt$$

$$f(t)\, e^{-j\omega t}$$

$$= \int_{-2}^{-1} (1)\, e^{-j\omega t}\, dt + \int_{-1}^{0} (2)\, e^{-j\omega t}\, dt$$

$$+ \int_{0}^{1} (2)\, e^{-j\omega t}\, dt + \int_{1}^{2} (1)\, e^{-j\omega t}\, dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-2}^{-1} + \left[\frac{2e^{-j\omega t}}{-j\omega}\right]_{-1}^{0} + \left[\frac{2e^{-j\omega t}}{-j\omega}\right]_{0}^{1}$$

$$+ \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{1}^{2}$$

$$= \left(\frac{-1}{j\omega}\right)\left\{ (e^{j\omega} - e^{2j\omega}) + (2 - 2e^{j\omega}) + (2e^{-j\omega} - 2) + (e^{-2j\omega} - e^{-j\omega}) \right\}$$

$$= \left(\frac{-1}{j\omega}\right)\left\{ e^{j\omega} - e^{2j\omega} + 2(e^{-j\omega} - e^{j\omega}) + (e^{-2j\omega} - e^{j\omega}) \right\}$$

$$= \left(\frac{-1}{j\omega}\right)\left\{ -e^{j\omega} + e^{-j\omega} \right\} + \left(\frac{-1}{j\omega}\right)\left[\frac{e^{-2j\omega} - e^{+2j\omega}}{1}\right]$$

$$\Rightarrow F(\omega) = \frac{e^{-j\omega} - e^{j\omega}}{-j\omega} + \left(\frac{2}{\omega}\right)\left[\frac{e^{2j\omega} - e^{-2j\omega}}{2j}\right]$$

Now, $\dfrac{e^{j\omega} - e^{-j\omega}}{j\omega} = \dfrac{(\cos\omega + j\sin\omega) - (\cos\omega - j\sin\omega)}{j\omega}$

$$= 2\sin\omega/\omega \longrightarrow (1)$$

We know

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta \rightarrow (2)$$

So, from (1) & (2)

$$\Rightarrow F(\omega) = \frac{2}{\omega}\left[\sin(\omega) + \sin 2(\omega)\right]$$

Ans

Problem Prove $\mathcal{F}(1) = 2\pi\,\delta(\omega)$

Sol^n:- $\displaystyle\int_{-\infty}^{\infty}(1)\,dt = t\Big|_{-\infty}^{\infty}\Big\} \Rightarrow$ doesn't converge

i.e, integral doesn't have finite value

So, FT of such a f^n by 1st principle can't possible (Dirichlet's cond^n fails)
→ sufficient cond^n

However, to solve it, limiting parameter can be applied to evaluate such FT &, this is as follows:

LIMITING PARAMETER METHOD

So, $\displaystyle \mathcal{F}(1) = \lim_{a\to 0}\int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\,dt \rightarrow (1)$

$e^{-a|t|}$ → converging factor

$\displaystyle \because \lim_{a\to 0}\int_{-\infty}^{\infty} e^{-at}\,dt \equiv \int_{-\infty}^{\infty}(1)\,dt$ as $a \to 0$

Now, $|t| = \begin{cases} -t & ; t<0 \\ t & ; t>0 \end{cases} \rightarrow (2)$

Using (2) in (1), we get,

* Note : In this proof, see :
We can put limit only on the constt a.
We cannot put it on t, as its varying from $-\infty$ to ∞.

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$$\Rightarrow \mathcal{F}(1) = \lim_{a\to 0} \left\{ \int_{-\infty}^{0} e^{-a(-t)} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-a(t)} e^{-j\omega t} dt \right\}$$

$$= \lim_{a\to 0} \left\{ \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-(a+j\omega)t} dt \right\}$$

$$\Rightarrow \mathcal{F}(1) = \lim_{a\to 0} \left\{ \left. \frac{e^{(a-j\omega)t}}{a-j\omega} \right]_{-\infty}^{0} \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_{0}^{\infty} \right\} \longrightarrow (3)$$

$$= \lim_{a\to 0} \left(\frac{1}{a-j\omega} + \frac{1}{a+j\omega} \right)$$

$$\Rightarrow \mathcal{F}(1) = \lim_{a\to 0} \left(\frac{2a}{a^2+\omega^2} \right) \longrightarrow (4).$$

With reference to eqn (4), the observations :-
→ When $\omega \neq 0$, $\mathcal{F}(1) = 0$
→ When $\omega = 0$.

$$\mathcal{F}(1) = \lim_{a\to 0} \left. \frac{2a}{a^2+\omega^2} \right|_{\omega=0}$$

$$\left(\equiv \frac{0}{0} \text{ form} = \text{Indeterminate form} \right)$$

Now, at $\omega = 0$

$$\mathcal{F}(1) = \lim_{a\to 0} \frac{2a}{a^2} \longrightarrow \infty$$ . . . continued

not a part of problem — General Concept

Consider $f(\omega) = \lim_{a\to 0} \frac{2a\omega^2}{a^2+\omega^2}$. At $\omega = 0$, its $\frac{0}{0}$ form.

Background of L' Hospital's Rule.

$f(\omega) = \frac{f_1(\omega)}{f_2(\omega)}$; $\lim_{\omega\to 0} f(\omega) = \frac{0}{0}$ form.

Then, by Rule, $\lim_{\omega\to 0} f(\omega) = \left. \frac{\frac{d}{d\omega} f_1(\omega)}{\frac{d}{d\omega} f_2(\omega)} \right|_{\omega=0}$

not part of problem.

hence,

$$\mathcal{F}(\omega)\Big|_{\omega=0} = \lim_{a\to 0} \frac{4a\omega}{a^2+2\omega}.$$

↳ with $a \to 0$ & $\omega = 0$, its again $\frac{0}{0}$ form.

So, applying L'Hospital's rule again

$$\therefore \mathcal{F}(\omega)\Big|_{\omega=0} = \lim_{a\to 0} \frac{4a}{a^2+2} \to 0$$

continued in problem.

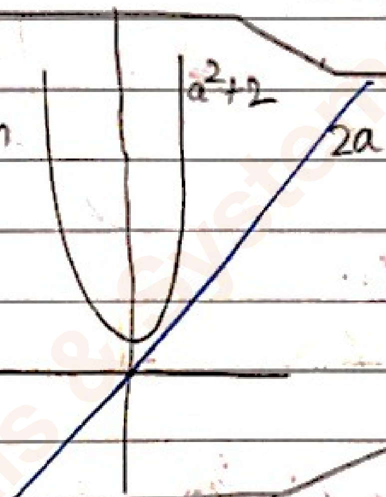
$$\text{So, } \mathcal{F}(1) = \lim_{a\to 0} \frac{2a}{a^2+\omega^2}$$

↳ $\omega \neq 0$; $\mathcal{F}(1) \to 0$

↳ $\omega = 0$; $\mathcal{F}(1) \to \infty$ → (5)

not part of problem.

Reason



$a^2+2 \to \infty$ faster than $2a \to \infty$, as seen from graph. So, $\frac{2a}{a^2+2} \to 0$.

Continued in problem

From (5),

$\mathcal{F}(1)$ must be a $\delta(\omega)$ function. But, its strength is not known

∴ Strength of Delta function in ω :-

let $I = \lim_{a\to 0} \int_{-\infty}^{\infty} \left(\frac{2a}{a^2+\omega^2}\right) d\omega$. (for strength, find its area in ω-plane ∵ its a f^n of ω

Let $\omega = a\tan\theta$
$\Rightarrow d\omega = a\sec^2\theta\, d\theta$
$\Rightarrow a^2+\omega^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$

So, $$I = \lim_{a\to 0}\int_{-\pi/2}^{\pi/2}\left(\frac{2a\,(a\sec^2\theta)}{a^2\sec^2\theta}\right)d\theta$$

$$= \lim_{a\to 0}(2)\int_{-\pi/2}^{\pi/2}\frac{a^2\sec^2\theta}{a^2\sec^2\theta}\,d\theta$$

$$= 2\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = 2\pi$$

So, finally,
Strength of $f^n = 2\pi$
Nature of $f^n = \delta(\omega)$.
So, value,
$$\mathcal{F}(1) = 2\pi\,\delta(\omega)$$

Note :- In the previous problem, a General Concept of L'Hospital's Rule was shown.
It is applicable only when $\exists\ \frac{0}{0}$ form in limit. So,
If $\exists$ the variable in both numerator & denominator, differentiate it to remove $\frac{0}{0}$ form & then evaluate limit.
In previous problem, f^n of ω wasn't there in numerator. So, this rule was NOT APPLICABLE.

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Q Prove :- $\mathcal{F}(e^{j\omega_0 t})\Big|_{\omega_0=0} = 2\pi\delta(\omega)$

We have proved in previous problem,

$\mathcal{F}(1) = 2\pi\delta(\omega)$.

Now, for $\mathcal{F}(e^{j\omega_0 t})\Big|_{\omega_0=0}$, we have

$= 2\pi\delta(\omega-0)$

$= 2\pi\delta(\omega)$.

= same as what was proved.

Q. Prove: $\mathcal{F}(u(t)) = \pi\delta(\omega) + \frac{1}{j\omega}$.

$$\mathcal{F}(u(t)) = \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0}(0)\,e^{-j\omega t}\,dt + \int_{0}^{\infty}(1)\,e^{-j\omega t}\,dt$$

$$= \frac{e^{-j\omega t}}{-j\omega}\Big|_{0}^{\infty}$$

$= \frac{1}{j\omega}$: NOT THE ANSWER (not complete)

Now, given the f^n, $u(t)$, we considered the effect at $t>0$. But due to discontinuity, $t<0$ wasn't considered. So, for $t<0$ we define a rel^n:-

$$u(t) = \frac{1}{2}\left[1 + sgn(t)\right]$$

→ signum f^n.

$$sgn(t) = \begin{cases} 1, & t>0 \\ -1, & t<0 \end{cases}$$

where $u(t) = \begin{cases} 1, & t>0 \\ 0, & t<0. \end{cases}$

So, $\mathcal{F}(u(t)) = \frac{1}{2}\mathcal{F}(1) + \frac{1}{2}\mathcal{F}(\mathrm{sgn}(t)) \longrightarrow (1)$

Now $\int_{-\infty}^{\infty} \mathrm{sgn}(t)\,dt = \int_{-\infty}^{0}(-1)\,dt + \int_{0}^{\infty} 1\,dt \longrightarrow \infty$

So, $\left|\int_{-\infty}^{\infty}\mathrm{sgn}(t)\,dt\right| \cancel{<} \infty$

So, Dirichlet's condn is not valid.
So, limiting parameter approach should be applied to find $\mathcal{F}(\mathrm{sgn}(t))$

So,

$$\mathcal{F}(\mathrm{sgn}(t)) = \lim_{a\to 0}\int_{-\infty}^{\infty} e^{-a|t|}\,\mathrm{sgn}(t)\,e^{-j\omega t}\,dt$$

$$\Rightarrow \mathcal{F}(\mathrm{sgn}(t)) = \lim_{a\to 0}\left[\int_{-\infty}^{0} e^{at}(-1)e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}\cdot e^{-j\omega t}(1)\,dt\right]$$

$$= \lim_{a\to 0}\left[\int_{-\infty}^{0} -e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt\right]$$

$$= \lim_{a\to 0}\left\{\left[-\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}\right\}$$

$$= \lim_{a\to 0}\left\{\frac{-1}{a-j\omega} - 0 + 0 - \left(\frac{1}{-(a+j\omega)}\right)\right\}$$

$$= \lim_{a\to 0}\left(\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right)$$

$$= \lim_{a\to 0}\left(\frac{-2j\omega}{a^2+\omega^2}\right)$$

as $a \longrightarrow 0$

$$\Rightarrow \mathcal{F}(\mathrm{sgn}(t)) = \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}.$$

From (1)

$$\mathcal{F}(u(t)) = \frac{1}{2}\mathcal{F}(1) + \frac{1}{2}\mathcal{F}(sgn(t))$$

$$= \frac{1}{2}\left(2\pi\delta(\omega)\right) + \frac{1}{2}\left(\mathcal{F}(sgn(t))\right)$$

$$\Rightarrow \mathcal{F}(u(t)) = \pi\delta(\omega) + \frac{1}{j\omega}. \quad Ans$$

Q. Find FT of $e^{j\omega_0 t}$

We know :-

$$\mathcal{F}(e^{j\omega_0 t}) = \int_{-\infty}^{\infty} e^{-j\omega t} \cdot e^{j\omega_0 t}\, dt$$

$$\Rightarrow \mathcal{F}(e^{j\omega_0 t}) = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\, dt \longrightarrow (1)$$

We knows

$$\mathcal{F}(1) = \int_{-\infty}^{\infty} e^{-j\omega t}(1)\, dt = 2\pi\delta(\omega) \rightarrow (2)$$

* By theory of function, observing eq^n (2), eq^n (1) can be defined as

$$\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\, dt = 2\pi\delta(\omega-\omega_0)$$

H.P

Q. Find out $\mathcal{F}(\cos\omega_0 t)$

$$\mathcal{F}(\cos\omega_0 t) = \int_{-\infty}^{\infty} e^{-j\omega t} \cdot \cos\omega_0 t\, dt$$

Now, $e^{j\theta} = \cos\theta + j\sin\theta$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\Rightarrow e^{j\theta} + e^{-j\theta} = 2\cos\theta$$

$$\Rightarrow \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \; ; \; \theta = \omega_0 t$$

$$\Rightarrow \mathcal{F}(\cos(\omega_0 t)) = \frac{1}{2}\mathcal{F}(e^{j\omega_0 t}) + \frac{1}{2}\mathcal{F}(e^{-j\omega_0 t})$$

$$= \frac{1}{2}\left(2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right)$$

$$\because \mathcal{F}(e^{j\omega_0 t}) = 2\pi\delta(\omega-\omega_0)$$

$$\Rightarrow \mathcal{F}(\cos(\omega_0 t)) = \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

Ans

Q. Find $\mathcal{F}(\sin\omega_0 t)$

Let $\omega_0 t = \theta$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\Rightarrow \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \; ; \; \theta = \omega_0 t$$

$$\Rightarrow \mathcal{F}(\sin\omega_0 t) = \frac{1}{2j}\mathcal{F}(e^{j\omega_0 t}) - \frac{1}{2j}\mathcal{F}(e^{-j\omega_0 t})$$

$$= \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

$$\Rightarrow \mathcal{F}(\sin\omega_0 t) = \pi j\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

Ans

* $f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau)\, d\tau$
* convolution $\equiv$ shifted f^n integration

§ CONVOLUTION IN THE SENSE OF F·T

$\hookrightarrow \mathcal{F}[f_1(t) * f_2(t)] = F_1(\omega) \cdot F_2(\omega)$

$\hookrightarrow F_1(\omega) = \mathcal{F}(f_1(t))$
$F_2(\omega) = \mathcal{F}(f_2(t))$

In the reverse,

$$f_1(t) \cdot f_2(t) = \frac{1}{2\pi}[F_1(\omega) * F_2(\omega)]$$

Note :- '$*$' : symbol for convolution in each frame, time (t) & ω.
$\hookrightarrow$ pure frequency

Proof :- Let $f_1(t) \rightleftharpoons F_1(\omega)$ (i.e FT & inverse exists)
$f_2(t) \rightleftharpoons F_2(\omega)$.

So, $\mathcal{F}(\,|\, f_1(t) * f_2(t)\,|\,) = \cancel{\int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau\right]}$

$$= \int_{-\infty}^{\infty} e^{-j\omega t}\left\{\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau\right\} dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\Big(\underbrace{\int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega t}\, dt}_{\to (1)}\Big)\, d\tau \quad \to (A)$$

(changing order of variables)

Solving further :- (M1) Use shifting property direct ~~as~~ result
(M2) Do substitution & solve.

* FT, by defin^n shows that its a transform^n from time domain to complex frequency domain. But, that need not always be w.r.t time domain

↳ see the problem below.

(M2) Let $t - \tau = x \Rightarrow t = \tau + x \Rightarrow dt = dx$

As $t \to -\infty$, $x \to -\infty$

$t \to +\infty$, $x \to +\infty$.

So, $$\int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f_2(x) e^{-j\omega(x+\tau)} dx$$

$$= e^{-j\omega\tau} \int_{-\infty}^{\infty} f_2(x) e^{-j\omega x} dx$$

Now, by defin^n of FT, $\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = F(\omega)$

in both cases, FT is applicable HERE.

{ So, from time domain (t) → frequency domain (ω)

& ||ly, space domain (x) → frequency domain (ω)

$$= e^{-j\omega\tau} F_2(\omega) \longrightarrow (2)$$

↳ result of shifting property.

Substituting (2) in (1), we get

$$\mathcal{F}(|f_1(t) * f_2(t)|) = \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau} F_2(\omega) d\tau$$

$$= F_2(\omega) \underbrace{\int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau} d\tau}_{\text{again, by defin}^n \text{ of FT, from } \tau \text{ domain to } \omega \text{ domain}}$$

$$\Rightarrow \mathcal{F}(|f_1(t) * f_2(t)|) = F_2(\omega) \cdot F_1(\omega)$$ H.P.

* FT forms the basis of Wavelet theory & (S)-transformⁿ : Modern Applicⁿˢ
↳ Stock hole #

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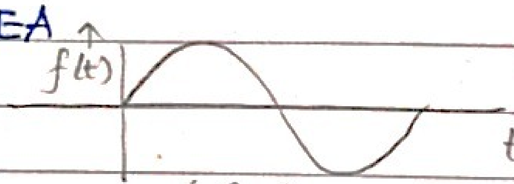
Q. $\mathcal{F}[f_1(t) f_2(t)] = 1$.
Prove the above

§. SAMPLING THEOREM (main applicⁿ in communicⁿ engineering)

- BASIC IDEA

① 
: f(t) is a cts time signal.

(fig 1)

②

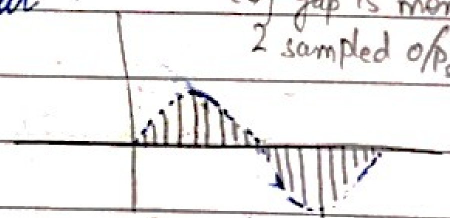
Discretize f(t) signal
↳ cut at diff pts. to find instt value.
So, in cts signal → ∃ SPAN.
in discrete signal → ∃ INSTANT

③ Physical significance : to find the 'instant'
time instant

④ Sampled output :
(If gap is more, we won't know how 2 sampled o/pₛ are related | Ø)

eg :

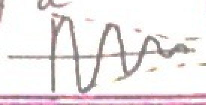
{for better sampled o/p, the distance (gap) should be very less}



(fig 2)

⑤ Graphical meaning of fig (2) :
Meaning : The locus of the tips of the sampled o/pₛ or lines must merge/lead to the cts original waveform in time domain

⑥ Mathematical expression of sampling.
Quantity = (No.) x (Gap b/w successive no.)
$\Rightarrow t = n(T_s) = n(\Delta T)$
↳ $T_s = \Delta T$ will be less as $n \to \infty$
(with t: finite)

* Sampling done : mainly done where the f^n is non linear ($\because$ in non linear, value of a neighbour is not clearly known). eg: 

Note: $t = n\,\Delta T$
$\hookrightarrow n \to \infty = 1/0$
$\Delta T \to 0$,
So, $t \to \frac{0}{0}$ form.
(That's why we write t : finite)

(i) eg.
Signal exists for 3 sec. (t)
Sampled at 1 ms rate (T_s)
No. of samples (n) = ?
$\Rightarrow n = \frac{3}{10^{-3}} = 3000$ samples.

* THEORY BEHIND SAMPLING :

• Main Points

1. Original signal : $f(t)$
2. $t = n T_s$ → sampling time; n → no. of samples; t ← quantized time in c/s domain
3. T_s can also be expressed as
$T_s = \frac{1}{F_s}$; F_s : sampling frequency
↳ should be more, so that T_s is low & thus, gap b/w samples is low.
4. Which is the imp. mathematical tool to implement the concept of sampling?
Idea :- many values, s.t, one value exists at only one pt. on graph & rest everywhere zero.
So, ans :- TRAIN of IMPULSE SIGNALS (Delta f^n)

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5. Mathematical expression of pt. (4).

$$\bar{f}(t) = f(t)\,\delta_T(t) = \sum f(nT_s)\,\delta(t-nT_s)$$

$\bar{f}(t)$ → Sampled o/p from $f(t)$

$f(t)$ → Original, cts time domain signal

$\delta_T(t)$ → Train of Impulses.
T → Period

6. $$\frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\theta} = \frac{1}{T_0}\left[\sum_{n=-\infty}^{-1} e^{jn\theta} + e^{jn\theta}\Big|_{n=0} + \sum_{n=1}^{\infty} e^{jn\theta}\right]$$

expressing → exponential Fourier Series of Delta f^n.

$\frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\theta} \equiv \delta_T(t)$ (with $T = T_0$)

$$= \frac{1}{T_0}\left\{\begin{aligned}&\left[(\cos\theta - j\sin\theta) + (\cos 2\theta - j\sin 2\theta) + (\cos 3\theta - j\sin 3\theta) + \dots\right]\\ &+ (\cos 0 + j\sin 0)\\ &+ \left[(\cos\theta + j\sin\theta) + (\cos 2\theta + j\sin 2\theta) + (\cos 3\theta + j\sin 3\theta) + \dots\right]\end{aligned}\right\}$$

$$\Rightarrow \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\theta} = \delta_T(t) = \frac{1}{T_0}\left\{\underbrace{1}_{\cos 0} + 2(\cos\theta + \cos 2\theta + \cos 3\theta + \dots)\right\}$$

$\delta_T(t)$ → with $T = T_0$; $\theta = \omega_s t$

7. $$f(t)\,\delta_T(t) = \frac{1}{T}\left[f(t) + 2f(t)\cos\omega_s t + 2f(t)\cos 2\omega_s t + 2f(t)\cos 3\omega_s t + \dots\right] \to (1)$$

$f(t)\,\delta_T(t) = \bar{f}(t)$ (from (5))

8. To find Fourier Transform of eq^n (1), (the temporary objective)

So,

$$\mathcal{F}(\text{LHS of eq}^n (1)) = \mathcal{F}(\bar{f}(t)) = \bar{F}(\omega)$$

$\bar{F}(\omega)$ → the way its defined

Now, we are left with finding FT of RHS of eq^n (1).

So find FT of each term on RHS of eq^n (1)

9. If $f(t) \xrightarrow{\mathcal{F}} F(\omega)$

Then, $\mathcal{F}[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} e^{j\omega_0 t} \, dt$

$$= \int_{-\infty}^{\infty} f(t) e^{-j(\omega-\omega_0)t} \, dt$$

$$= F(\omega - \omega_0)$$

(by theory of f^n)

So, Illy, $\mathcal{F}(f(t)e^{-j\omega_0 t}) = F(\omega + \omega_0)$

10. Now, seeing the RHS of eq^n (1), it is

$$\frac{1}{T}\left[f(t) + 2f(t)\cos\omega_s t + 2f(t)\cos 2\omega_s t + \cdots\right]$$

Now, $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$, $\theta = \omega_s t$

So, $\mathcal{F}(f(t)\cos(\omega_s t)) = \mathcal{F}\left(f(t)\dfrac{e^{j\theta}}{2}\right)$

$$+ \mathcal{F}\left(f(t)\frac{e^{-j\theta}}{2}\right)$$

$$= \frac{1}{2}\mathcal{F}(\omega - \omega_s) + \frac{1}{2}\mathcal{F}(\omega + \omega_s)$$

Illy, $\mathcal{F}(\underbrace{2f(t)\cos(\omega_s t)}) = 2\left[\frac{1}{2}\mathcal{F}(\omega - \omega_s) + \frac{1}{2}\mathcal{F}(\omega + \omega_s)\right]$

$$= \mathcal{F}(\omega - \omega_s) + \mathcal{F}(\omega + \omega_s)$$

one of the terms in RHS of eq^n (1).

Proceeding in same way, we get

$$\mathcal{F}\left(\underbrace{2f(t)\cos 2\omega_s t}_{\text{another term in RHS of eq}^n \text{ ①}}\right) = F(\omega - 2\omega_s) + F(\omega + 2\omega_s)$$

11. Finally, the applicⁿ of FT to both sides of eqⁿ ① leads to :

From pts ⑧ & ⑩

$$\bar{F}(\omega) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} F(\omega - \omega_s)$$

$$\rightarrow \frac{1}{T}\Big[F(\omega - \omega_s) + F(\omega - 2\omega_s) + \dots F(\omega - n\omega_s) + F(\omega)$$
$$F(\omega + \omega_s) + F(\omega + 2\omega_s) + \dots F(\omega + n\omega_s)$$

→ Physical meaning of −ve frequency (like $F(\omega - \omega_s)$)

Explanation ① $\omega = \frac{d\theta}{dt}$

$\theta \uparrow \Rightarrow \omega$ +ve
$\theta \downarrow \Rightarrow \omega$ −ve.
} So, ω −ve is feasible

Explanation ② :- ω is not −ve. The counting is −ve. i.e,

we have	
$F(\omega + (-1)\omega_s)$	$F(\omega + (1)(-\omega_s))$
$F(\omega + (-2)\omega_s)$	$F(\omega + 2(-\omega_s))$
⋮	⋮
$F(\omega + (-n)\omega_s)$ ✓	$F(\omega + n(-\omega_s))$ ✗

12. Successful Sampling :-

$$F_s \geq 2B$$

→ when $f(t)$ has a spectrum limited to B bandwidth.

Chapter : LAPLACE TRANSFORM

MAIN POINTS :

(1) → Defin^n :- (Transform^n from time domain → complex frequency)
Let $f(t)$: a time varying signal (f^n)
Then, $F(s) = \mathcal{L}[f(t)]$

$$\Rightarrow F(s) = \int_0^\infty e^{-st} f(t)\,dt$$: first principle formula
→ an Integral Transform

(2) → W.r.t point (1), s = complex frequency
$= \sigma + j\omega$

σ unit : neper/sec
ω unit : rad/sec.

(3) → Physical significance of kernel (s) :

$S = \sigma + j\omega$

(a) → σ : related to (or, produced due to) energy dissipating element in a physical sys, leading to damping phenomena.
electrical or mechanical.

ex:- σ is responsible (or related) to R (resistance) of R-L series circuit.

$e^{-Rt/L}$: damping component $\equiv \sigma$ (relation)

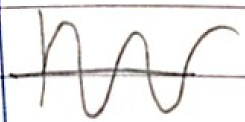
Hence, exponential (or log) $\equiv \sigma$.

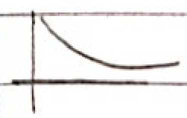
$\Rightarrow \sigma$: units neper. (another name for log : neper)

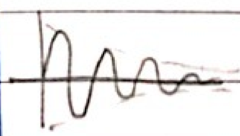
(b) → ω : imaginary part of 's' is produced due to (or related to) energy storing element (in a sys) leading to sustained oscill^n.

So, consider an R-L sys having i/p f^n as

$$f^n = \underbrace{e^{-Rt/L}}_{\text{exponential } f^n \text{ (leads to decay of the overall } f^n)} \underbrace{\sin \omega t}_{\text{a constt oscill}^n \text{ (sustained)}}$$

$\sin\omega t$ = 

$e^{-Rt/L}$ = 

$e^{-Rt/L}\sin\omega t$ = 

⇓

damping (due to R) $\equiv \sigma$.

sustainance (due to sine) $\equiv \omega$

- L & C : energy storing elements.
  ↳ in the forms like $\frac{1}{2}LI^2$, $\frac{1}{2}CV^2$.

⇓

Sustained oscill^n : due to energy storing.

- J (Moment of Inertia) : mechanical elemt. for energy storing.
  ↳ as $\frac{1}{2}J\omega^2$.

  where we have friction as damping factor.

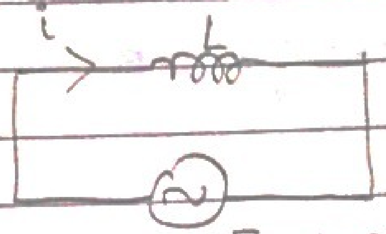
(c) → Rel^n (Interpret^n) of Sustained Oscill^n
↳ from view point of L or C.

(PTO)

Note :-

L is having sustained behaviour.

↳ eg :



$e = E_m \sin\omega t$ (= i/p)

Now,

$$|e| = L\frac{di}{dt}$$

$$\Rightarrow \frac{di}{dt} = \frac{|e|}{L} = \frac{E_m}{L}\sin\omega t$$

$$\Rightarrow i = \frac{E_m}{L}\int \sin\omega t\, dt$$

$$i = -\frac{E_m}{(L\omega)}\cos\omega t + C_1$$

↳ Integrⁿ constt.

(= o/p)

So, sustained oscillⁿ at i/p gives sustained o/p.

So, L shows sustained behaviour.

||ly, for capacitor.

(d) → "j" term in s is appearing due to :-

DELAY (in time)

↳ obeyed by 'L' or 'C'.

how? → see above :-

i/p : $\sin\omega t$, o/p → $\cos\omega t$

So, delay of $\pi/2$ (can be seen in Argon diagram)

* L & C obey Lenz's law.
that means, they depend on dt

$\because |e| = N\frac{d\phi}{dt}$

(by lenz's law)

$\because$ it depends on dt; giving an i/p of $E_m \sin\omega t$, o/p (current) won't come immediately. It'll come only after 'dt' time.
So, from 0 to dt, the behaviour of inductance is not shown.

why?

($\because e = L\frac{di}{dt}$
giving emf, we should get current)
Only possible if L gives value immediately.
(But, due to delay, it won't)

So, the property from 0 to dt is imaginary. Hence, "j" is there, (with ω)

(4) * Rules of LTI :-

(i) linearity

If $f_1(t) \xrightarrow{L} F_1(s)$

$f_2(t) \xrightarrow{L} F_2(s)$

Then,
$$\mathcal{L}\left[a f_1(t) \pm b f_2(t)\right]$$
$$= a F_1(s) \pm b F_2(s)$$
s : Laplace domain.

(ii) Time shifting property.

If $f(t) \xrightarrow{\mathcal{L}} F(s)$

Then,
$$\mathcal{L}\left[f(t-a)\, u(t-a)\right] = e^{-as} F(s)$$

(iii) Frequency shifting property :-

If $f(t) \xrightarrow{\mathcal{L}} F(s)$

Then
$$\mathcal{L}\left[e^{s_0 t} f(t)\right] = F(s-s_0)$$

Or, if $f(t) \xrightarrow{\mathcal{L}} F(s)$

then, $\mathcal{L}\left[e^{-at} f(t)\right] = F(s+a)$

(iv) Time differentiation :-

If $f(t) \xrightarrow{\mathcal{L}} F(s)$

then,
$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = sF(s) - f(0^-)$$
$$= sF(s) - f(0)$$

$\hookrightarrow f(0) = f(t)\Big|_{t=0}$

& $\mathcal{L}\left(\frac{d^2 f(t)}{dt^2}\right) = s^2 F(s) - s f(0) - f'(0)$

$\hookrightarrow f'(0) = \left.\frac{df(t)}{dt}\right|_{t=0}$.

||ly, $\mathcal{L}\left(\frac{d^n f(t)}{dt^n}\right) = s^n F(s) - s^{n-1} f(0)$

$- s^{n-2} f'(0) - s^{n-3} f''(0)$

$- \cdots$

→ (V) Time Integration property :-

If $f(t) \xrightarrow{\mathcal{L}} F(s)$

Then,

(i) $\mathcal{L}\left[\int f(t)\,dt\right] = \frac{1}{s} F(s) + \frac{1}{s} f^{-1}(0)$

$\hookrightarrow f^{-1}(0) = \left.\int f(t)\,dt\right|_{t=0}$

(ii) $\mathcal{L}\left[\int_{-\infty}^{t} f(t)\,dt\right] = \frac{1}{s} F(s)$

Proof:-

Let $\int f(t)\,dt = g(t)$

$\therefore \frac{d}{dt} g(t) = f(t)$

* Laplace inverse notations:- L^{-1} or $\mathcal{L}^{-1}$

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Proof: Applying LT.
continued

$\Rightarrow \mathcal{L}\left[\frac{d}{dt}g(t)\right] = F(s)$

Using Differentiation theorem of LT:

$sG(s) - g(0) = F(s)$

↳ when $G(s) = \mathcal{L}\, g(t)$.

$\Rightarrow G(s) = \frac{1}{s}F(s) + \frac{1}{s}g(0)$

$= \frac{1}{s}F(s) + \frac{1}{s}\left[\int f(t)dt\Big|_{t=0}\right]$

$= \frac{1}{s}F(s) + \frac{1}{s}f^{-1}(0)$

→ (vi) Regarding Inverse LT:-

If $f(t) \xrightarrow{L} F(s)$

Then $F(s) = L[f(t)]$

& $f(t) = \mathcal{L}^{-1}F(s)$

$= L^{-1}[F(s)]$

→ (a) If

$L(f_1(t)) = F_1(s)$

& $L(f_2(t)) = F_2(s)$.

Then,

$L^{-1}(aF_1(s) \pm bF_2(s))$

$= a\, f_1(t) \pm b f_2(t)$

↳ a, b : constts.

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(b) If $L^{-1}(F_1(s)) = f_1(t)$
& $L^{-1} F_2(s) = f_2(t)$

then, $L^{-1}[F_1(s) \pm F_2(s)]$
$= f_1(t) \pm f_2(t)$

(c) If $L^{-1} F(s) = f(t)$

then, $L^{-1}(a F(s)) = a f(t)$
↳ a : constt.

(d) If $L^{-1} F_1(s) = f_1(t)$
& $L^{-1} F_2(s) = f_2(t)$

then,
$L^{-1}[F_1(s) \cdot F_2(s)] \neq f_1(t) \cdot f_2(t)$

$L^{-1}[F_1(s) \cdot F_2(s)] = f_1(t) * f_2(t)$

↳ * : convolution property in the SENSE OF LT

$\Rightarrow L^{-1}[F_1(s) \cdot F_2(s)] = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$

$= \int_0^t f_1(t-\tau) f_2(\tau) d\tau$

$\rightarrow$ * $f_1(t) \cdot f_2(t) = \frac{1}{2\pi j}[F_1(s) * F_2(s)]$

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example Find $L^{-1}\left(\frac{1}{s(s+2)}\right)$

Method 1 : Using Partial fraction :

By Inspection. or solve using $\frac{A}{s} + \frac{B}{s+2}$.

$$\text{Let } F(s) = \frac{1}{s(s+2)}$$

$$= \frac{1}{2}\left[\frac{2}{s(s+2)}\right]$$

$$= \frac{1}{2}\left[\frac{(s+2)-s}{s(s+2)}\right]$$

$$\Rightarrow F(s) = \frac{1}{2s} - \frac{1}{2(s+2)}$$

$$\text{So, } L^{-1}(F(s)) = \frac{1}{2}(u(t)) - \frac{1}{2}e^{-2t}$$

↳ using property (vi)(b)

Method 2 : Using Convolution Integral .

$$\text{Let } F(s) = \frac{1}{s(s+2)} = F_1(s) \cdot F_2(s), \text{ say,}$$

$$\text{where } F_1(s) = \frac{1}{s} \xrightarrow{\mathcal{L}^{-1}} f_1(t) = 1$$

$$F_2(s) = \frac{1}{s+2} \xrightarrow{\mathcal{L}^{-1}} f_2(t) = e^{-2t}.$$

$$\text{Now } \mathcal{L}^{-1} F(s) = f_1(t) * f_2(t)$$

$$= \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau.$$

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$$= \int_0^t (1) e^{-2\tau} d\tau$$

$$= \left[\frac{e^{-2\tau}}{-2}\right]_0^t$$

$$= \left[\frac{e^{-2t}}{-2} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[1 - e^{-2t}\right]$$

$$\Rightarrow \mathcal{L}^{-1}[F(s)] = \frac{1}{2}\left[u(t) - e^{-2t}\right]$$

$\hookrightarrow$ same as in M1

Q. Find $L^{-1}\left(\frac{1}{(s+2)^2+3}\right)$

Method 1 : Applying std. theorem.

Std. thm

If $f(t) \xrightarrow{L} F(s)$

then, $\mathcal{L}\left[e^{-at} F(t)\right] = F(s+a)$

Let, in the above problem,

$a = 2,\ b = \sqrt{3}$

hence, $L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right] = ?$

From above expression,

$$F(s) = \frac{1}{s^2+b^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+b^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{b}\right)\left(\frac{b}{s^2+b^2}\right) = \frac{1}{b}\mathcal{L}^{-1}\left(\frac{b}{s^2+b^2}\right)$$

☆ Formulas of LT are there in the next half of book (control systems)

Using table, $\frac{1}{\omega} \mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{1}{\omega} \sin \omega t$

So, $\mathcal{L}^{-1}\left(\frac{1}{(s+a)^2+b^2}\right) = e^{-at}\left(\frac{1}{b}\right)\sin bt$.

So, $\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+(\sqrt{3})^2}\right] = e^{-2t}\left(\frac{1}{\sqrt{3}}\right)\sin\sqrt{3}t$

Method 2 : Using Partial fraction.

Now, $(s+2)^2+3 = s^2+4s+4+3$
$= s^2+4s+7$

Now, $s^2+4s+7=0$

$\Rightarrow s = \frac{-4 \pm \sqrt{16-28}}{2}$

$= \frac{-4 \pm j2\sqrt{3}}{2}$

$\Rightarrow s = -2 \pm j\sqrt{3}$.

So, $s_1 = -2+j\sqrt{3}$
$s_2 = -2-j\sqrt{3}$. } complex conj roots (or POLES)

$$\Rightarrow \frac{1}{(s+2)^2+3} = \frac{1}{(s-s_1)(s-s_2)}$$

$$= \frac{K_1}{s-s_1} + \frac{K_2}{s-s_2}$$

$$\text{or} = \frac{1}{(s_2-s_1)}\left[\frac{(s-s_1)-(s-s_2)}{(s-s_1)(s-s_2)}\right]$$

$$= \frac{1}{-2j\sqrt{3}} \left[\frac{1}{s - s_2} - \frac{1}{s - s_1} \right]$$

$$= \frac{-1}{j2\sqrt{3}} \left[\frac{1}{s - s_2} - \frac{1}{s - s_1} \right]$$

So, $\mathcal{L}^{-1}\left(\frac{1}{(s-s_1)(s-s_2)} \right) = \frac{-1}{2j\sqrt{3}} \left[\mathcal{L}^{-1}(s-s_2) - \mathcal{L}^{-1}(s-s_1) \right]$

$$= \frac{-1}{2j\sqrt{3}} \left[e^{s_2 t} - e^{s_1 t} \right]$$

Now $s_1 = -2 + j\sqrt{3}$

& $s_2 = -2 - j\sqrt{3}$

$\Rightarrow f(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-s_1)(s-s_2)} \right) = \frac{j}{2\sqrt{3}} \left[e^{-2t} \cdot e^{-j\sqrt{3}t} - e^{-2t} \cdot e^{j\sqrt{3}t} \right]$

$\hookrightarrow$ say

$\Rightarrow f(t) = \frac{j}{2\sqrt{3}} e^{-2t} \left[e^{-j\theta} - e^{j\theta} \right]$

$\hookrightarrow \theta = \sqrt{3}t$

Now, $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow f(t) = \frac{j}{2\sqrt{3}} \cdot e^{-2t} \left[-2j \sin\theta \right]$

$\Rightarrow f(t) = \frac{e^{-2t}}{\sqrt{3}} (\sin\sqrt{3}t)$

Hence, verified

☆ ~~SPECIAL NOTE:~~
~~(Ref.~~

Q. Find $L^{-1}\left[\frac{8s+10}{(s+1)(s+2)^3}\right]$

here, $s=-1$: discrete pole
$s=-2$: repeated poles, order 3

Continued ↓

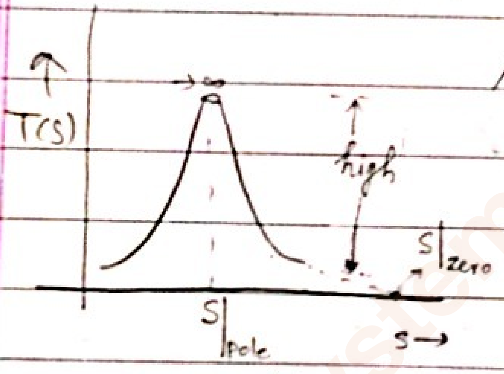
History of Concepts.

☆ Concept of zero & pole

Let TF be $T(s)$, s.t

$$T(s) = \frac{\text{Numerator}(s)}{\text{Denominator}(s)} = \frac{N(s)}{D(s)}$$

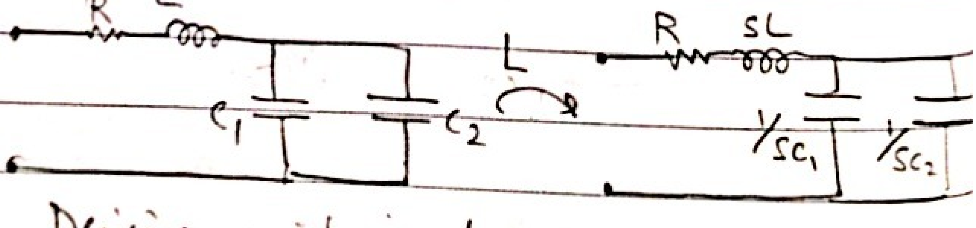
↳ If $D(s)=0$, $T(s) \rightarrow \infty$.



At '0' value of $D(s)$, $T(s)$ tends to infinity, so, geographically, it reaches at top, so, its roots are called POLES & IIly, $N(s)=0$ gives ZEROS.

☆ BASIC Concept,

Consider the circuit



Driving point impedance:-

when LT applied to a time varying circuit, the transformⁿ occurs as:

Impedance $\left(\frac{\text{Voltage}}{\text{current}}\right)$ • $R \rightarrow R$

$$\left[\begin{aligned} &\because v = iR \\ &\Rightarrow V(s) = I(s)R \end{aligned}\right]$$

$\Rightarrow R \rightarrow R$

Parts of Driving point impedance.

- $L \rightarrow sL$

  $\left(\because e = L\frac{di}{dt}\right.$

  Applying LT (initial)

  $\Rightarrow E(s) = Ls\,I(s) - 0$

  $\Rightarrow \frac{E(s)}{I(s)} = Ls$

  So, $L \rightarrow sL$)

- $C \rightarrow 1/sC$

  $\left(\because i = C\frac{dv}{dt}\right.$

  $\Rightarrow$ Appling LT.

  $\Rightarrow I(s) = Cs\,V(s) - 0$

  $\Rightarrow \frac{V(s)}{I(s)} = \frac{1}{Cs}$

  $\Rightarrow C \rightarrow 1/sC$)

* $\boxed{\text{Network } f^n}$

  → A f^n which is a TF, Driving pt impedance or Driving point Admittance

  → Any network f^n of the form $\frac{N(s)}{D(s)}$

Ques. continued

$$F(s) = \frac{8s+10}{(s+1)(s+2)(s+2)(s+2)} \longrightarrow (1)$$

$$\Rightarrow F(s) = \frac{k_1}{s+1} + \frac{a_0}{(s+2)^3} + \frac{a_1}{(s+2)^2} + \frac{a_2}{(s+2)} \longrightarrow (2)$$

Multiplying by $(s+1)(s+2)^3$ both sides, we get

$$\Rightarrow 8s+10 = k_1(s+2)^3 + a_0(s+1) + a_1(s+1)(s+2) + a_2(s+1)(s+2)^2$$

Put $s = -1$

$$\Rightarrow k_1 = \left.\frac{8s+10}{(s+2)^3}\right|_{s=-1}$$

$$\Rightarrow k_1 = \frac{-8+10}{(-1+2)^3} = 2$$

Put $s = -2$

$$\Rightarrow a_0 = \left.\frac{8s+10}{s+1}\right|_{s=-2}$$

$$\Rightarrow a_0 = \frac{-16+10}{-2+1} = 6$$

Now, for finding a_1 & a_2, direct putting of any value of s not possible

So,

M1 Equating coeff on both sides

$$8s+10 = 2(s+2)^3 + 6(s+1) + a_1(s+1)(s+2) + a_2(s+1)(s+2)^2$$

$$\Rightarrow 8s+10 = 2(s^3+6s^2+12s+8) + 6s+6 + a_1(s^2+3s+2) + a_2(s+1)(s^2+4s+4)$$

$$= 2s^3+12s^2+24s+16 + 6s + 6 + a_1s^2 + 3a_1s + 2a_1 + a_2s^3 + 4a_2s^2 + 4a_2s + a_2s^2 + 4a_2s + 4a_2$$

$$\Rightarrow 8s+10 = s^3(2+a_2) + s^2(12+a_1+4a_2+a_2) + s(24+6+3a_1+8a_2) + (16+6+2a_1+4a_2)$$

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$\Rightarrow 10 = 22 + 2a_1 + 4a_2$

$\Rightarrow 5 = 11 + a_1 + 2a_2$

&

$8 = 30 + 3a_1 + 8a_2$

$0 = 2 + a_2 \Rightarrow \boxed{a_2 = -2}$

$0 = 12 + a_1 + 5a_2$

$\Rightarrow 12 + a_1 - 10 = 0$

$\Rightarrow \boxed{a_1 = -2}$ — (3)

Using (3) in (2).

$$\Rightarrow F(s) = \frac{2}{s+1} + \frac{6}{(s+2)^3} - \frac{2}{(s+2)^2} - \frac{2}{s+2}$$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{2}{s+1}\right) + 6\,\mathcal{L}^{-1}\left(\frac{1}{(s+2)^3}\right) - 2\,\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)$$
$$- 2\,\mathcal{L}^{-1}\left(\frac{1}{(s+2)}\right)$$

$$\Rightarrow f(t) = 2e^{-s} + 6\left(\frac{t^2 e^{-2t}}{2!}\right) - 2te^{-2t} - 2e^{-2t}$$

Ans

Sth like chain rule in LT.

$\because L(t^n) = \frac{n!}{s^{n+1}}$

Also, If $f(t) \xrightarrow{\mathcal{L}} F(s)$
then $e^{-at} f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s+a)$

So, given $\frac{1}{(s+2)^3} = F(s+a)$

$\Rightarrow F(s) = \frac{1}{s^3}$

Now, $\frac{1}{(s+2)^3} = \left(\frac{1}{2!}\right) \frac{2!}{(s+2)^3}$

$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s+2}\right)^3 = \frac{1}{2!}\,\mathcal{L}^{-1}\left(\frac{2!}{(s+2)^3}\right)$

$\Rightarrow \mathcal{L}^{-1}(1/s+2)^3 = (1/2)\, e^{-2t}\, (t^2)$

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Q. If $f(t) \xrightarrow{L} F(s)$

Then, $\mathcal{L}\{t f(t)\} = ?$

Sol^n :- $F(s) = \int_0^\infty e^{-st} f(t)\,dt \longrightarrow (1)$

Idea :- we want to bring t on RHS of (1).

Now, $\frac{d}{dt} e^{at} = a e^{at} \Rightarrow \frac{d}{ds} e^{-st} = -t e^{-st}$ ★.

So, differentiate both side w.r.t s.

So, LHS :- $\frac{d}{ds} F(s) = \frac{\partial}{\partial s} F(s)$

$$\Rightarrow \frac{\partial}{\partial s} F(s) = \int_0^\infty \left(\frac{\partial}{\partial s} e^{-st}\right) f(t)\,dt$$

(using Lebnitz's thm : Topic : Differentiation under sign of integration

$$= -\int_0^\infty e^{-st} (t f(t))\,dt$$

$$\Rightarrow \frac{\partial}{\partial s} F(s) = -\mathcal{L}[t f(t)]$$

$$\Rightarrow L[t f(t)] = (-1)^1 \frac{\partial}{\partial s} F(s) \text{ or } (-1)^1 \frac{d}{ds} F(s)$$

$\hookrightarrow$ ||ly, $L[t^2 f(t)] = (-1)^2 \frac{\partial^2}{\partial s^2} F(s)$

Imp ★ $\boxed{L[t^n f(t)] = (-1)^n \frac{\partial^n F(s)}{\partial s^n} \quad \hookrightarrow n \in \mathbb{Z}}$

✱ APPLIC^n OF LT TO D.E.

Q. Solve :- $(D^2+5D+6)\,y(t) = (D+1)\,f(t)$

Given :-
$$\left.\begin{aligned} f(t) &= e^{-4t}u(t) \\ y(t=0) = y(0) &= 2 \\ \frac{dy}{dt}(t=0) = \dot{y}(0) &= 1 \end{aligned}\right\} \rightarrow (1)$$

Sol^n :- (M1) LHS.

$$(D^2+5D+6)\,y(t) = \frac{d^2y(t)}{dt^2} + 5\frac{d}{dt}y(t) + 6y(t)$$

Applying $\mathcal{L}$(LHS terms), we get

$$= \mathcal{L}\left[\frac{d^2y(t)}{dt^2}\right] + 5\mathcal{L}\left[\frac{dy(t)}{dt}\right] + 6\mathcal{L}[y(t)]$$

let $\mathcal{L}(y(t)) = Y(s)$

$$= \left[s^2Y(s) - s\,y(0) - y'(0)\right] + 5\left[sY(s) - y(0)\right] + 6\,Y(s) \longrightarrow (2)$$

$$\left(\begin{aligned} &\text{✱ If } f(t) \xrightarrow{\mathcal{L}} F(s) \\ &\mathcal{L}\,\frac{d^2f(t)}{dt^2} = s^2F(s) - sf(0) - f'(0) \\ &\mathcal{L}\,\frac{df(t)}{dt} = sF(s) - f(0) \end{aligned}\right)$$

RHS

$$\mathcal{L}[(D+1)f(t)] = \mathcal{L}\left[\frac{df(t)}{dt} + f(t)\right] \rightarrow (3)$$

$$= \mathcal{L}\left[\frac{d}{dt}f(t)\right] + \mathcal{L}(f(t))$$

Now, $f(t) = e^{-t} u(t)$.

So, $\mathcal{L}[f(t)] = \mathcal{L}[e^{-4t} u(t)]$

$$\Rightarrow F(s) = \frac{1}{s+4} \rightarrow (4)$$

& $$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$= s\left(\frac{1}{s+4}\right) - \left[e^{-4t} u(t)\right]_{t=0}$$

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = \frac{s}{s+4} - 1 . \underbrace{u(0)}_{\rightarrow 0}$$

$$= \frac{s}{s+4} \rightarrow (5)$$

Using (4) & (5) in (3)

$$\Rightarrow \mathcal{L}\left[(D+1) f(t)\right] = \frac{s}{s+4} + \frac{1}{s+4} = \frac{s+1}{s+4} \rightarrow (6)$$

Using (1) in (2)

$$\Rightarrow LHS = \left[s^2 Y(s) - s(2) - 1\right] + 5\left[sY(s) - 2\right] + 6Y(s)$$

$$\Rightarrow LHS = Y(s)\left[s^2 + 5s + 6\right] - 2s - 11$$

Now

LHS = RHS

$$\Rightarrow Y(s)\left[s^2 + 5s + 6\right] - 2s - 11 = \frac{s}{s+4}$$

terms are coming due to initial value ⇒ fictitious forcing fn

$$\Rightarrow Y(s) = \left[\frac{s+1}{s+4} + 2s + 11\right] \times \frac{1}{s^2 + 5s + 6}$$

$$\Rightarrow (s^2 + 5s + 6) Y(s) = \left(\frac{s+1}{s+4}\right) + (2s + 11)$$

4 Energy dissipation & Energy storing

→ 2 essential components of any phenomena in universe

→ wherever ∃ differentiation term, it is related to energy dissipation & energy storing

delay

→ whatever was starting value is taken

⇓

Extra

The significance of initial value in D.E

$$\text{Eg, } (s^2+5s+6)\,Y(s) = s+1 + \frac{(2s^2+19s+44)}{(s+4)}$$

$$\Rightarrow (s^2+5s+6)\,Y(s) = \frac{2s^2+20s+45}{(s+4)}$$

$$\Rightarrow Y(s) = \frac{2s^2+20s+45}{(s+2)(s+3)(s+4)}$$

$$\left[\equiv \frac{N(s)}{D(s)} \right]$$

$N(s) = 0$: zeros

$D(s) = 0$: Poles

$$= \frac{k_1}{s+2} + \frac{k_2}{s+3} + \frac{k_3}{s+4}$$

Solving By Partial Fraction,

$$k_1 = \frac{13}{2},\ k_2 = -3,\ k_3 = -\frac{3}{2}$$

$\Rightarrow y(t) = L^{-1}(Y(s))$ = Req^d Sol^n of D.E

$$\Rightarrow y(t) = k_1 e^{-2t} + k_2 e^{-3t} + k_3 e^{-4t}$$

$$\left(\because L\left(\frac{k_1}{s+2}\right) = k_1\, L\left(\frac{1}{s+2}\right) = k_1 e^{-2t} \right)$$

$$\Rightarrow y(t) = \frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t}.$$

Ans

BLOCK DIAGRAM & SYSTEM REALIZATION using LAPLACE DOMAIN

*** BASIC POINTS :**

(i) Laplace Domain : means → 's' domain. (complex frequency $= \sigma + j\omega$)

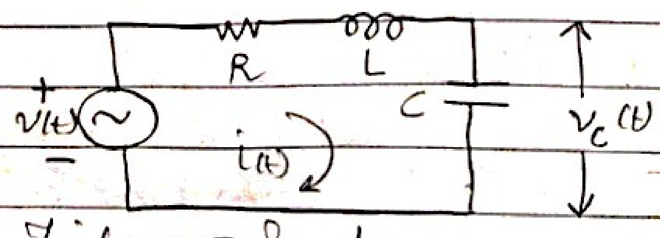
(ii) Block diagram : o/p & i/p rel^nship after applying LT to the D.E of particular physical sys. & assuming that initial values are relaxed.

(iii) Such block diagrams as explained in pt(ii) will ultimately lead to the concept of Transfer fn (TF) which is defined as

$$\text{TF}, \boxed{T_{(s)} = \frac{\text{LT of o/p or response}}{\text{LT of i/p or forcing } f^n}}.$$

↳ subject to the cond^n : initial values are relaxed (zero).

(iv) On the basis of pts. (ii) & (iii) following example is considered :-



i/p = v(t)

o/p = $V_c(t)$

Find : TF & draw concerned block diagram.

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Solⁿ:- We want relⁿ b/w $v(t)$ & $v_c(t)$. For that:

Step 1) To get $i(t)$ from $v(t)$

KVL

$$v(t) = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

Applying LT & relaxing initial condⁿˢ

$$V(s) = R\,I(s) + Ls\,I(s) - Li(0) + \frac{1}{sC}\int i\,dt\Big|_{t=0} + \frac{1}{sC}\int idt \longrightarrow (1)$$

$$\left(\begin{array}{l}\text{If } f(t) \xrightarrow{\mathcal{L}} F(s), \text{ then} \\ \mathcal{L}\left\{\int f(t)\,dt\right\} = \dfrac{F(s)}{s} + \dfrac{1}{s}f^{-1}(0) \\ \qquad \hookrightarrow f^{-1}(0) = \int f(t)\,dt\Big|_{t=0}\end{array}\right)$$

where $L(i_{(t)}) = I(s)$

& $L(v(t)) = V(s)$

Relaxing initial condⁿˢ:-

So, $i(0) = 0 \longrightarrow$ Initial condⁿ (1) IC

$$\frac{1}{C}\int i\,dt\Big|_{t=0} = 0 = \frac{q(0)}{C} = v_c(0)$$

$\hookrightarrow$ IC (2)

Substituting IC (1) & IC (2) in eqⁿ (1), we get

$$V(s) = \left(R + sL + \frac{1}{sC}\right)I(s) - L(0) + \frac{1}{s}(0)$$

or $$V(s) = \left(R + sL + \frac{1}{sC}\right)I(s)$$

or $$I(s) = \left[\frac{sC}{RCs + LCs^2 + 1}\right]V(s)$$

TF of 1st stage

$$\Rightarrow \frac{I(s)}{V(s)} = T_1(s) = \left[\frac{sC}{RCs + LCs^2 + 1}\right]$$

Open loop Block diagram

$$V(s) \longrightarrow \boxed{T_1(s)} \longrightarrow I(s)$$

(v) Step 2: Once i(t) has been developed/produced/generated, it will act as i/p to the capacitor, to produce a voltage $= V_c(t)$.

Now,

$$V_c(t) = \frac{1}{C}\int i(t)\,dt \longrightarrow (2)$$

$\therefore$ Applying LT to eq^n (2)

$$\Rightarrow V_c(s) = \frac{1}{CS} I(s) - \frac{1}{CS} q(0)$$

$$= \frac{1}{CS} I(s) - \frac{V_c(0)}{S}$$

0

$$\Rightarrow V_c(s) = \frac{1}{CS} I(s)$$

$$\Rightarrow \frac{V_c(s)}{I(s)} = \frac{1}{sC} = T_2(s)$$

So, final block diagram:-

open loop block diagram

$$V(s) \longrightarrow \boxed{T_1(s)} \xrightarrow{I(s)} \boxed{T_2(s) = \frac{1}{sC}} \xrightarrow{V_c(s)}$$

Hence, final TF →

$$\frac{V_C(s)}{V(s)} = T(s)$$

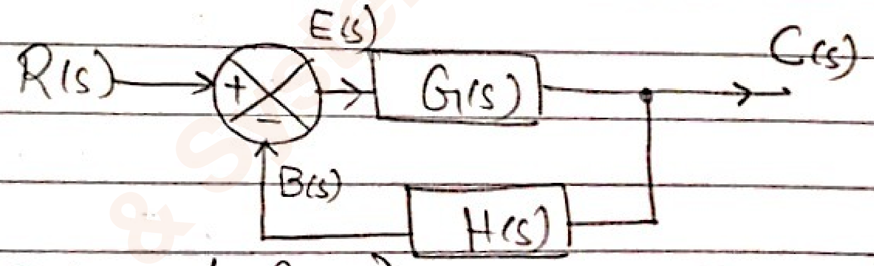
Applying cascading rule to block diagram (Rule 2 : control system).

So, $T(s) = \dfrac{V_C(s)}{V(s)} = \dfrac{V_C(s)}{I(s)} \times \dfrac{I(s)}{V(s)}$

$\Rightarrow T(s) = T_1(s) \times T_2(s)$

(vi) Block diagram in pt. (V) is open loop block diagram.

(vii) Std. Closed loop block diagram.



$R(s) = \mathcal{L}\,(r(t))$ $\qquad B(s) = C(s)\,H(s)$

$C(s) = \mathcal{L}\; c(t)$ $\qquad C(s) = G(s)\,E(s)$

$E(s) = R(s) - B(s)$: for −ve feedback

↳ $\cancel{e(t) = r(t) - b(t)}$

↳ Block diagram approach only existing in LT domain. Not possible in time domain. ($\because$ initial values relaxed)

Ultimately, by algebraic manipulation, we get

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}.$$

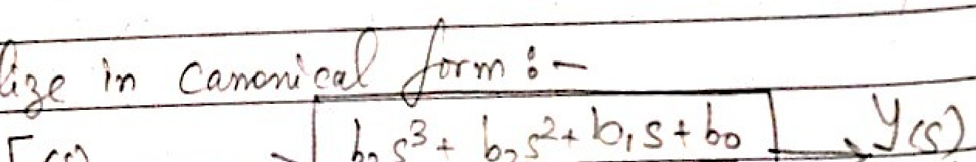
↳ Rule 9 of control system.

✩ SYSTEM REALIZATION USING CANONICAL FORM :-

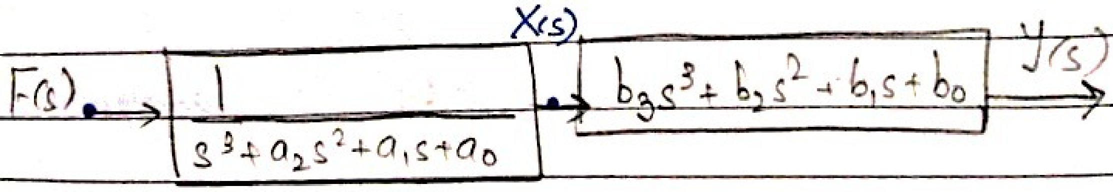
* LTIC system :-

$$H(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

eg (1) - Realize in canonical form :-



Sol^n



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Realizn of (Part A)

Now,

$$Y(s) = (b_3 s^3 + b_2 s^2 + b_1 s + b_0) X(s) \longrightarrow (a)$$

$$\Rightarrow X(s) = \frac{1}{(b_3 s^3 + b_2 s^2 + b_1 s + b_0)} \times Y(s)$$

Also,

Realizn of (Part B)

$$X(s) = \left(\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}\right) F(s) \longrightarrow (b)$$

PART B

$$\Rightarrow F(s) = s^3 X(s) + a_2 s^2 X(s) + a_1 s X(s) + a_0 X(s)$$

$$\Rightarrow f(t) = (D^3 + a_2 D^2 + a_1 D + a_0)\, x(t)$$

$$\left[\begin{array}{l} \text{If } f(t) \xrightarrow{\mathcal{L}} F(s) \\ \text{Then, } \mathcal{L}\left(\frac{d^3 f(t)}{dt^3}\right) = s^3 F(s) \; \underbrace{- s^2 f(0) - s f'(0) - f''(0)}_{\to 0} \\ \text{(Relaxing initial cond}^{ns}) \to \\ \Rightarrow \mathcal{L}(D^3 f(t)) = s^3 F(s) \\ \text{|||}^{ly}, \mathcal{L}(D^2 f(t)) = s^2 F(s) \\ \& \; \mathcal{L}(D f(t)) = s F(s) \end{array}\right]$$

Extra Idea:

$$\mathcal{L}\left(\int f(t)\,dt\right) = \frac{1}{s} F(s) - \frac{1}{s} f^{-1}(0)$$

Relaxing initial cond^{ns}:

$$\Rightarrow \mathcal{L}\left(\int f(t)\,dt\right) = \frac{1}{s} F(s)$$

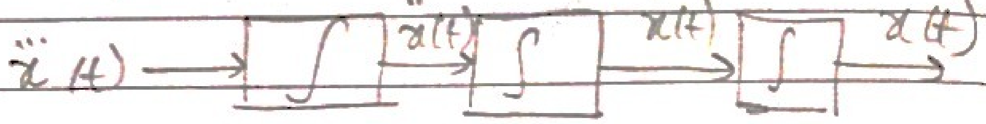
$$\mathcal{L}\left(\iint f(t)\,dt\right) = \frac{1}{s^2} F(s)$$

$$\Rightarrow f(t) = \dddot{x}(t) + a_2 \ddot{x}(t) + a_1 \dot{x}(t) + a_0 x(t)$$

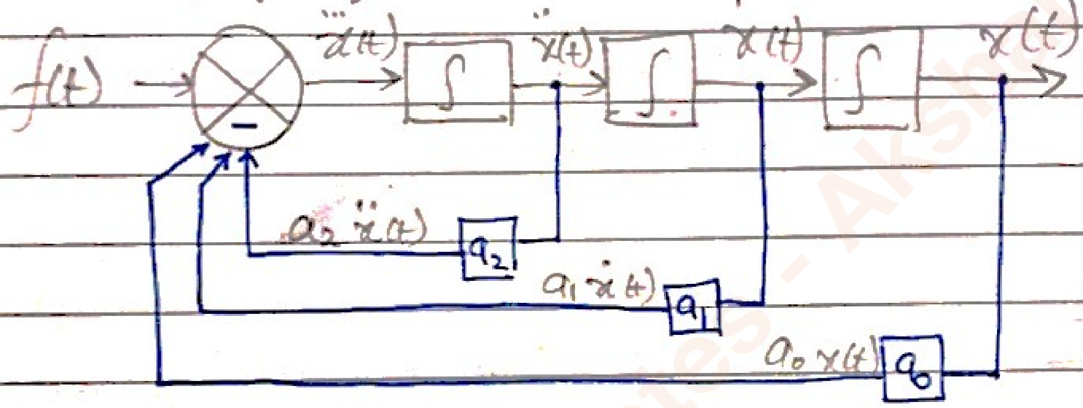
$$\left[\begin{array}{l} D^3 x(t) = \dddot{x}(t) \\ D^2 x(t) = \ddot{x}(t) \\ D\, x(t) = \dot{x}(t) \end{array}\right]$$

$$\Rightarrow \dddot{x}(t) = -a_2 \ddot{x}(t) - a_1 \dot{x}(t) - a_0 x(t) + f(t) \longrightarrow (1)$$

Now, Idea to generate $x(t)$ from $\dddot{x}(t)$ is as shown,



Realizing eq^n (1)



how to do?

↳ In canonical form realiz^n,

- See highest power of DE (D^3 above)
- write its term in terms of other terms.
- write i/p first & then, using a summing pt, make final circuit.

☆ Note :- We don't have $\dddot{x}(t)$, $\ddot{x}(t)$ & $\dot{x}(t)$ directly. So, they are generated as shown above.

☆ Note :- In canonical form :- Always the initial cond^{ns} are relaxed (by defin^n).

Now, using same idea as before,

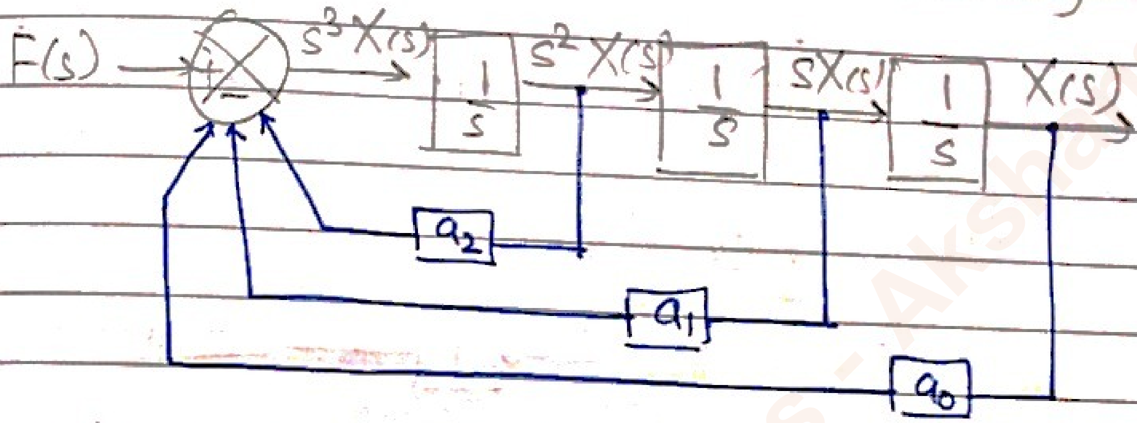
$$\mathcal{L}\left(\dddot{x}(t)\right) = \underbrace{s^3X(s)} - \underbrace{s^2\cancel{x(0)} - s\cancel{x(0)}}_{\text{(neglected)}} - s^2\cancel{x(0)}$$

Canonical form realizⁿ is involved with first realizing eqⁿ ① as shown ↙, then, realizing eqⁿ ⓑ.

for that eqⁿ ⓑ :-

$$F(s) = s^3 X(s) + a_2 s^2 X(s) + a_1 s X(s) + a_0 X(s)$$

$$\Rightarrow s^3 X(s) = (+F(s)) + (-a_2 s^2 X(s) - a_1 s X(s) - a_0 X(s))$$



Basic Idea — highest order is taken as i/p. Then using integrator, the order is reduced & with this, reqᵈ eqⁿ is reached.

PART A :- Just like above, it can also be solved using eqⁿ ⓐ.

—x—

Q. Find the canonical realizⁿ :-

$$X(s) = \frac{5}{s+2}.$$

Solⁿ $\frac{5}{s+2} = \left(\frac{1}{s+2}\right)5 \longrightarrow$ eqⁿ (A)

Solving for 1st part.

$$X(s) = \left(\frac{1}{s+2}\right) F(s)$$

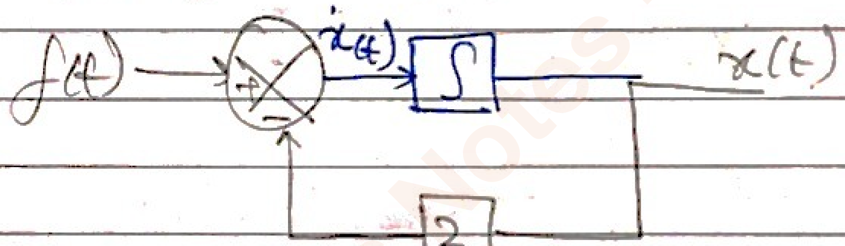
$$\Rightarrow (s+2) X(s) = F(s) \longrightarrow (1)$$

$$\Rightarrow \dot{x}(t) + 2x(t) = f(t).$$

$$\Rightarrow \dot{x}(t) = f(t) - 2x(t) \longrightarrow (2)$$

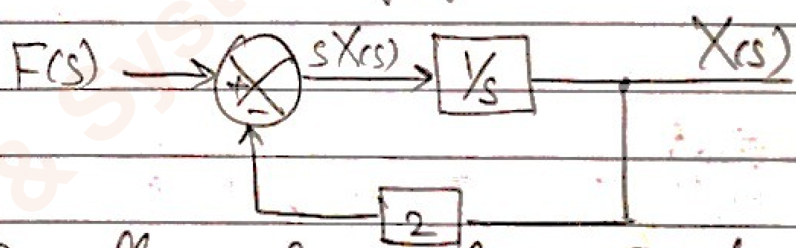
Realizing eq^n. (2).

fig (1)



↓ Changing it to s domain

fig (2)



(Basically, making figure (1) & applying the following to get fig (2) :-

$$\int \longrightarrow 1/s.$$

$$x(t) \longrightarrow X(s)$$

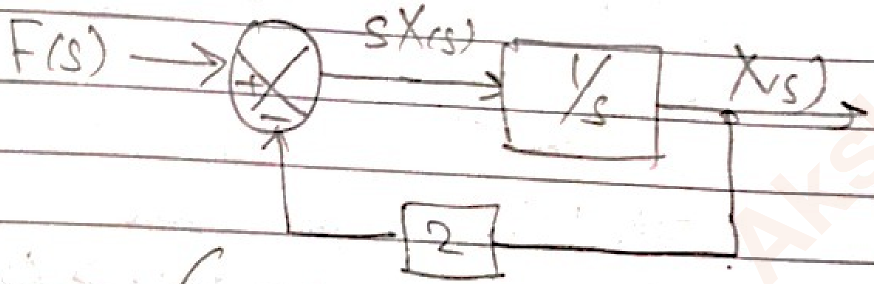
$$\dot{x}(t) \longrightarrow sX(s)$$

$$\ddot{x}(t) \longrightarrow s^2 X(s)$$
)

Now, we can realize eq^n ① directly as, from eq^n ①,

$$sX(s) = F(s) - 2X(s)$$

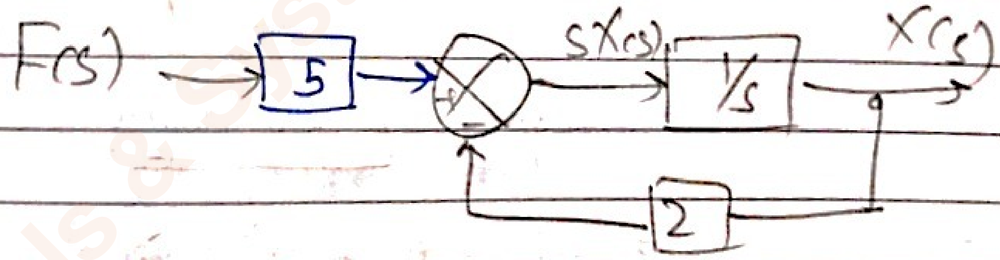
↓ using it.



final eq^n realiz^n. → eq^n (A).

(i.e) X(5) of F(s)

=



* $s = \sigma + j\omega$; σ : responsible for damping.
Fourier Trans.: ether medium → damping → 0.
so, only ω is there.

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Ch - Z-TRANSFORM

↳ only in Discrete domain eg. lift, crane
(Cts domain :- eg. motor → running continuously)

MAIN PTS.

* INTRODUCTION

↳ (1) Transformⁿ from Differential eqⁿ → Algebraic Eqⁿ
(Time domain)

(2) Applicⁿˢ : Mainly in Discrete Time Sys.
(LT : In Cts. time sys)

(3) Advantages : (a) Initial condⁿˢ are directly incorporated.
(b) System dynamics :- analysis better.

(4) Expression or definⁿ :-

Z Transform or Direct Transform : $F(z) = \sum_{K=0}^{\infty} f(K) z^{-k}$

→ k : sampling or Instance or Cutting.
→ Relⁿ with cts domain : $t = kT$
→ $f(k)$: sequence

(5) Inverse z-transform :-
$f(K) = Z^{-1}[F(z)]$

* Fourier Transform assumes resistance/damping $\to 0$
In TV transmission, from operator to us, $\exists$ no resistance. So, FT is seen there.

(6) Examples:

(a) find $F(z)$ for $f(k) = \delta(k)$

Soln:- $F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$

$= \sum_{k=0}^{\infty} \delta(k) z^{-k}$

Now, $\boxed{\delta(k) = \begin{cases} 1 \to k=0 \\ 0 \to \text{otherwise} \end{cases} \quad \hookrightarrow \delta(k) = \delta(t)\big|_{t=k}}$ *

$\Rightarrow F(z) = \delta(0) z^0 + \delta(1) z^{-1} + \delta(2) z^{-2} + \dots$

$= \delta(0) z^0 + 0 + 0 + \dots$

$= 1.$

(7) (b) $Z(u(k)) = ?$; $\boxed{u(k) = u(t)\big|_{t=k} = \begin{cases} 1, k \geq 0 \\ 0, k < 0 \end{cases}}$ *

$\Rightarrow F(z) = \sum_{k=0}^{\infty} u(k) z^{-k}$

$= u(0) z^0 + u(1) z^{-1} + u(2) z^{-2} + \dots$

$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$

M3: Taylor Series.

M2: $z = 1/x$

So, by Binomial expansion $= (1-x)^{-1}$

M1: Its Geometric Progression to infinity; Sum $= \frac{a}{1-r}$

$\hookrightarrow$ a: First term

r: common ratio

$= \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$; $\underline{|z| > 1}$

* $e^{j\theta}$: rotating phasor.
: constt magnitude → 1 $(\cos^2\theta + \sin^2\theta)$

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(c) $f(k) = e^{\pm j\beta k}$; find $F(z)$

Sol^n:- $F(z) = \dfrac{z}{z - e^{\pm j\beta}}$; s.t $|z| > |e^{\pm j\beta}|$

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} e^{\pm j\beta k} z^{-k}$$

$$= 1 + \frac{e^{\pm j\beta}}{z} + \frac{e^{\pm 2j\beta}}{z^2} + \frac{e^{\pm 3j\beta}}{z^3} + \dots$$

let $e^{\pm j\beta} = a$.

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \dots$$

let $\dfrac{a}{z} = \dfrac{1}{x}$.

$$= 1 + x + x^2 + x^3 + \dots$$

$$= (1 - x)^{-1}$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left(\frac{z - a}{z}\right)^{-1} = \frac{z}{z - a}$$

$$= \frac{z}{z - e^{\pm j\beta}} ; \quad |z| > |e^{\pm j\beta}|$$

Q. Find z-transform of $\cos\beta k$.

Sol^n:- $\cos\beta k = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$, $\theta = \beta k$

So, $Z[\cos\beta k] = \frac{1}{2} Z[e^{j\beta k}] + \frac{1}{2} Z[e^{-j\beta k}]$

* FT $t \longrightarrow \omega$

* LT $t \longrightarrow s$

* Z-T $k \longrightarrow z$

kernel

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So, $Z[\cos\beta k] = \frac{1}{2}\left[\frac{z}{z-e^{j\beta}}\right] + \frac{1}{2}\left[\frac{z}{z-e^{-j\beta}}\right]$

from 6 (c)

$$= \frac{z[z-\cos\beta]}{z^2-2z\cos\beta+1} \quad ; |z|>1$$

how?

$$\frac{1}{2}\left[\frac{z}{z-e^{j\beta}} + \frac{z}{z-e^{-j\beta}}\right]$$

$$= \frac{1}{2}z\left[\frac{z-e^{-j\beta}+z-e^{j\beta}}{(z-e^{j\beta})(z-e^{-j\beta})}\right]$$

$$\left[\begin{array}{l} e^{j\theta} = \cos\theta + j\sin\theta \\ e^{-j\theta} = \cos\theta - j\sin\theta \\ \Rightarrow e^{j\theta}+e^{-j\theta} = 2\cos\theta \\ \Rightarrow \boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}} \\ \& \; e^{j\theta}-e^{-j\theta} = 2j\sin\theta \\ \Rightarrow \boxed{\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}} \end{array}\right]$$

$$= \frac{1}{2}z\left[\frac{2z-2\cos\beta}{z^2+1-z(e^{j\beta}+e^{-j\beta})}\right]$$

$$= \frac{1}{2}z\left[\frac{2z-2\cos\beta}{z^2+1-z[2\cos\beta]}\right]$$

$$\Rightarrow Z[\cos\beta k] = \frac{z[z-\cos\beta]}{z^2-2z\cos\beta+1}$$

Ans

Z-Trans : Discrete control engineering model

LT applic^{ns} : Cts sys control engineering model & sol^n of DE

Q(1) Find Z-transform of $\sin\beta k$.

Ans :- $\dfrac{z\sin\beta}{z^2-2z\cos\beta+1}$

$$\sin\beta k = \frac{e^{j\beta k}-e^{-j\beta k}}{2j}$$

$$Z[\sin\beta k] = Z\left[\frac{e^{j\beta k}-e^{-j\beta k}}{2j}\right]$$

$$= \frac{1}{2j}\left(\frac{z}{z-e^{j\beta}}\right)-\frac{1}{2j}\left(\frac{z}{z-e^{-j\beta}}\right)$$

$\hookrightarrow z > |e^{\pm j\beta}|$

$$= \frac{z}{2j}\left(\frac{z-e^{-j\beta}-z+e^{j\beta}}{(z-e^{j\beta})(z-e^{-j\beta})}\right)$$

$$= \frac{z}{2j}\left[\frac{2j\sin\beta}{z^2-z(e^{j\beta}+e^{-j\beta})+1}\right]$$

$$= \frac{z\sin\beta}{z^2-z(2\cos\beta)+1}$$

$$\Rightarrow Z[\sin\beta k] = \frac{z\sin\beta}{z^2-2z\cos\beta+1} \quad \text{H.P}$$

PROPERTIES

(1.) LINERITY : If

$f_1(k) \longrightarrow F_1(z)$

& $f_2(k) \longrightarrow F_2(z)$

Then,

$$[af_1(k) \pm bf_2(k)] \underset{Z^{-1}}{\overset{Z}{\rightleftharpoons}} aF_1(z) \pm bF_2(z)$$

* Seeing similarity of P② in LT

$\mathcal{L} f(t) = \int_0^\infty e^{-st} f(t)\, dt$

So, $\mathcal{L}\, t f(t) = \frac{-\partial}{\partial s} F(s) = -\frac{d}{ds} F(s)$

kernel

② Multiplicⁿ by "k"

If $f(k) \longleftrightarrow F(z)$

Then,

$$[k f(k)] \longleftrightarrow -z \frac{d}{dz} F(z)$$

→ Proof :-

$Z[k f(k)]$

$= \sum_{k=0}^{\infty} k f(k) z^{-k}$

$= -\sum_{k=0}^{\infty} (-1) k f(k) z^{-k-1} z^{+1}$

why ?

Idea :- We have $k f(k) z^{-k}$

Now, ∵ $k f(k)$ is there, ∃ match of differentiation:

eg. $\frac{d}{dx} x^n = n x^{n-1}$

So, $\frac{d}{dz} z^{-k} = -k z^{-k-1}$

So, thinking like this makes us see the possibility of bringing differentiation fⁿ

★ Thought process on seeing a fⁿ : $a x^{-a}$ $(-k z^{-k})$

$a \cdot x^{-a} = (-1)(-a x^{-a}) = (-1)[-a x^{-a-1}][x^1] = (-1)\left(\frac{d}{dx} x^{-a}\right)(x) = -x \frac{d}{dx} x^{-a}$

$= -\sum_{k=0}^{\infty} f(k) \frac{d}{dz} z^{-k} (z^{+1})$

$= -z \frac{d}{dz} \sum_{k=0}^{\infty} f(k) z^{-k}$

$= -z \frac{d}{dz} F(z)$

H.P

③ Multiplicⁿ by a^k (Scale Change)

If $f(k) \longleftrightarrow F(z)$

then,

$$Z[a^k f(k)] = F\left(\frac{z}{a}\right)$$

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✯ Standard Table in z-transform

$f(k)$	$f(z)$
$\delta(k-j)$	z^{-j}
$k\,u(k)$	$\frac{z}{(z-1)^2}$
$y^k u(k)$	$\frac{z}{z-y}$
$y^{k-1} u(k-1)$	$\frac{1}{z-y}$
$\frac{k(k-1)(k-2)\ldots(k-m+1)}{y^m \cdot m!} y^k u(k)$	$\frac{z}{(z-y)^{m+1}}$

Problem

Find $z^{-1}\left[\frac{8z-19}{(z-2)(z-3)}\right]$ → discrete roots (poles)

$$\frac{8z-19}{(z-2)(z-3)} = \frac{k_1}{z-2} + \frac{k_2}{z-3}$$ (Partial fraction)

$$k_1 = \left.\frac{8z-19}{z-3}\right|_{z=2} = \frac{16-19}{-1} = 3$$

$$k_2 = \left.\frac{8z-19}{z-2}\right|_{z=3} = \frac{24-19}{3-2} = 5$$

$$\Rightarrow \frac{8z-19}{(z-2)(z-3)} = \frac{3}{z-2} + \frac{5}{z-3}$$

$(a^2-2ab+b^2)(a-b)$

$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$

$a^3 - b^3 - 3a^2b + 3ab^2$

Hence, $F(z) = \dfrac{3}{z-2} + \dfrac{5}{z-3}$

$\Rightarrow f(k) = \underbrace{Z^{-1}[F(z)]}_{\text{using std. table.}} = \left[3\cdot(2)^{k-1} + 5(3)^{k-1}\right]u(k-1)$

Problem $F(z) = \dfrac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$

Find $f(k)$ i.e $Z^{-1}F(z)$

The nature of problem: combin^n of discrete & repeated real roots

(M1) Method of equating coeff

Sol^n: $\dfrac{F(z)}{z} = \dfrac{2z^2-11z+12}{(z-1)(z-2)^3}$

Using partial fraction

$\Rightarrow \dfrac{2z^2-11z+12}{(z-1)(z-2)^3} = \dfrac{k_1}{z-1} + \dfrac{k_2}{(z-2)^3} + \dfrac{k_3}{(z-2)^2} + \dfrac{k_4}{z-2}$

$k_1 = \left.\dfrac{2z^2-11z+12}{(z-2)^3}\right|_{z=1} = \dfrac{2-11+12}{(1-2)^3} = -3 \longrightarrow (a)$

Also, $2z^2-11z+12 = k_1(z-2)^3 + k_2(z-1) + k_3(z-1)(z-2) + k_4(z-1)(z-2)^2$

$$\Rightarrow \left.\begin{aligned} 0z^3 &= z^3(k_1+k_4) \\ +2z^2 &\quad + z^2(-6k_1+k_3-5k_4) \\ -11z &\quad + z(+12k_1+k_2-3k_3+8k_4) \\ +12 &\quad + (-8k_1-k_2+2k_3-4k_4) \end{aligned}\right\} \rightarrow (1)$$

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Now $k_2 = \left.\dfrac{2z^2-11z+12}{z-1}\right|_{z=2} = \dfrac{8-22+12}{1} = -2 \rightarrow$ (b)

Also, equating the coeff. from eq^n (1)

$\Rightarrow k_1 + k_4 = 0 \Rightarrow -3 + k_4 = 0 \Rightarrow \boxed{k_4 = 3} \rightarrow$ (c)

$-6k_1 + k_3 - 5k_4 = 2$

$\Rightarrow 18 + k_3 - 5(3) = 2$

$\Rightarrow 3 + k_3 = 2 \Rightarrow \boxed{k_3 = -1} \rightarrow$ (d)

From (a), (b), (c) & (d),

$$\frac{2z^2-11z+2}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$$

$$\Rightarrow F(z) = -3\left[\frac{z}{z-1}\right] - 2\left[\frac{z}{(z-2)^3}\right] - 1\left(\frac{z}{(z-2)^2}\right) + 3\left(\frac{z}{z-2}\right)$$

$$\Rightarrow z^{-1}(F(z)) = u(k)\left\{\left[-3(1)^k\right] - 2\left[\frac{k(k-1)}{2^2 \cdot 2!}(2^k)\right] - (1)\frac{k\,2^k}{2^1 \cdot 1!} + 3(2^k)\right\}$$

$$\Rightarrow f(k) = u(k)\left[-3 - \frac{k(k-1)}{4}\cdot 2^k - \frac{k\,2^k}{2} + 3\cdot 2^k\right]$$

Ans

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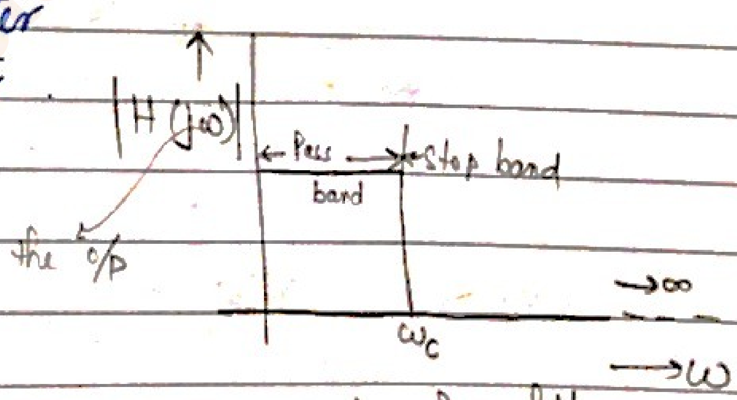
8 INTRO TO ANALOG FILTER DESIGN

to be read through from provided course material

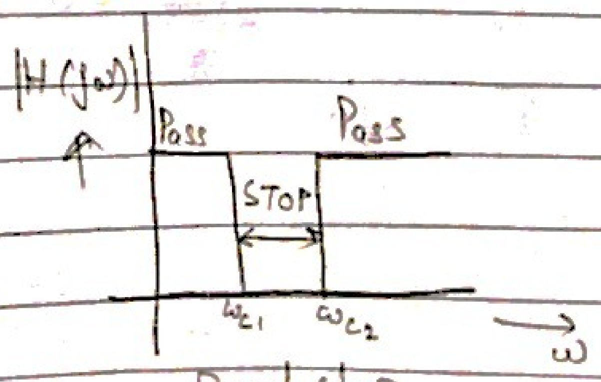
FILTERS: Selection of frequency ~~filter~~.
allows i/p signal to pass or block

Low Pass — High Pass — Band Pass — Band Stop.

Low Pass
- ① Butterworth filter
- ② Chebyshev's filter
- ③ ~~Electrical~~ Elliptical filter.



Low Pass filter



Band stop

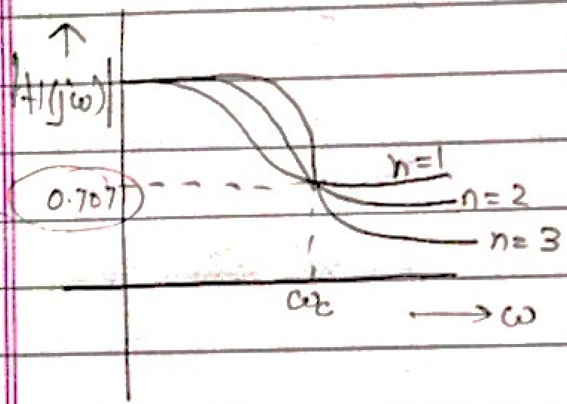
H is coming $\exists$ impulse i/p (i/p : $\delta(t)$)

So, $\frac{C(s)}{\mathcal{L}(\delta(t))} = H(s) \Rightarrow \frac{C(s)}{1} = H(s)$. So, response $= H(s)$

$$\star \; H(j\omega) = H(s)\Big|_{s=j\omega} = \text{response}$$

& " $|H(j\omega)|$ vs ω " : Amplitude response

① Butterworth filter



: n^{th} order Butterworth filter

$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

→ Background $\left(0.707 = \frac{1}{\sqrt{2}}\right)$

Consider an RLC series ckt

Driving point impedance $= Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

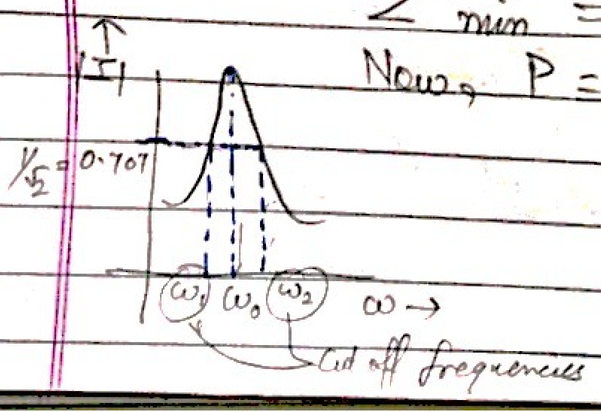
↳ when $\omega L = \frac{1}{\omega C}$, resonance condⁿ,

$\Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$

So, at resonance, Z is min. $(= R)$

Z min $\Rightarrow |I|$ shoots up.

Now, $P = |I|^2 R$ & so, $\frac{1}{2} P = \frac{1}{2} |I|^2 R$



$= \left(\frac{|I|}{\sqrt{2}}\right)^2 R$

$= \left(\frac{1}{\sqrt{2}} |I|\right)^2 R$

★ The factor that comes

end of course